Local topological toughness and local factors

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April 20, 2006

Abstract

We localize and strengthen Katona's idea of an edge-toughness to a local topological toughness. We disprove a conjecture of Katona concerning the connection between edge-toughness and factors. For the topological toughness we prove a theorem similar to Katona's $2k$-factor-conjecture, which turned out to be false for his edge-toughness. We prove, that besides this the topological toughness has nearly all known nice properties of Katona's edge-toughness and therefore is worth to be considered.

1 Preliminaries and Results

For notations not defined here we refer to [2]. Unless otherwise stated, t is an arbitrary non negative real number, k is an arbitrary integer, G is an arbitrary finite graph (loops and multiple edges allowed), U is an arbitrary subgraph of G, X and H are arbitrary disjoint subsets of $V(G)$, Y is an arbitrary subset of $E(G - X - H)$, and f is an arbitrary function that maps H into the positive integers. A cycle covering H is called *H-cycle*. The union of internally disjoint H-paths is called *H-local k-factor*, if all vertices of H have degree k in it, *partial H-local k-factor*, if all vertices of H have at most degree k in it, *H-local f-factor* if each vertex h of H has degree $f(h)$ in it, and *partial H-local f-factor* if each vertex h of H has at most degree $f(h)$ in it. The *size* of H-local factors is the number of its H-paths. The maximum number of internally disjoint H-paths we denote by $p_G(H)$. With $G[X]$ we denote the subgraph of G induced by X, $[Y]$ denotes the graph with edge set Y whose vertex set is the set of all vertices incident with edges of Y. Instead of $G[V([Y])]$ we shortly write $G[Y]$. $E'(G)$ denotes the set of all edges in G except the loops. Let $\mathcal{C}(G)$ denote the set of components of G and $\partial_G(U)$ denote the set of vertices of U incident with edges of $G - E(U)$. For $V(U) - \partial_G(U)$ we will write shortly $in_G(U)$. According to [10] we define the *permeability*

*Research supported by the "Mathematics in Information Society" project carried out by Alfréd Rényi Institute of Mathematics - Hungarian Academy of Sciences, in the framework of the European Community's "Confirming the International Role of Community Research" programme. e-mail: frank.goering@mathematik.tu-chemnitz.de

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of a pair (X, Y) by:

$$\mathrm{perm}_G(X,Y) = |X| + \sum_{C \in \mathcal{C}([Y])} \left\lfloor \frac{|\partial_{G-X}(C)|}{2} \right\rfloor$$

The following definitions generalize this concept:

Let G be a graph, and f be a function mapping $H^* \subseteq V(G)$ into the set of positive integers. An *f-separator* of G is a pair (X, Y) with $X \subseteq V(G)$, $Y \subseteq E(G-X)$ and $\partial_{G-X}Y$ disjoint to H^* such that $G - X - Y$ has no H^*-paths.

The *permeability* of an f-separator is

$$\mathrm{perm}_{G,f}(X,Y) = |X \setminus H^*| + \sum_{v \in X \cap H^*} f(v) + \sum_{C \in \mathcal{C}([Y])} \left\lfloor \frac{1}{2} \left(|\partial_{G-X} C| + \sum_{v \in V(C) \cap H^*} f(v) \right) \right\rfloor$$

In 1997 the second coauthor introduced the concept of edge-toughness (see [8]). It is strengthening the concept of toughness introduced by Chvátal in 1971. We will define these concepts.

G is *t-tough* (in the sense of Chvátal, cf. [1]) if deleting of k vertices of G results in at most $\max\{1, \frac{k}{t}\}$ components. The *toughness of* G (denoted by $t(G)$) is the supremum over all reals t such that G is t-tough. It turns out that $t(G) = \infty$ if any two vertices of G are adjacent, and $t(G) = \min\left\{ \frac{|X|}{|\mathcal{C}(G-X)|} \;\middle|\; 2 \leq |\mathcal{C}(G-X)| \right\}$ otherwise.

G is *t-edge-tough* if for all (X, Y) with $[Y] = G[Y]$ we have:

$$|\mathcal{C}(G - X - Y - in_{G-X}([Y]))| \leq \max\{1, \frac{\mathrm{perm}_G(X,Y)}{t}\}$$

The *edge-toughness of* G (denoted by $te(G)$) is the supremum over all reals t such that G is t-edge-tough.

The ideas are as follows: Every path passing a vertex of X needs another one of these vertices, every path not counted by this and passing a component C of [Y] needs two more vertices of the boundary of C in $G-X$. If G is hamiltonian, then deleting X, Y and $in_{G-X}(Y)$ therefore results in at most $\mathrm{perm}_G(X,Y)$ components incident with edges not contained in Y. This idea gives Chvatal's toughness if we restrict (X, Y) to that pairs with $Y = \emptyset$. Otherwise – as defined before – we get edge-toughness.

We are interested in a local toughness concept, since the topic of the existence of cycles through prescribed vertices of a graph seems to be of interest (cf. [6, 7, 5, 3, 10, 11]).

Local versions of Katona's edge toughness and Chvatal's toughness are naturally defined as follows:
H is called *k-edge-tough* (or *k-tough*) *in* G if for all (X, Y) with $[Y] = G[Y]$) (or (X, Y) with $Y = \emptyset$) the graph $G - X - Y$ has at most $\max\{1, \frac{\mathrm{perm}_G(X,Y)}{k}\}$ components containing a vertex of H. The local version of Chvatal's toughness occurs for instance in [5].

What would be useful properties we expect from a local version of toughness? First of all, we should be sure that in a graph G for a subset H of its vertices not being 1-tough in G the graph G contains no H-cycle. Second, if a set H is k-tough in G, then every at least two element subset of H should be k-tough, too. Third, the toughness of H in G should not depend on the length of paths in G, the inner vertices of which have degree 2 in G.

Obviously the local versions of the mentioned toughnesses fulfill the first and the second condition, but break the third. The latter is easy to see (e.g. intersect each edge of a complete graph).

The toughness concepts we have discussed so far deal with disconnecting graphs. Our idea is complementary - it deals with connecting vertices.

Every cycle for every k element subset H of vertices has exactly k internally disjoint H-paths. This simple observation leads to the following definition: H is *topological t-tough in G* iff for all $H' \subseteq H$ with $|H'| \geq 2$ the graph G contains (at least) $t|H'|$ internally disjoint H'-paths. We chose this name because subdividing edges has obviously no effect on this value. Moreover, this definition ensures that the topological toughness fulfills all our three conditions.

If $V(G)$ itself is topological t-tough in G, we will say shorter that G is topological t-tough. The *topological toughness* of H in G is the maximal t such that H is t-tough in G, the topological toughness of G is the topological toughness of $V(G)$ in G.

We want to compare the ideas of edge toughness and topological toughness. For this we need Mader's theorem about the number of internally disjoint H-paths (cf. [12]) in G. We use it in the version of [2].

Theorem 1 (Mader, 1978) $p_G(H) = |E'(G[H])| + \min\{perm_G(X,Y) \mid \forall C \in \mathcal{C}(G - X - Y - E'(G[H])) : |V(C) \cap H| \leq 1\}$

This provides a new tool in order to prove non-hamiltonicity of some graphs. If we can present sets X, Y and H such that X, Y separates H and $p_G(H) < |H|$ then the graph cannot be hamiltonian.

Mader's theorem is often understood as a generalization of Menger's theorem (cf. [14]).

Theorem 2 (Menger, 1927) *Let a and b be nonadjacent vertices of G. The maximum number $p_G(\{a,b\})$ of internally disjoint ab-paths in G equals the minimum number of vertices of $G - \{a,b\}$ separating a from b in G.*

In [10] the concept of A-separators is introduced. In our notation we will replace A by H and call it small H-separator. For an independent set H a pair (X,Y) is called *H-separator* G has no H-path avoiding X and Y, and *small H-separator*, if additionally $\mathrm{perm}_G(X,Y) < |H|$ holds. Obviously, G can't have an H-cycle if it has a small H-separator. Our topological toughness by Theorem 1 generalizes the idea of small H-separators: An independent set H is topological 1-tough in G if and only if G has no H-separator.

For a graph G being t-edge-tough means having a system of at least $t|H|$ H-paths for certain (but not all!) subsets H of the vertex set of G. Especially using Mader's theorem one can prove easily the following lemma which classifies the edge-toughness in a connector-language:

Lemma 3 *If for each independent set H with $|H| \geq 2$ there are at least $t|H|$ internally disjoint H-paths in G, then G is t-edge-tough. If G is t-edge-tough, then for each induced subgraph U there are at least $|\mathcal{C}(U)|t$ subgraphs of G being U-paths or cycles not disjoint to U which are disjoint out of U.*

Obviously Lemma 3 combined with the definition of topological toughness leads to

Corollary 4 *Every topological t-tough graph G is t-edge-tough and therefore t-tough in Chvátal's sense.*

All the toughnesses are constructed to detect non-hamiltonicity by a toughness value less than one (which one can prove by presenting a single separator). Let NC be the set of graphs not being 1-tough, NE be the set of graphs not being 1-edge-tough and NT be the set of graphs not being topological 1-tough. The following observation tells us, that beyond the mentioned versions of toughness, topological toughness detects non-hamiltonicity best:

Observation 1 *The following holds:* $NC \subset NE \subset NT$

Replacing edge-toughness by the topological toughness unfortunately doesn't preserve the strong (linear) connection to Chvátal's toughness:

Observation 2 *Let* G *be a complete graph on* $2k^2+1$ *vertices after deleting one of it's edges. Then the toughness of* G *is* $k^2 - \frac{1}{2}$ *and the topological toughness is* $2k - \frac{3}{2}$*.*

The toughness of G is $k^2 - \frac{1}{2}$ because by deleting vertices one can only separate the endvertices of the missing edge and has to delete all other vertices for this purpose. For an h-element subset H of the vertex set of G we find at least $(2k^2+1-h) + \binom{h}{2} - 1$ internally disjoint H-paths and this bound is tight (if H contains the endvertices of the missing edge this becomes obvious). For the topological toughness of G we therefore get

$$\begin{aligned} t &= \min\left\{ \frac{\binom{h}{2} + (2k^2+1-h) - 1}{h} \;\middle|\; h = 2,\ldots,k^2+1 \right\} && (1) \\ &= -\frac{3}{2} + \min\left\{ \frac{h}{2} + \frac{2k^2}{h} \;\middle|\; h = 2,\ldots,k^2+1 \right\} && (2) \\ &= -\frac{3}{2} + \left[\frac{h}{2} + \frac{2k^2}{h} \right]_{h=2k} && (3) \\ &= 2k - \frac{3}{2} && (4) \end{aligned}$$

However, we prove the following:

Theorem 5 *If* H *is* $(4t^2+2t)$*-tough in* G *and* $|H| \geq \frac{(t+\frac{3}{2})^2}{2}$*, then* H *is topological* t*-tough in* G*.*

Here $4t^2 + 2t$ is not best possible, as it is seen in the next theorem for small values of t.

Theorem 6 *If* $t \leq 1$*,* $|H| \geq 3$ *and* H *is* $2t$*-tough in* G*, then* H *is topological* t*-tough in* G*.*

Unfortunately this implies that there is no graph which is not topological 1-tough (and therefore non-hamiltonian) but its toughness is at least 2. However, we think that further refining the ideas may lead to such constructions, even for higher toughness.

The connection between k-factors and toughness first was proved in [4]:

Theorem 7 (Enomoto, Jackson, Katerinis, Saito, 1985) *Let* k *be a positive integer and* G *be a* k*-tough graph such that* $k|V(G)|$ *is even. Then* G *has a* k*-factor.*

We know that a graph being 1-tough may have a hamiltonian cycle – which is a 2-factor (more precisely, we know that a graph not being 1-tough cannot be hamiltonian). Therefore the idea of toughness creates the conjecture that every k-?-tough graph has a $2k$-factor. This was conjectured by the second coauthor for the edge-toughness (see [9]) and proved for $k = 1$.

Unfortunately, this is not true in general.

Let C_q^p denote the p^{th} power of a cycle on q vertices. (In the p^{th} power of a cycle on vertices $v_1, \ldots, v_q$ the vertices v_i and v_j are connected iff $|i-j| \leq p$ or $|i-j| \geq q-p$.) Then take m disjoint copies of $C_{k^2+k-1}^{k-1}$ and denote these by $H_1, \ldots, H_m$. Moreover, take a complete graph K_x, where x is the largest integer satisfying $x < m\frac{k^2+k-1}{k}$ and connect each vertex of K_x to each vertex of each H_i. The resulting graph is denoted by $G_{k,m}$. It is worth of mentioning that the smallest such construction for $k=2$ is obtained from a K_7 by deleting the edges of a cycle of length 5.

Theorem 8 *If $m \geq 1$ and $k \geq 2$ are integers, then $G_{k,m}$ is k-edge-tough but has no $2k$-factor.*

Even a local version of Theorem 7 is not true.

Observation 3 *Let G be the graph obtained deleting an edge e from K_{24}, and H be a 6 element set of vertices of G containing both ends of e. Then H is 11-tough in G but G has no H-local 11-factor.*

Proof. Since there are only two independent vertices in G, clearly $\tau(G) = \frac{22}{2} = 11$.

Since H forms a nearly complete subgraph, there are $\binom{6}{2} - 1 = 14$ edges spanned by H. All other internally disjoint H-paths must use a different vertex of $V(G) \setminus H$. There are 18 such vertices, so there are only 32 internally disjoint H-paths instead of the required 33. $\square$

However, the situation changes if we consider the topological toughness:

Theorem 9 *Every topological k-tough graph has a $2k$-factor.*

This is a consequence of our main result:

Theorem 10 *A set H^* of vertices of a graph G is topological k-tough in G if and only if for every $H \subseteq H^*$ with $|H| \geq 2$ the graph G has an H-local $2k$-factor.*

This theorem is a little surprising because it says that it is sufficient to have enough H-paths for each $H \subseteq H^*$, $|H| \geq 2$, to be able to arrange them in a regular way for each such H.

To prove Theorem 10 we use the following theorem, which is equivalent to Theorem 2 in [13]:

Theorem 11 *Let G be a graph, $H^* \subseteq V(G)$ be independent in G, and f be a function that maps H^* to the positive integers. Then the maximal size of a partial H^*-local f-factor equals the minimum of $perm_{G,f}(X,Y)$ taken over all f-separators (X,Y) of G.*

Theorem 11 also has the following corollary, which provides a necessary and sufficient condition for the existence of H^*-local f-factors:

Corollary 12 *For G, H^* and f defined as in Theorem 11, G has an H^*-local f-factor if and only if for each f-separator (X,Y) we get:*

$$perm_{G,f}(X,Y) \geq \frac{1}{2} \sum_{h \in H^*} f(h)$$

Corollary 12 has the following special case ($f(h) = 2k$ for each $h \in H^*$):

Corollary 13 *Let G and H^* be defined as above. Then G has an H^*-local $2k$-factor if and only if if for each (X,Y) such that $X \subseteq V(G)$, $Y \subseteq E(G-X)$, $\partial_{G-X}Y \subseteq V(G-H^*)$, and $G-X-Y$ has no H^*-path, we get:*

$$|X \setminus H^*| + 2k|H^* \cap X| + k|H^* \cap V([Y])| + \sum_{C \in \mathcal{C}([Y])} \left\lfloor \frac{1}{2} |\partial_{G-X} C| \right\rfloor \geq k|H^*|$$

2 Proofs

We only need to prove Theorems 5, 6, 8 and 10. Since the equivalence of Theorem 11 and Theorem 3 in [13] is not easy to deduce, we will give a proof of Theorem 11, too.

Proof of Theorem 5. The proof is indirect. Suppose H is not topological t-tough in G. Then there is a set $H' \subseteq H$ with $|H'| \geq 2$ such that the maximum number of internally disjoint H'-paths in G is less than $t|H'|$. By Mader's Theorem (Theorem 1) there is a total separator (X', Y') of H' in $G - E(G[H'])$ satisfying $\mathrm{perm}_G(X', Y') + |E'(H')| < t|H'|$. Let α be the independence number of $G[H']$ and set $x = \frac{|H'|}{\alpha}$. Let $\overline{G}$ be the simple graph on H' in which two vertices are adjacent exactly if they are not adjacent in G. Clearly $\overline{G}$ is $K_{\alpha+1}$-free. By Turán's Theorem (cf. [2]) it has at most as many edges as a complete α-partite graph on $|H'|$ vertices with nearly equal partition classes, such that the sizes of classes differ by at most one. Thus $G[H']$ has at least $\alpha\frac{x(x-1)}{2}$ edges. Therefore we have $\alpha\frac{x(x-1)}{2} \leq |H'|t = \alpha x t$. This leads to $x \leq 2t+1$.

Consider first the case $\alpha \geq 2$. Let H'' be an independent subset of H' in G with $|H''| = \alpha$. Then in (X', Y'), we can replace each component C of $\mathcal{C}(Y')$ by all but one vertices of $\partial_{G-X}(C)$, and replace the edges induced by H' by the vertices of $H' \setminus H''$. This operation leads to a total separator $(X'', \emptyset)$ of H'' with $|X''| \leq 2\left(\mathrm{perm}_G(X', Y') + |E(G[H'])|\right)$. For the Chvátal-toughness t_c of G we get:

$$t_c \leq \frac{|X''|}{|H''|} = \frac{|X''|}{\alpha} = \frac{x|X''|}{|H'|} \leq \frac{2(\mathrm{perm}_G(X', Y') + |E(G[H'])|)x}{|H'|} < 2tx \leq 4t^2 + 2t.$$

Consider now the other case $\alpha = 1$. If every vertex of $H \setminus H'$ is contained in X', then $\mathrm{perm}_G(X', Y') \geq |H'| + 1$ and since $\alpha = 1$ implies that H' spans a complete graph, we obtain $t|H'| > |H| - |H'| + \binom{|H'|}{2}$. This leads to:

$$|H| < \frac{1}{2}(2t + 3 - |H'|)|H'| \leq \frac{(t + \frac{3}{2})^2}{2}.$$

Therefore, such H is not considered in the theorem we have to prove. Hence we may and shall assume that H contains a vertex v not contained in $H' \cup X'$. Furthermore, we may assume that for every component C of $[Y']$ we have $\left\lfloor \frac{|\partial_{G-X'}(C)|}{2} \right\rfloor \geq 1$, that is, every component has a nonzero contribution to the permeability of (X', Y') in G. Let X^* contain all vertices of X' and for each component C of $[Y']$ all but one element of $\partial_{G-X'}(C)$, chosen such that X^* do not contain v. Clearly, $G - X^* - E(G[H'])$ has no component containing two vertices of H', but it may have a component containing v and a vertex w of H', but no other vertex of H'. If this is the case, set $X'' = X^* \cup \{w\}$, otherwise set $X'' = X^*$. In both cases let H'' consist of

v and one element of $H' \setminus X''$. Then X'' separates H''. We get $|X''| \leq 2\mathrm{perm}_G(X', Y') + 1 < 2(t|H'| - |E'(G[H'])|) + 1 \leq 2(t|H'| - \binom{|H'|}{2}) + 1 = (2t + 1 - |H'|)|H'| + 1 < (t + \frac{1}{2})^2 + 1$. Finally, the toughness of H' in G is at most $\frac{|X''|}{2} < 4t^2 + 2t$. □

Proof of Theorem 6. It suffices to prove the following for all H: If $|H| \geq 3$ and H contains a set H' with $|H'| \geq 2$ such that G has no $t|H'|$ internally disjoint H'-paths, then G contains an independent subset H'' of at least two vertices, such that there is a set $X'' \subseteq V(G - H'')$ being a total separator of H'' with $2t|H''| > |X''|$.

If $G[H']$ is connected, then it has at least $|H'| - 1$ edges. Its edges are internally disjoint H'-paths. Therefore it also has at most $|H'| - 1$ edges. Thus it is a tree and $t > \frac{1}{2}$ holds. Furthermore, $G - E(G[H'])$ has no H'-path.

If, furthermore, $|H'| = 2$, then there is a vertex $h \in H \setminus H'$. In this case let $H' = \{h_1, h_2\}$. Clearly, either $G - h_1$ has no $\{h, h_2\}$-path or $G - h_2$ has no $\{h, h_1\}$-path. Suppose w.l.o.g. the latter is the case. Then with $H'' = \{h, h_1\}$ and $X'' = \{h_2\}$ we are done.

If, otherwise, $|H'| > 2$, then $G[H']$ has a cutvertex x. Let H'' consist of two vertices of different components of $G[H'] - x$. We are done with $X'' = \{x\}$.

So we may suppose that $G[H']$ contains at least 2 components. Choose for H'' one vertex of each of these components.

By Theorem 1 there is a set $X' \subseteq V(G - H')$ and a set $Y' \subseteq V(G - H' - X')$ such that $\mathrm{perm}_G(X', Y') + |E'(G[H'])| < t|H'|$ and $G - X' - Y' - E'(G[H'])$ has no H'-paths. Therefore, by the construction of H'', the graph $G - X' - Y'$ has no H''-paths. Furthermore, we get $\mathrm{perm}_G(X', Y') < t|H'| - |E'(G[H'])| < t|H''|$. Let X'' consist of all vertices of X' and all but one vertices of $\partial_{G-X'}(C)$ for all $C \in \mathcal{C}([Y])$. Then $|X''| \leq 2\mathrm{perm}_G(X', Y') < 2t|H''|$ but $G - X''$ has no H''-path. □

Proof of Theorem 8. First we prove that $G_{k,m}$ has no $2k$-factor. Since each H_i is $2k-2$ regular, and they are disjoint from each other, each vertex of each H_i in a $2k$-factor must send at least 2 edges to K_x, so there must be at least $2m(k^2 + k - 1)$ edges ending in K_x. However, x is given such that this is not possible, because in a $2k$-factor each vertex of K_x is incident to at most $2k$ of these edges.

Now we prove that $G_{k,m}$ is k-edge-tough. By Lemma 3 it is enough to prove that for any independent vertex set $H \subset V(G_{k,m})$ with $|H| \geq 2$ there are at least $k|H|$ internally disjoint H-paths. It is clear that H cannot contain any vertex of K_x. Let $h_i = |H \cap V(H_i)|$. Since all H_i are copies of $C^{k-1}_{k^2+k-1}$, there are at most k independent vertices in it, so $h_i \leq k$. On can easily see that if $i \leq j - k$, then there are $k - 1$ disjoint paths from v_i to v_j using only vertices $v_{i'}$ with $i < i' < j$. This implies that in each H_i there are at least $(k-1)h_i$ internally disjoint H-paths. Thus we have $d := x + \sum_{i=1}^{m}(k-1)h_i$ internally disjoint H-paths in $G_{k,m}$. Since $x > m\frac{k^2+k-1}{k} - 1$ and $\sum_{i=1}^{m} h_i \leq km$ straightforward calculation gives that $d \geq k\sum_{i=1}^{m} h_i$ holds if $m \geq 2$, which proves our claim in this case. If $m = 1$, then $x = k$ and similar argument shows that our claim holds, so the proof is complete. □

Before starting to prove Theorem 11, we add some notation and provide some Lemmas.

The *neighborhood* of a vertex v in a graph G is the set of v and all vertices adjacent to v in G and we denote it by $N_G(v)$. Call (X, Y) an *optimal H^*-separator* if (X, Y) is an H^*-separator and $\mathrm{perm}_G(X, Y) = p_G(H^*)$. If H^* is independent in G, Theorem 1 guarantees the existence of an optimal H^*-separator. The following Lemma refines this statement.

Lemma 14 *Let G' be a graph and H^* be an independent subset of $V(G')$. Then G' has an optimal H^*-separator (X, Y) satisfying the following conditions:*

1. *If $u, v, w \in V(G - H^* - X)$, $w \in N_G(v) = N_G(w)$, and $\{u, v\} \in Y$, then $\{u, w\} \in Y$.*

2. *If $u, v \in V(G - H^*)$, $N_G(u) \subseteq N_G(v)$, and $u \in X$, then $v \in X$.*

3. *For each component C of $[Y]$ the graph G' obtained from G by deleting all elements of $X \cup Y \setminus E(C)$ and contracting C to a vertex c has no pair G'_1, G'_2 of subgraphs such that $H^* \cap V(G'_2) = \emptyset$, $|V(G'_1 \cap G'_2)| = 1$, and $G'_1 \cup G'_2 = G'$ (G' has no endblock disjoint from H^*).*

Proof of Lemma 14. We give a constructive proof of the conditions, starting with an arbitrary optimal separator (X, Y). In each step of the construction we apply a transformation, which changes X and Y while $\mathrm{perm}_G(X, Y)$ is nonincreasing and (X, Y) remains an H^*-separator of G. Hence, $\mathrm{perm}_G(X, Y)$ is constant, and (X, Y) remains an optimal H^*-separator of G.

This is the general step with its three cases:

Case 1)

If (X, Y) infringes condition 1 we proceed as follows: Since having the same neighborhood in $G-H^*-X$ is an equivalence relation, it induces a partition of $V(G-H^*-X)$ into a finite set $\mathcal{A}$ of classes. All $A \in \mathcal{A}$ for which there are vertices $v, w \in A$ and a vertex $u \in N_G(v) = N_G(w)$, such that $\{u, v\} \in Y$, and $\{u, w\} \notin Y$ we call *asymmetric classes.* The vertices u we call the *asymmetric neighbors* of A, the elements of $N_G(v) \setminus v$ we call the *neighbors* of A.

Infringement of condition 1 yields at least one asymmetric class A. If there is a neighbor of A, which is not contained in $V([Y])$, we delete all edges from Y, which connect an asymmetric neighbor of A with an element of A. Otherwise we add all edges incident with vertices in A to Y.

In the first situation, some components of $[Y]$ may split, but ∂Y only may loose vertices. Hence $\mathrm{perm}_G(X, Y)$ will not increase. Furthermore, the ends of the deleted edges were connected in $G - X - Y$ before this step. Hence, an H^*-path of $G - X - Y$ after this step yields an H^*-path of $G - X - Y$ before this step. Thus, if before this step (X, Y) was an optimal H^*-separator of G, in this step it remains an optimal H^*-separator.

In the second situation, all at most $|A|$ components of $[Y]$ containing vertices of A glue together, but if there is more than one such component, then $\partial_{G-X} Y$ looses at least $|A|$ vertices.

Hence, $\mathrm{perm}_G(X, Y)$ will not increase, too. Thus, clearly, (X, Y) remains an optimal H^*-separator of G, too.

Both variants of our transformation cannot produce asymmetric neighbors of another class $A' \in \mathcal{A}$.

Hence, this step decreases the number of asymmetric classes.

Case 2)

If, otherwise, (X, Y) infringes condition 2, then there are vertices $u, v \in V(G - H)$ such that $N_G(u) \subseteq N_G(v)$, $u \in X$, and $v \notin X$. In this case, we delete u from X and add all edges to Y that connect u to a neighbor of v in Y.

In $G - X - Y$ this transformation adds u to the component which contains v, but all components remain separated. Hence (X, Y) stays an H^*-separator of G.

In $\mathrm{perm}_G(X, Y)$ the term $|X|$ decreases by one, while u is added $\partial_{G-X} C$ with C being the component of $[Y]$ containing v (which must exist, because otherwise we have a contradiction to the optimality of (X, Y) before this step). Since nothing else changes in $\mathrm{perm}_G(X, Y)$, the H^*-separator (X, Y) stays optimal.

In this case, condition 1 obviously remains satisfied, whereas $|X|$ increases.

Case 3)

If, finally, (X,Y) only infringes condition 3, the graph G' obtained from G by deleting all elements of $X \cup Y \setminus E(C)$ and contracting C to a vertex c has a pair G'_1, G'_2 of subgraphs such that $H^* \cap V(G'_2) = \emptyset$, $|V(G'_1 \cap G'_2)| = 1$, and $G'_1 \cup G'_2 = G'$.

We add all edges of G to Y which have a corresponding edge in G'_2. This may glue some components $C_1, \dots C_k$ of $[Y]$ together, resulting in a component C'. Hence, in $\mathrm{perm}_G(X,Y)$ the only change is, that the part $\sum\limits_{i=1}^{k} \left\lfloor \frac{1}{2}|\partial_{G-X} C_i| \right\rfloor$ of the sum-term will be replaced by $\left\lfloor \frac{1}{2}|\partial_{G-X} C'| \right\rfloor$. Furthermore, $\partial_{G-X} C' \subseteq \bigcup\limits_{i=1}^{k} \partial_{G-X} C_i$. If $C_i \neq C$ we get additionally $|\partial_{G-X} C_i \setminus \partial_{G-X} C'| \geq 1$. Hence,

$$\begin{aligned} \left\lfloor \frac{1}{2}|\partial_{G-X} C'| \right\rfloor &\leq \left\lfloor \frac{1}{2} 1 - k + \sum_{i=1}^{k} |\partial_{G-X} C_i| \right\rfloor \\ &\leq \sum_{i=1}^{k} \left\lfloor \frac{1}{2}|\partial_{G-X} C_i| \right\rfloor \end{aligned}$$

Consequently, (X,Y) stays an optimal H^*-separator of G. In this last case, condition 1 remains satisfied, while $|X|$ stays constant and $|Y|$ increases.

This algorithm will stop after a finite number of steps, because G is finite. The resulting optimal H^*-separator (X,Y) of G obviously proves the Lemma. □

The next Lemma is a consequence of Lemma 14.

Lemma 15 *Let G be a graph and H^* be an independent subset of $V(G)$. Then, G has an optimal H^*-separator (X,Y) such that for each $h \in H^*$ with the property, that the neighborhoods of the neighbors of h are identical, one of the following conditions holds:*

1. *The neighborhood of h is disjoint to X and no edge of Y is incident to a vertex of it.*
2. *Each vertex adjacent to h is contained in X.*
3. *Each edge incident with a neighbor of h but not incident with h is in Y.*

Proof of Lemma 15. Let (X,Y) be an optimal H^*-separator of G satisfying the conditions of Lemma 14 and let h be an arbitrary vertex of H^* with the property, that the neighborhoods of the neighbors of h are identical.

Suppose there is a vertex of X in the neighborhood of an h. By condition 2 of Lemma 14 condition 2 of Lemma 15 holds.

If, otherwise, condition 1 of Lemma 15 is violated, then Y contains an edge connecting a neighbor v of h with a vertex $w \neq h$.

By condition 1 of Lemma 14, each neighbor of h is connected with w in $[Y]$. If, furthermore, condition 3 of Lemma 15 is violated, too, then $G - X - Y$ contains an edge e connecting v with a vertex $u \neq h$.

Let C be the component of $[Y]$ containing w and D be the component of $G - X - Y$ containing e. We have $V(D) \cap H^* = \{h\}$. After contracting C in D to a vertex c, this vertex becomes a cutvertex of D. This contradicts condition 3 of Lemma 14, and the proof is done. □

The following Lemma is a big step toward the proof of Theorem 11.

Lemma 16 *Let G be a graph, H^* be an independent subset of $V(G)$, and f be a function mapping H^* into the positive integers. Let G^* be obtained from G by deleting each edge incident with a vertex of H^* and, for each $h \in H^*$, adding $f(h)$ new vertices and connecting them to h and all vertices being in G adjacent to h.*

Then the maximal size of a partial H^*-local f-factor of G^* is $p_G(H^*)$.

Proof of Lemma 16. If we have a partial H^*-local f-factor of G^* of size s, it is obvious, that G has s internally disjoint H^*-paths.

For the other direction, consider a set S of $p_G(H^*)$ internally disjoint paths of G^*. By construction of G^*, S is a partial H^*-local f-factor of G^*.

Beyond all possibilities choose S with a minimal number of edges in the union of its paths. This additional condition yields, that a path P containing a neighbor of a vertex $h \in H^*$ also contains the edge connecting it to h and ends there. Hence, if we contract in G^* each edge incident with an element of H^*, G^* becomes G and S becomes a partial H^*-local f-factor of G of the same size $|S|$. This completes the proof. $\square$

Proof of Theorem 11. Let G^* be the graph obtained from G as described in Lemma 16.

First, let (X^*, Y^*) be an optimal H^*-separator of G^* satisfying the conditions of Lemma 14 and set $X = (V(G) \cap X^*) \cup \{h \in H^* \mid N_{G^*}(h) \subseteq X\}$ and $Y = (E(G) \cap Y^*) \cup \{\{h, x\} \in E(G) \mid N_{[Y^*]}(x) = N_{G^*}(h)\}$. Then, by Lemma 15, (X, Y) is an f-separator of G with $\mathrm{perm}_{G,f}(X, Y) = \mathrm{perm}_{G^*}(X^*, Y^*)$.

Second, let (X, Y) be an f-separator of G and set $X^* = (X \setminus H) \cup \bigcup\limits_{x \in X \cap H} N_{G^*}(x)$ and $Y^* = E([Y] - H) \cup \{\{v, w\} \in E(G^*) \mid \exists h \in H^* : v \in N_{G^*}(h) \wedge w \neq h\}$. Then (X^*, Y^*) is an H^*-separator of G^* with $\mathrm{perm}_{G^*}(X^*, Y^*) = \mathrm{perm}_{G,f}(X, Y)$. Thus, the maximum number of internally disjoint H^*-paths equals the minimum of $\mathrm{perm}_{G,f}(X, Y)$ taken over all f-separators of G. With Lemma 16, the proof is complete. $\square$

Proof of Theorem 10. If all the local $2k$-factors exist, the assertion is trivial. Hence we only have to prove the other direction. Therefore H^* is assumed to be topological k-tough in G. Consequently, every subset of H^* is topological k-tough in G and thus it suffices to prove that there is an H^*-local $2k$-factor.

We'll do this indirectly, i.e. in the sequel we assume, that there is no H^*-local $2k$-factor, and we have to show, that there is an $H \subseteq H^*$, such that $p_G(H) < k|H|$ (i.e. G has no $k|H|$ internally disjoint H-paths).

Because intersecting edges by additional vertices will neither destroy H^*-local $2k$-factors, nor change the maximum number of internally disjoint H^*-paths, we may and will assume in the sequel, that H^* is independent in G.

By Corollary 13 there is a pair (X, Y) such that

1. $X \subseteq V(G)$,
2. $Y \subseteq E(G - X)$,
3. $\partial_{G-X}[Y] \subseteq V(G - H^*)$
4. $G - X - Y$ has no H^*-path, and
5. $|X \setminus H^*| + 2k|X \cap H^*| + k|V([Y]) \cap H^*| + \sum\limits_{C \in \mathcal{C}([Y])} \left\lfloor \frac{1}{2} |\partial_{G-X} C| \right\rfloor < k|H^*|$.

From property 5, we deduce $|H^* \setminus (X \cup V([Y]))| \geq 1$.

First we study the case when $|H^* \setminus (X \cup V([Y]))| = 1$. In this case let H be the set of the unique vertex $h \in H^* \setminus (X \cup V([Y]))$, and an arbitrary other vertex h' from H^*. Note, that here Y must (and will) not be contained in $G - X - H$.

Property 5 in this case yields $|X \cap H^*| = 0$. Hence $[Y]$ has a component C_h containing h'. By property 3 we get that $X \cup \partial_{G-X} C_h$ in G separates h from h'. Finally, property 5 yields $|X| + \left\lfloor \frac{|\partial_{G-X} C|}{2} \right\rfloor \leq k - 1$, and hence $|X \cup \partial_{G-X} C_h| \leq 2k - 1$, which by Theorem 2 completes the proof in this case.

In the remaining case we set $H = H^* \setminus (X \cup V([Y]))$. Here $X \subseteq V(G - H)$ and $Y \subseteq E(G - H - X)$ hold, and $G - X - Y$ has no H-path. Finally, Theorem 1 together with property 5 yield

$$\begin{aligned}
p_G(H) &\leq |X| + \sum_{C \in \mathcal{C}([Y])} \left\lfloor \frac{1}{2} |\partial_{G-X} C| \right\rfloor \\
&\leq |X \setminus H^*| + 2k|X \cap H^*| + k|V([Y]) \cap H^*| - k|H^* \cap (X \cup V([Y]))| + \\
&\quad + \sum_{C \in \mathcal{C}([Y])} \left\lfloor \frac{1}{2} |\partial_{G-X} C| \right\rfloor \\
&< k(|H^*| - |H^* \cap (X \cup V([Y]))|) = k|H^* \setminus (X \cup V([Y]))| = k|H|.
\end{aligned}$$

□

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