Michael Quellmalz

Reconstructing Functions on the Sphere from Circular Means

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Abstract

The present thesis considers the problem of reconstructing a function f that is defined on the d-dimensional unit sphere from its mean values along hyperplane sections. In case of the two-dimensional sphere, these plane sections are circles. In many tomographic applications, however, only limited data is available. Therefore, one is interested in the reconstruction of the function f from its mean values with respect to only some subfamily of all hyperplane sections of the sphere. Compared with the full data case, the limited data problem is more challenging and raises several questions. The first one is the injectivity, i. e., can any function be uniquely reconstructed from the available data? Further issues are the stability of the reconstruction, which is closely connected with a description of the range, as well as the demand for actual inversion methods or algorithms.

We provide a detailed coverage and answers of these questions for different families of hyperplane sections of the sphere such as vertical slices, sections with hyperplanes through a common point and also incomplete great circles. Such reconstruction problems arise in various practical applications like Compton camera imaging, magnetic resonance imaging, photoacoustic tomography, Radar imaging or seismic imaging. Furthermore, we apply our findings about spherical means to the cone-beam transform and prove its singular value decomposition.

Contents

1 Introduction

The reconstruction of an unknown function from indirect measurements plays a key role in various areas of pure and applied mathematics. Since the second half of the last century, different imaging modalities have been developed and brought advances in fields like medical examinations or nondestructive testing. Mathematically, the main challenge in many imaging devices is the reconstruction of a function f that is defined on a subset of $\mathbb{R}^3$ from measurements

$$g(L) = \int_L f(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}, \tag{1.1}$$

along certain submanifolds L of $\mathbb{R}^3$, where $\mathrm{d}\boldsymbol{x}$ denotes the surface measure on L. The submanifolds L can be different geometric objects such as lines, planes, circles [And88], spheres [FPR04, Kun07, Hal11] or cones [TKK18], depending on the imaging modality.

The computerized tomography (CT) is a well-known imaging modality, where one captures X-ray images of an object, often the human body, from several directions and reconstructs the object from these images. Cormack [Cor63] won the 1979 Nobel Price in Medicine for the development of CT. The mathematical model is the Radon transform, which assigns to a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined in the Euclidean plane $\mathbb{R}^2$ its integrals along all lines $L \subset \mathbb{R}^2$, i. e.,

$$\mathcal{R}f(L) := \int_L f(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}, \tag{1.2}$$

where $\mathrm{d}\boldsymbol{x}$ denotes the line integral along the line L. The main task in computerized tomography is to reconstruct the function f from its Radon transform $\mathcal{R}f$. The transform (1.2) was first described in 1917 by Radon [Rad17], who showed an inversion formula of $\mathcal{R}$. The generalization to hyperplanes in $\mathbb{R}^d$ is due to Mader [Mad27].

New imaging modalities give raise to the investigation of Radon-like transforms (1.1) on other manifolds. In this thesis, we focus on functions f that are defined on the unit sphere $\mathbb{S}^{d-1} := \{\boldsymbol{\xi} \in \mathbb{R}^d \,;\, \|\boldsymbol{\xi}\| = 1\}$ and their integrals along hyperplane sections. We denote the section of the sphere $\mathbb{S}^{d-1}$ with the hyperplane with normal vector $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$

and signed distance $t \in [-1, 1]$ to the origin by

$$C(\boldsymbol{\xi}, t) := \{\boldsymbol{\eta} \in \mathbb{S}^{d-1} \,;\, \boldsymbol{\xi}^\top \boldsymbol{\eta} = t\}.$$

Any hyperplane section $C(\boldsymbol{\xi}, t)$ is a $(d-2)$-dimensional subsphere of $\mathbb{S}^{d-1}$. In the practically most relevant case of the two-dimensional sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, the plane section $C(\boldsymbol{\xi}, t)$ is a circle. We define the mean operator $\mathcal{M}$ that integrates a continuous function $f\colon \mathbb{S}^{d-1} \to \mathbb{C}$ along all sections $C(\boldsymbol{\xi}, t)$ by

$$\mathcal{M} f(\boldsymbol{\xi}, t) := \int_{C(\boldsymbol{\xi},t)} f(\boldsymbol{\eta}) \, \mathrm{d}\mu(\boldsymbol{\eta}), \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1},\ t \in [-1, 1], \tag{1.3}$$

where $\mathrm{d}\mu$ denotes the surface measure on $C(\boldsymbol{\xi}, t)$ that is normalized to one.

The reconstruction of a function f from integrals (1.3) along circles of the sphere $\mathbb{S}^2$ has been studied since the early twentieth century in convex geometry to investigate bodies of constant width [Min05], star bodies [Gro98] or intersection bodies [Gar06, Chapter 8]. This problem also arises in many practical applications such as cone-beam tomography [Lou16, QHL18], Compton camera imaging [Ter15, Moo17, TKK18], magnetic resonance imaging [Tuc04], photoacoustic tomography [ZS10, HMS16], Radar imaging [YY11] and seismic imaging [AMS08]. While our motivation comes from imaging problems on $\mathbb{S}^2$, we consider the general situation $\mathbb{S}^{d-1}$ for $d \geq 3$ in this thesis.

In many applications, we want to reconstruct the function f given limited data of the mean operator $\mathcal{M} f(\boldsymbol{\xi}, t)$, where $(\boldsymbol{\xi}, t)$ is in some subset $D \subset \mathbb{S}^{d-1} \times [-1, 1]$. Since f is defined on $\mathbb{S}^{d-1}$, it seems reasonable that also the set D is $(d-1)$-dimensional. However, there are several questions and challenges attached to this problem. Fixing a set D, which functions f can be uniquely reconstructed from given data $\mathcal{M} f$ on D? Furthermore, we are interested in a description of the range of $\mathcal{M}$: Given some function g on D, does there exist a function f such that $g = \mathcal{M} f$? In case the reconstruction of f is possible, we would like to have an inversion formula or algorithm and investigate the stability of the inverse.

A well-studied case of a restriction of the mean operator $\mathcal{M}$ is the Funk–Radon transform

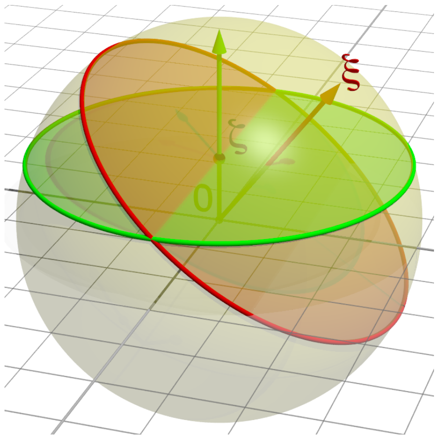
$$\mathcal{F} f(\boldsymbol{\xi}) := \mathcal{M} f(\boldsymbol{\xi}, 0), \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1}, \tag{1.4}$$

which was originally posted by Funk [Fun11] in 1911, preceding Radon's paper [Rad17] by six years. The Funk–Radon transform $\mathcal{F}$ on the two-dimensional sphere $\mathbb{S}^2$ takes the integrals along all great circles $C(\boldsymbol{\xi}, 0)$. We note that both the lines in $\mathbb{R}^2$, which are used for the Radon transform (1.2), and the great circles of the sphere $\mathbb{S}^2$ are geodesics, i. e., the shortest curves connecting two points.

A powerful tool to investigate such integral operators like the Funk–Radon transform is provided by the singular value decomposition (SVD). Not only does the SVD allow for characterizations of the kernel and range of an operator as well as the stability of the inversion, which becomes less stable as the singular values decay faster; but it also gives a direct and straightforward approach to compute the inverse numerically. For the

Funk–Radon transform $\mathcal{F}$ on $\mathbb{S}^2$, the SVD essentially goes back to Minkowski [Min05]; the spherical harmonics are the eigenfunctions of $\mathcal{F}$. It was used to show that the kernel of $\mathcal{F}$ consists of the odd functions $f(\boldsymbol{\xi}) = -f(-\boldsymbol{\xi})$. Furthermore, since its singular values asymptotically decay as $n^{-\frac{d-2}{2}}$, the Funk–Radon transform $\mathcal{F}$ is a continuous, one-to-one map from $L^2_{\mathrm{even}}(\mathbb{S}^{d-1})$ to the Sobolev space $H^{\frac{d-2}{2}}_{\mathrm{even}}(\mathbb{S}^{d-1})$, see [Str81, Paragraph 4]. This decay rate implies that the inversion of $\mathcal{F}$ is a mildly ill-posed problem.

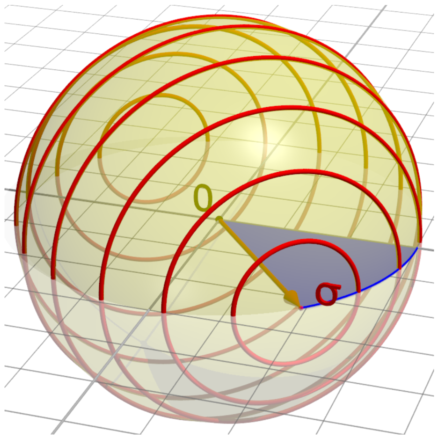
While many facts about the Funk–Radon transform are well-known, the situation becomes more difficult when we turn to other families of circles or hyperplane sections on the sphere. The main contribution of this thesis is that we characterize those families for which the restriction of $\mathcal{M}$ is injective and that we extend the classical theory of the Funk–Radon transform $\mathcal{F}$ to more general families of circles or hyperplane sections on the sphere. We prove uniqueness theorems, singular value decompositions and Sobolev space estimates. In particular, we are interested in the following cases.



Sections through a common point. We consider the sections of the sphere with all the hyperplanes that have a common point $\boldsymbol{\zeta}$ inside the sphere, i.e., $\|\boldsymbol{\zeta}\| < 1$. Formally, we write for a fixed vector $\boldsymbol{\zeta}$ with $\|\boldsymbol{\zeta}\| < 1$ the mean operator $\mathcal{U}f(\boldsymbol{\xi}) = \mathcal{M}f(\boldsymbol{\xi}, \boldsymbol{\xi}^\top\boldsymbol{\zeta})$. This operator is dubbed the spherical transform or the non-geodesic Funk transform [Pal17]. If $\boldsymbol{\zeta} = \mathbf{0}$ is the origin, we obtain the classical Funk–Radon transform $\mathcal{F}$. The first description of this spherical transform as well as an inversion formula on $\mathbb{S}^2$ was shown in [Sal16] and extended to the d-dimensional case in [Sal17]. They make use of the stereographic projection to reduce the problem to a certain circular Radon transform in the equatorial plane $\mathbb{R}^2$. It seems worth mentioning that these inversion formulas avoided the non-injectivity by restricting the support of the function f.

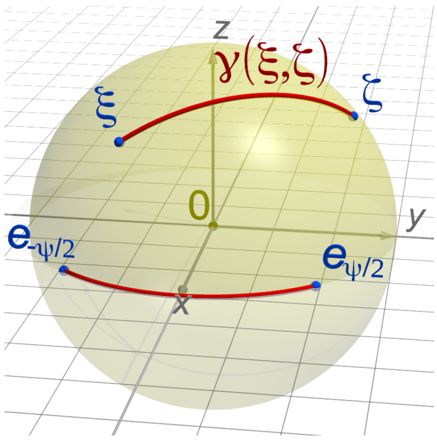
We establish a geometric connection of the spherical transform with the classical Funk–Radon transform $\mathcal{F}$. This approach allows us to transfer much of the classical theory to the spherical transform. In particular, we will show that the nullspace consists of the functions f on $\mathbb{S}^{d-1}$ that are odd with respect to the point reflection in $\boldsymbol{\zeta}$ and the multiplication of some weight. Furthermore, the range is the same Sobolev space $H^{\frac{d-2}{2}}_{\mathrm{even}}(\mathbb{S}^{d-1})$ as for the Funk–Radon transform. However, the spherical transform $\mathcal{U}$ behaves differently if the common point $\boldsymbol{\zeta}$ is located on the sphere, i. e., $\|\boldsymbol{\zeta}\| = 1$, which was considered in [AD93]. Then $\mathcal{U}$ is injective for all bounded functions, which was shown via stereographic projection, see [Hel99, Rub17b].

Vertical slices of the sphere. In the next setting, we restrict the normal vectors $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$ of the hyperplanes to be on the equator $\xi_d = 0$ of the sphere. The respective sections $C(\boldsymbol{\xi}, t)$ of such hyperplanes with the sphere are parallel to the ξ_d–axis. Thus, the mean operator on these sections is called the vertical slice transform. The study of this problem is motivated by a setup of photoacoustic tomography [ZS10].



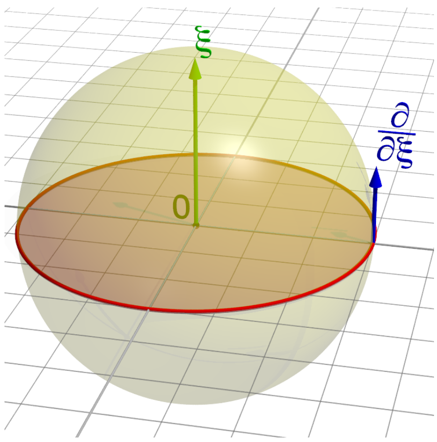
It is rather obvious that only functions f that are even in the last coordinate could be reconstructed given the integrals along all vertical slices. We prove an SVD and show that the vertical slice transform is indeed injective for functions that are even in the last coordinate. The asymptotic decay of the singular values is more elaborate than for those of the Funk–Radon transform. Nevertheless, we are able to obtain tight upper and lower bounds. Furthermore, the SVD forms the basis of a reconstruction scheme, which does not depend on a projection to the unbounded plane as the method suggested in [ZS10]. Recently, an alternative method of showing the SVD has been proposed in [Rub18]: The vertical slices are projected orthogonally to the equatorial hyperplane, where they become straight lines (or planes if $d > 3$), and the integrals along these lines are treated with the Radon transform. We note that the latter approach on $\mathbb{S}^2$ goes back to [GRS94].

More generally, we take a look at what happens if the normal vectors $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$ are on a different circle of latitude $\xi_d = z$ of the sphere for some $z \in (-1, 1)$, which includes the vertical slices for $z = 0$. It turns out that this transform is injective for all except countably many values of z. In particular, it is non-injective on all zeros of the associated Legendre functions of dimension d. For a specific value of z, however, it is usually not easy to decide whether this problem is injective. The case that the normal vectors $\boldsymbol{\xi}$ are in some arbitrary subset $S \subset \mathbb{S}^{d-1}$ was investigated in [AQ96, AVZ99], where it was shown that the mean operator $\mathcal{M}$ restricted to $S \times [-1, 1]$ is injective if and only if S is not contained in the zero set of any spherical harmonic.



Arcs of great circles. In seismic imaging or, more specifically, spherical surface wave tomography, one has to deal with the integrals along arcs of great circles instead of full circles of the sphere $\mathbb{S}^2$, see [WD84, AMS08]. Given two distinct points on the sphere $\mathbb{S}^2$, the shortest curve that connects these points on the sphere is always a great circle arc. If the two points are not antipodal, the shortest arc connecting them is unique. Conversely, any great circle arc that is smaller than a semicircle is determined by its two endpoints. Hence, the manifold of all great circle arcs on $\mathbb{S}^2$ is four-dimensional. As for the mean operator $\mathcal{M}$, the inversion problem of the arc transform is overdetermined. However, there are only few results about the injectivity of the arc transform when restricted to certain families of great circle arcs. A result from [Ami07] shows that, from the integrals along all great circle arcs connecting two subsets of $\mathbb{S}^2$, any function can be uniquely reconstructed on the closure of the two sets. We will further see that any function on a convex subset of $\mathbb{S}^2$ that is contained in a hemisphere is uniquely determined by its integrals along all great circle arcs connecting two boundary points of this set.

An interesting case is the family of great circle arcs with fixed arc-length, which forms a three-dimensional manifold. As for the case with full data, we show an SVD that implies the injectivity of the arc transform with fixed arc-length. If the arc-length is π, we have the set of all great semicircles. The injectivity of the semicircle transform has already been proven in [Gro98], where it was used to show a uniqueness result about half-plane sections of star bodies.



Derivatives perpendicular to hyperplane sections. Like the classical Funk–Radon transform in (1.4), the so-called generalized Funk–Radon transform is also about great circle integrals, but this time we take the directional derivative $\frac{\partial}{\partial \boldsymbol{\xi}}$ of j-th order, which acts perpendicular to the circle of integration. This transform was defined by [Lou16] in the context of cone-beam tomography, an equivalent definition for $j = 1$ is due to [MMÓ00]. We show an SVD of the generalized Funk–Radon transform, which turns out to be injective only on the subset of either the even functions or the odd functions on $\mathbb{S}^{d-1}$, depending on whether the order j of the derivative is even or odd, respectively. For $j > 0$, The singular values decay with a faster rate than those of the Funk–Radon transform, which is explained by the additional differentiation of f. Formally, the generalized Funk–Radon transform extends to negative j; for $j = -1$ we have the hemispherical transform.

The motivation of studying this problem comes from the cone-beam tomography, where one measures the integrals of a function on the Euclidean space $\mathbb{R}^d$, usually with compact support, along all rays that start in a certain scanning set. Some reconstruction formulas for the cone-beam tomography rely on the generalized Funk–Radon transform, see [Lou16]. Grangeat's formula gives a connection between the cone-beam transform and the Radon transform. This connection can be expressed as follows: the generalized Funk–Radon transform applied to the cone-beam transform is exactly a derivative of the Radon transform of the same function, see (4.5). We will use Grangeat's formula together with our previous findings in order to obtain an SVD of the cone-beam transform, where the scanning set is the unit sphere and the function is supported inside the unit ball of $\mathbb{R}^d$.

Outline of this thesis

Apart from this introduction, the present thesis is divided into four chapters.

Chapter 2: Harmonic analysis on the sphere and the rotation group

We introduce the fundamental theory of harmonic analysis on the sphere $\mathbb{S}^{d-1}$ and the rotation group SO(3). The chapter provides the notation and the essential tools that will be utilized throughout this thesis. We start with studying the sphere in the framework of smooth submanifolds of $\mathbb{R}^d$ in Section 2.1.1. The spherical harmonics $Y_{n,d}^k$, which form an orthonormal basis of polynomials on the sphere $\mathbb{S}^{d-1}$, are introduced recursively on the dimension d in Section 2.1.3. Spherical Sobolev spaces give a characterization of the smoothness of functions defined on $\mathbb{S}^{d-1}$. The respective Sobolev norm is defined by the decay of the spherical Fourier coefficents or, equivalently, by powers of the Laplace–Beltrami operator, see Section 2.1.5. We prove the continuity of the multiplication and the composition operator with a smooth function in the spherical Sobolev spaces in the Theorems 2.6 and 2.7, respectively. These theorems form generalizations of continuity results from [IKT13].

On the rotation group SO(3), the role of orthogonal basis polynomials is played by the rotational harmonics or Wigner D-functions, see Section 2.2. They are closely related with the spherical harmonics on $\mathbb{S}^2$, see (2.65). Moreover, we collect some useful identities about double factorials and the Gamma function in Section 2.3.

Chapter 3: Circular means on the sphere

This chapter serves the most important part of the present thesis. It starts with the definition and basic properties of the mean operator $\mathcal{M}$, which maps to a function on the sphere $\mathbb{S}^{d-1}$ its mean values along all hyperplane sections. We state a proof of its singular value decomposition in Theorem 3.4. For $\mathbb{S}^2$, it states that $\mathcal{M}Y_n^k(\boldsymbol{\xi}, t) = Y_n^k(\boldsymbol{\xi})\, P_n(t)$, where Y_n^k is a spherical harmonic of degree $n \in \mathbb{N}_0$ and P_n is the Legendre polynomial of degree n.

In Theorem 3.7, we show that every function $\mathcal{M}f$ in the range of the mean operator $\mathcal{M}$ satisfies the partial differential equation (3.13), which constitutes a generalization of John's equation for the mean operator $\mathcal{M}$. We find a description of the range of the mean operator in terms of Sobolev spaces on the product manifold $\mathbb{S}^{d-1} \times (-1, 1)$ in Section 3.1.2. The John-type equation (3.13) turns out to provide a condition that is not only necessary but also sufficient for a function to be in the range of the mean operator $\mathcal{M}$, provided that the function f is sufficiently smooth with respect to Sobolev spaces, see Theorem 3.11. In the rest of the chapter, we take a closer look at different restrictions of the mean operator.

In Section 3.2, we collect some classical results about the Funk–Radon transform $\mathcal{F}$, which is the restriction $\mathcal{F}f(\boldsymbol{\xi}) = \mathcal{M}f(\boldsymbol{\xi}, 0)$ to great circles. We state the singular value decomposition and a Sobolev estimate, which follows from the asymptotic behavior of the singular values, in the Theorems 3.12 and 3.13, respectively. These results build the foundation of different generalizations we consider later on. We provide an overview about inversion methods of the Funk–Radon transform and explain how the singular value decomposition is utilized to obtain a computationally efficient inversion algorithm in Section 3.2.2. Lastly, in Section 3.2.3, we present different both inner-mathematical and practical applications of the Funk–Radon transform.

Fixing the second argument of the mean operator $\mathcal{M}$ to some value $z \in [-1,1]$, we arrive at the spherical section transform $\mathcal{T}_z(\boldsymbol{\xi}) = \mathcal{M}f(\boldsymbol{\xi}, z)$ in Section 3.3. It takes the integrals of the function f along all $(d-2)$-dimensional subspheres of $\mathbb{S}^{d-1}$ with the radius $\sqrt{1-z^2}$. The injectivity of this transform depends of course on the value z. The so-called "Freak Theorem" by Schneider, Proposition 3.19, states that the spherical section transform $\mathcal{T}_z$ is injective for all but countably many values of $z \in [-1,1]$.

In Section 3.4, we come to a variation of the Funk–Radon transform, where we take the j-th order directional derivative of the function f perpendicular to the great circle along which we integrate f. Formally, we can write the generalized Funk–Radon transform of order $j \in \mathbb{N}_0$ as

$$\mathcal{S}^{(j)} f(\boldsymbol{\xi}) = \int_{\boldsymbol{\xi}^\top \boldsymbol{\eta} = 0} \left(-\frac{\partial}{\partial \boldsymbol{\xi}}\right)^j f(\boldsymbol{\eta}) \, \mathrm{d}\boldsymbol{\eta}, \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1},$$

see (3.31). For $j = 0$, we have $\mathcal{S}^{(0)} = \mathcal{F}$. Apart from this case, the generalized Funk–Radon transform $\mathcal{S}^{(j)}$ is not directly a restriction of the mean operator $\mathcal{M}$. We obtain a singular value decomposition of $\mathcal{S}^{(j)}$ in Theorem 3.24. As for the Funk–Radon transform $\mathcal{F}$, the spherical harmonics are eigenfunctions of $\mathcal{S}^{(j)}$. Furthermore, this transform $\mathcal{S}^{(j)}$ is injective on a subset of either the even or the odd functions on $\mathbb{S}^{d-1}$, depending on the order j. We show a tight Sobolev estimate for the generalized Funk–Radon transform $\mathcal{S}^{(j)}$ in Theorem 3.26. We point out special cases of j, in which $\mathcal{S}^{(j)}$ coincides with the hemispherical transform for $j = -11$ or the spherical cosine transform for $j = -2$ in Section 3.4.3. The section closes by considering other similarly defined Radon-like transforms that coincide with $\mathcal{S}^{(j)}$ in certain situations, see Section 3.4.4.

In Section 3.5, we consider the sections of $\mathbb{S}^{d-1}$ with hyperplanes whose normal vectors $\boldsymbol{\xi}$ are in a fixed set $S \subset \mathbb{S}^{d-1}$, i.e., we consider the restriction of the mean operator $\mathcal{M}$ to the set $S \times [-1,1]$. It is known that this restriction is injective if and only if S is not contained in the zero set of any spherical harmonic, see Proposition 3.33. We find in Theorem 3.35 that the injectivity of the mean operator on the set S is equivalent to the question whether the radial functions with centers in S are dense in $L^2(\mathbb{S}^{d-1})$. For the case that the set S itself is a hyperplane section of $\mathbb{S}^{d-1}$, we obtain the injectivity for all but countably many values of the distance in Theorem 3.39, which resembles Schneider's Freak Theorem. If S is the equator $\{(\xi_1, \dots, \xi_d) \in \mathbb{S}^{d-1} \,;\, \xi_d = 0\}$, the respective circles of integration are vertical slices of the sphere. Formally, we have the vertical slice transform $\mathcal{V}f(\boldsymbol{\sigma}, t) = \mathcal{M}f(\binom{\boldsymbol{\sigma}}{0}, t)$ for $\boldsymbol{\sigma} \in \mathbb{S}^{d-2}$ and $t \in [-1,1]$. The singular value decomposition of $\mathcal{V}$ is presented in Theorem 3.43. The reconstruction of a function f from these vertical slices is unique only if f is even with respect to the last coordinate.

Furthermore, we see in Theorem 3.44 that the singular values of $\mathcal{V}$ decay a little slower than those of the Funk–Radon transform $\mathcal{F}$. Afterwards, we show in Theorem 3.46 that, with the help of orthogonal projection, the vertical slice transform $\mathcal{V}$ can be reduced to the Radon transform in the equatorial hyperplane. The respective connection of the inverse problem is given in Theorem 3.47.

In Section 3.6, the circles are obtained by the intersection of the sphere with hyperplanes that have a common point inside the sphere. Formally, we take for $z \in (-1, 1)$ the common point $(0, \dots, 0, z)$ inside the sphere and write the restriction of the mean operator

$$\mathcal{U}_z f(\boldsymbol{\xi}) = \mathcal{M} f(\boldsymbol{\xi}, z\xi_d), \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1}.$$

If $z = 0$, which means that the common point is the origin, we have the classical Funk–Radon transform $\mathcal{F} = \mathcal{U}_0$. In Section 3.6.1, we find two geometric transformations that connect $\mathcal{U}_z$ with the classical Funk–Radon transform $\mathcal{F}$. Then the transform $\mathcal{U}_z$ is factorized into the Funk–Radon transform $\mathcal{F}$ and two simple operators that correspond to a weighted south- or northwards drift on the sphere, respectively, see Theorem 3.58. With the help of this factorization, we are able to characterize the nullspace of $\mathcal{U}_z$ as the set of functions that are symmetric with respect to the point reflection in $(0, \dots, 0, z)$ and the multiplication of some weight, see Theorem 3.60. Furthermore, in Theorem 3.64, we obtain a Sobolev estimate and a description of the range of this operator $\mathcal{U}_z$, which behaves, up to some constants depending only on z, almost like the Funk–Radon transform $\mathcal{F}$. Moreover, an inversion formula of $\mathcal{U}_z$ is shown in Theorem 3.65 that forms a generalization of an inversion formula of the Funk–Radon transform $\mathcal{F}$ by Helgason [Hel90].

Chapter 4: Applications

In this chapter, we consider two particular applications of the spherical transforms that are investigated in the previous chapter. In Section 4.1, we are going to take a look at the cone-beam transform, which integrates a function defined on the Euclidean space $\mathbb{R}^d$ along all rays that start in a certain set. The cone-beam transform provides the mathematical background of a common setup in the three-dimensional X-ray computed tomography. Grangeat's formula gives a connection of the cone-beam transform with the generalized Funk–Radon transform $\mathcal{S}^{(j)}$ and the Radon transform $\mathcal{R}$, see (4.5). We utilize the results obtained in Section 3.4 about the transform $\mathcal{S}^{(j)}$ in order to show the singular value decomposition of the cone-beam transform in Theorem 4.4, where the function is supported on the d-dimensional unit ball with d odd and the sources are located on the sphere. We present upper and lower bounds of the singular values in Theorems 4.8 and 4.10, respectively. We state our results for the practically most relevant case $d = 3$ in Theorem 4.13.

In the second section, 4.2, we consider the integrals along incomplete great circles of the two-sphere $\mathbb{S}^2$. These integrals play an important role in the modeling of the spherical surface wave tomography. We call the operator that maps a function f on $\mathbb{S}^2$ to its integrals along incomplete great circles the arc transform on the sphere. We show the singular value decomposition of the arc transform with full data in Theorem 4.19. The task of recovering a function f from all great circle arc integrals is overdetermined. As for the mean operator $\mathcal{M}$, we can think of restrictions of the arc transform that uniquely determine the function f. In Section 4.2.3, we present two simple cases of injective restrictions of the arc transform. In Section 4.2.4, we focus on the special case that only the integrals along certain great circle arcs are known, namely those arcs having a fixed length. Even from this limited data, it is still possible to recover the original function on the sphere. We obtain a singular value decomposition for this case in Theorem 4.22.

Chapter 5: Conclusion

The last chapter summarizes the results of this thesis. Table 5.1 provides a nice overview about the notation, the injectivity, the range and the singular value decompositions of all the different restrictions of the mean operator $\mathcal{M}$ that we have investigated in the present thesis.

Publications by the author

Parts of this thesis have already been published in peer-reviewed publications by the author. Section 3.4 about the generalized Funk–Radon transform contains material from first part of our article [QHL18]. Section 3.6 about the spherical transform contains material that is submitted for publication [Que18], which forms the continuation and extension of the article [Que17]. Section 4.1 about the cone-beam transform contains, up to some editorial changes and additional remarks, material from the second part of our article [QHL18]. Section 4.2 about integrals along great circles arcs contains, up to some editorial changes and additional remarks, material from parts of our paper [HPQ18].

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2 Harmonic analysis on the sphere and the rotation group

In this chapter, we introduce our notation and the fundamental theory of harmonic analysis on the sphere and the rotation group, which will be utilized throughout this thesis. We start by reviewing some facts about integration on smooth manifolds with a special focus on the sphere. In Section 2.1.3, we show an explicit construction of the spherical harmonics, which form a basis of orthogonal polynomials on the sphere. Spherical Sobolev spaces, which are covered in Section 2.1.5, give a precise characterization of the smoothness of functions.

On the rotation group SO(3), the role of orthogonal basis functions is played by the rotational harmonics, see Section 2.2. Moreover, we collect some formulas about double factorials and the Gamma function in Section 2.3. We mostly skip the proofs of well-known results that can be found in the referenced literature.

2.1 Harmonic analysis on the sphere

We are going to summarize some basic facts about harmonic analysis on the sphere as it can be found, e. g., in [DX13, FGS98, AH12]. We denote with $\mathbb{Z}$ the set of integers, with $\mathbb{N}$ the positive integers, with $\mathbb{N}_0$ the nonnegative integers, and with $\mathbb{R}$ and $\mathbb{C}$ the real and complex numbers, respectively. We denote the unit vectors in $\mathbb{R}^d$ with

$$\begin{array}{rcl} \boldsymbol{\epsilon}^1 & := & (1,0,\dots,0)^\top \\ \vdots & & \vdots \\ \boldsymbol{\epsilon}^d & := & (0,\dots,0,1)^\top. \end{array}$$

Then any vector $\boldsymbol{x} = (x_1,\dots,x_d)^\top \in \mathbb{R}^d$ can be written by its components

$$\boldsymbol{x} = x_1\boldsymbol{\epsilon}^1 + \cdots + x_d\boldsymbol{\epsilon}^d.$$

We define the $(d-1)$-dimensional sphere

$$\mathbb{S}^{d-1} := \{\boldsymbol{\xi} \in \mathbb{R}^d \,;\, \|\boldsymbol{\xi}\| = 1\}$$

as the set of unit vectors $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_d)^\top$ in the d-dimensional Euclidean space $\mathbb{R}^d$. Throughout this thesis, we will use bold Greek letters to denote points on the sphere $\mathbb{S}^{d-1}$. We will reserve d for the dimension of the Euclidean space and, if not stated otherwise, we always assume that $d \geq 2$. In the case $d = 1$, the sphere consists only of two points, $\mathbb{S}^0 = \{-1, 1\}$.

2.1.1 The sphere as a smooth manifold

In order to give a proper meaning of the surface measure on the sphere $\mathbb{S}^{d-1}$ and on its subspheres, we will need to introduce some formalism. In this section, we give an introduction to the notion of a smooth manifold, more precisely, a smooth submanifold of $\mathbb{R}^d$. This brief introduction follows [Que18] and is based on [AF02].

Let $n \in \mathbb{N}_0$. We say that a function $\mathbb{R}^n \to \mathbb{R}^n$ is smooth if it has partial derivatives of arbitrary order. A diffeomorphism is a bijective, smooth function $f\colon \mathbb{R}^n \to \mathbb{R}^n$ whose inverse f^{-1} is also smooth. We denote the derivative or Jacobian of f by

$$J_f(\boldsymbol{x}) := \begin{pmatrix} \frac{\partial f_1(\boldsymbol{x})}{\partial x_1} & \cdots & \frac{\partial f_1(\boldsymbol{x})}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n(\boldsymbol{x})}{\partial x_1} & \cdots & \frac{\partial f_n(\boldsymbol{x})}{\partial x_n} \end{pmatrix}.$$

Manifolds A subset $M \subset \mathbb{R}^d$ is called an n-dimensional *smooth submanifold* without boundary if for every $\boldsymbol{\xi} \in M$ there exists an open neighborhood $N(\boldsymbol{\xi}) \subset \mathbb{R}^d$ containing $\boldsymbol{\xi}$, an open set $U \subset \mathbb{R}^d$, and a diffeomorphism $\tilde{m}\colon U \to N(\boldsymbol{\xi})$ such that

$$\tilde{m}(U \cap (\mathbb{R}^n \times \{0\}^{d-n})) = M \cap N(\boldsymbol{\xi}).$$

We denote by $m = \tilde{m}|_V$ the restriction of such diffeomorphism m to the set $V = U \cap (\mathbb{R}^n \times \{0\}^{d-n})$. Then $m\colon V \to M$ is called a *map* of the manifold M. A map can be seen as a local parameterization of the manifold. An *atlas* of the manifold M is a finite family

$$\{(m_i, V_i) \,;\, i = 1 \ldots, l\}$$

consisting of sets $V_i \subset \mathbb{R}^n$ and maps $m_i\colon V_i \to M$, $i = 1, \ldots, l$, such that the sets $m_i(V_i)$ cover M, i. e.,

$$\bigcup_{i=1}^{l} m_i(V_i) = M.$$

Tangent space The *tangent space* $T_{\boldsymbol{\xi}}M$ of the manifold M at $\boldsymbol{\xi} \in M$ is the set of vectors $\boldsymbol{x} \in \mathbb{R}^d$ for which there exists a smooth path $\gamma\colon [0,1] \to M$ satisfying $\gamma(0) = \boldsymbol{\xi}$ and $\gamma'(0) = \boldsymbol{x}$. With the help of a map $m\colon V \to M$, the tangent space $T_{m(\boldsymbol{x})}M$ at the point $m(\boldsymbol{x}) \in M$ for $\boldsymbol{x} \in V$ is expressed as the range of the Jacobian J_m, i. e.,

$$T_{m(\boldsymbol{x})}M = \operatorname{range} J_m(\boldsymbol{x}) = \{J_m(\boldsymbol{x})\,\boldsymbol{y}\ ;\ \boldsymbol{y} \in \mathbb{R}^n\}.$$

The last equation does not depend on the choice of the map m. The tangent space $T_{\boldsymbol{\xi}}M$ of an n-dimensional manifold M is a linear subspace of $\mathbb{R}^d$ with dimension n. The tangent space of the d-dimensional Euclidean space $\mathbb{R}^d$ is $T_{\boldsymbol{x}}\mathbb{R}^d = \mathbb{R}^d$ for any $\boldsymbol{x} \in \mathbb{R}^d$.

In order to define integrals of differential forms, we have to discuss the *orientation* of the tangent space. In the Euclidean space $\mathbb{R}^n$, we use the canonical orientation: we say that a basis $[\boldsymbol{v}^1, \ldots, \boldsymbol{v}^n]$ of $\mathbb{R}^n$ has a positive orientation if $\det(\boldsymbol{v}^1, \ldots, \boldsymbol{v}^n)$ is positive. Note that the latter condition already implies that the vectors $\boldsymbol{v}^i$ are linearly independent. With the help of a map m, we can push the orientation of $\mathbb{R}^n$ to the tangent space of a manifold M. We set that $[J_m\boldsymbol{v}^1, \ldots, J_m\boldsymbol{v}^n]$ of $T_{\boldsymbol{\xi}}M$ has positive orientation with respect to the map m if $[\boldsymbol{v}^1, \ldots, \boldsymbol{v}^n]$ has positive orientation in $\mathbb{R}^n$. We call an atlas $\{m_i\colon V_i \to M\}_{i=1}^l$ an orientation of the manifold M if for each point $\boldsymbol{\xi} \in M$, every map m_i with $\boldsymbol{\xi} \in m_i(V_i)$ gives the same orientation of the tangent space $T_{\boldsymbol{\xi}}M$.

Forms A k-form ω on the manifold M is a family $(\omega_{\boldsymbol{\xi}})_{\boldsymbol{\xi}\in M}$ of antisymmetric multilinear functionals

$$\omega_{\boldsymbol{\xi}}\colon (T_{\boldsymbol{\xi}}M)^k \to \mathbb{R},$$

where $(T_{\boldsymbol{\xi}}M)^k = T_{\boldsymbol{\xi}}M \times \cdots \times T_{\boldsymbol{\xi}}M$. We call the functional $\omega_{\boldsymbol{\xi}}$ antisymmetric if for any $\boldsymbol{v}^1, \ldots, \boldsymbol{v}^k \in T_{\boldsymbol{\xi}}M$ and $i, j = 1, \ldots, k$, we have

$$\omega_{\boldsymbol{\xi}}(\boldsymbol{v}^1, \ldots \boldsymbol{v}^i, \ldots, \boldsymbol{v}^j, \ldots, \boldsymbol{v}^k) = -\omega_{\boldsymbol{\xi}}(\boldsymbol{v}^1, \ldots \boldsymbol{v}^j, \ldots, \boldsymbol{v}^i, \ldots, \boldsymbol{v}^k),$$

and multilinear if all coordinate functions

$$T_{\boldsymbol{\xi}}M \ni \boldsymbol{x} \mapsto \omega_{\boldsymbol{\xi}}(\boldsymbol{v}^1, \ldots \boldsymbol{v}^{i-1}, \boldsymbol{x}, \boldsymbol{v}^{i+1}, \ldots, \boldsymbol{v}^k)$$

are linear.

Let $f\colon M \to N$ be a smooth mapping between the manifolds M and N and let ω be a k-form on N. The *pullback* of the form ω with respect to f is the k-form $f^*(\omega)$ on M that is defined for any $\boldsymbol{v}^1, \ldots, \boldsymbol{v}^k \in T_{\boldsymbol{\xi}}M$ by

$$(f^*(\omega))_{\boldsymbol{\xi}}([\boldsymbol{v}^i]_{i=1}^k) := \omega_{f(\boldsymbol{\xi})}\left(\left[\frac{\mathrm{d}}{\mathrm{d}t} f \circ \gamma_i(t)\bigg|_{t=0}\right]_{i=1}^k\right),$$

where $\gamma_i\colon [0,1] \to M$ are smooth paths satisfying $\gamma_i(0) = \boldsymbol{\xi}$ and $\gamma_i'(0) = \boldsymbol{v}^i$. If the smooth function $f\colon \mathbb{R}^d \to \mathbb{R}^d$ extends to the surrounding space $\mathbb{R}^d$, the pullback of ω can be expressed with the Jacobian J_f as

$$(f^*(\omega))_{\boldsymbol{\xi}}([\boldsymbol{v}^i]_{i=1}^k) = \omega_{f(\boldsymbol{\xi})}([J_f\,\boldsymbol{v}^i]_{i=1}^k). \tag{2.1}$$

On an n-dimensional manifold M, there exists only one n-form up to the multiplication of a constant real number depending only on $\boldsymbol{\xi} \in M$. Let $\boldsymbol{e}^1, \dots, \boldsymbol{e}^n$ be a positively oriented, orthonormal basis of the tangent space $T_{\boldsymbol{\xi}} M$. Then the *volume form* $\mathrm{d}M$ is defined as the unique n-form on M satisfying

$$(\mathrm{d}M)_{\boldsymbol{\xi}}(\boldsymbol{e}^1, \dots, \boldsymbol{e}^n) = 1.$$

For the volume form $(\mathrm{d}M)_{\boldsymbol{\xi}}$ at the point $\boldsymbol{\xi} \in M$, we also write $\mathrm{d}M(\boldsymbol{\xi})$. The volume form on the Euclidean space $\mathbb{R}^d$ is then just the determinant, i. e., at any point $\boldsymbol{x} \in \mathbb{R}^d$, we have for $\boldsymbol{v}^1, \dots, \boldsymbol{v}^d \in T_{\boldsymbol{x}} \mathbb{R}^d = \mathbb{R}^d$

$$\mathrm{d}\mathbb{R}^d(\boldsymbol{x})(\boldsymbol{v}^1, \dots, \boldsymbol{v}^d) = \det(\boldsymbol{v}^1, \dots, \boldsymbol{v}^d).$$

A set $\{\varphi_i\}_{i=1}^l$ of functions $\varphi_i \in C^\infty(M)$ is a partition of unity of the manifold M with respect to the oriented atlas $\{m_i \colon V_i \to M\}_{i=1}^l$ if $\operatorname{supp}(\varphi_i) \subset m_i(V_i)$ for all i and $\sum_{i=1}^l \varphi_i \equiv 1$ on M. Then the integral of a k-form ω on M is defined as

$$\int_M \omega := \sum_{i=1}^l \int_{V_i} m_i^*(\varphi_i \cdot \omega) = \sum_{i=1}^l \int_{V_i} c_i \, \mathrm{d}\mathbb{R}^n,$$

where the latter is the standard volume integral $\mathrm{d}\mathbb{R}^n$ on $V_i \subset \mathbb{R}^n$ and the functions $c_i \colon V_i \to \mathbb{R}$ are uniquely determined by the condition $m_i^*(\varphi_i \cdot \omega) = c_i \, \mathrm{d}\mathbb{R}^n$.

Let $f \colon M \to N = f(M)$ be a diffeomorphism between the n-dimensional manifolds M and $f(M)$, and let ω be an n-form on N. Then the substitution rule [Jän01, page 94] holds,

$$\int_{f(M)} \omega = \int_M f^*(\omega). \tag{2.2}$$

The volume form on the sphere Let $d \geq 2$. The $(d-1)$-dimensional unit sphere $\mathbb{S}^{d-1}$ is a $(d-1)$-dimensional manifold in $\mathbb{R}^d$ with tangent space

$$T_{\boldsymbol{\xi}} \mathbb{S}^{d-1} = \{\boldsymbol{x} \in \mathbb{R}^d \, ; \, \langle \boldsymbol{\xi}, \boldsymbol{x} \rangle = 0\}$$

for $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$. We choose an orientation on $\mathbb{S}^{d-1}$ by saying that a basis $[\boldsymbol{x}^1, \dots, \boldsymbol{x}^{d-1}]$ of the tangent space $T_{\boldsymbol{\xi}} \mathbb{S}^{d-1}$ is oriented positively if

$$\det[\boldsymbol{\xi}, \boldsymbol{x}^1, \dots, \boldsymbol{x}^{d-1}] > 0. \tag{2.3}$$

This choice implies that on the one-dimensional sphere $\mathbb{S}^1$, which is the unit circle in $\mathbb{R}^2$, the positive orientation is counterclockwise.

Let $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$ and $\boldsymbol{v}^1, \dots, \boldsymbol{v}^{d-1} \in T_{\boldsymbol{\xi}} \mathbb{S}^{d-1}$. Then the volume form on the sphere $\mathbb{S}^{d-1}$ is given by

$$\mathrm{d}\mathbb{S}^{d-1}(\boldsymbol{\xi})(\boldsymbol{v}^1, \dots, \boldsymbol{v}^{d-1}) := \det(\boldsymbol{\xi}, \boldsymbol{v}^1, \dots, \boldsymbol{v}^{d-1}).$$

The so-defined $(d-1)$-form $\mathrm{d}\mathbb{S}^{d-1}(\boldsymbol{\xi})$ is indeed the volume form on the sphere $\mathbb{S}^{d-1}$ since is antisymmetric and multilinear and that for a basis $\boldsymbol{e}^1, \dots, \boldsymbol{e}^{d-1}$ of $T_{\boldsymbol{\xi}} \mathbb{S}^{d-1}$ that is oriented positively according to (2.3), we have $\mathrm{d}\mathbb{S}^{d-1}(\boldsymbol{e}^1, \dots, \boldsymbol{e}^{d-1}) = 1$. We will occasionally write $\mathrm{d}\boldsymbol{\xi}$ instead of $\mathrm{d}\mathbb{S}^{d-1}(\boldsymbol{\xi})$ if the choice of the measure is unambiguous.

Proposition 2.1. *Let V be an open subset of $\mathbb{R}^{d-1}$, and let $\varphi\colon V \to \mathbb{S}^{d-1}$ be a map of the sphere $\mathbb{S}^{d-1}$. Then, at $\boldsymbol{x} \in V$, the pullback of the volume form on the sphere is*

$$[\varphi^*(\mathrm{d}\mathbb{S}^{d-1})](\boldsymbol{x}) = \det(\varphi(\boldsymbol{x}), J_\varphi(\boldsymbol{x}))\, \mathrm{d}\mathbb{R}^{d-1}(\boldsymbol{x}). \tag{2.4}$$

Proof. We follow the derivation of [Mül98, (§1.25)]. Let $\boldsymbol{x} \in V$, and let $\boldsymbol{u}^1, \dots, \boldsymbol{u}^{d-1} \in T_{\boldsymbol{x}}V = \mathbb{R}^{d-1}$. By (2.1), we have the pullback

$$\begin{aligned}
[\varphi^*(\mathrm{d}\mathbb{S}^{d-1})](\boldsymbol{x})(\boldsymbol{u}^1, \dots, \boldsymbol{u}^{d-1}) &= \mathrm{d}\mathbb{S}^{d-1}(\varphi(\boldsymbol{x}))(J_\varphi(\boldsymbol{x})\boldsymbol{u}^1, \dots, J_\varphi(\boldsymbol{x})\boldsymbol{u}^{d-1}) \\
&= \det\left(\varphi(\boldsymbol{x}), J_\varphi(\boldsymbol{x})\boldsymbol{u}^1, \dots, J_\varphi(\boldsymbol{x})\boldsymbol{u}^{d-1}\right).
\end{aligned}$$

Setting $M := (\varphi(\boldsymbol{x}), J_\varphi(\boldsymbol{x})) \in \mathbb{R}^{d\times d}$, we obtain

$$\begin{aligned}
[\varphi^*(\mathrm{d}\mathbb{S}^{d-1})](\boldsymbol{x})(\boldsymbol{u}^1, \dots, \boldsymbol{u}^{d-1}) &= \det\left(M\boldsymbol{\epsilon}^1, M\begin{pmatrix}0\\ \boldsymbol{u}^1\end{pmatrix}, \dots, M\begin{pmatrix}0\\ \boldsymbol{u}^{d-1}\end{pmatrix}\right) \\
&= \det(M)\, \det\left(\boldsymbol{\epsilon}^1, \begin{pmatrix}0\\ \boldsymbol{u}^1\end{pmatrix}, \dots, \begin{pmatrix}0\\ \boldsymbol{u}^{d-1}\end{pmatrix}\right) \\
&= \det(\varphi(\boldsymbol{x}), J_\varphi(\boldsymbol{x}))\, \det\left(\boldsymbol{u}^1, \dots, \boldsymbol{u}^{d-1}\right).
\end{aligned}$$

■

We can apply the last proposition to reduce the integration on the sphere $\mathbb{S}^{d-1}$ to the integration on the $(d-1)$-dimensional unit ball

$$\mathbb{B}^{d-1} := \{\boldsymbol{x} \in \mathbb{R}^{d-1} \,;\, \|\boldsymbol{x}\| < 1\}$$

as follows.

Proposition 2.2. *We denote by*

$$\mathbb{S}^{d-1}_{+} := \{\boldsymbol{\xi} \in \mathbb{S}^{d-1} \,;\, \xi_d > 0\}$$

and

$$\mathbb{S}^{d-1}_{-} := \{\boldsymbol{\xi} \in \mathbb{S}^{d-1} \,;\, \xi_d < 0\}$$

the upper and lower hemisphere of $\mathbb{S}^{d-1}$, respectively. We define the two maps

$$\varphi^\pm\colon \mathbb{B}^{d-1} \to \mathbb{S}^{d-1}_{\pm}, \quad \varphi^\pm(\boldsymbol{x}) := \begin{pmatrix}\boldsymbol{x}\\ \pm\sqrt{1-\|\boldsymbol{x}\|^2}\end{pmatrix}. \tag{2.5}$$

Then we have

$$\int_{\mathbb{S}^{d-1}_{\pm}} f(\boldsymbol{\xi})\, \mathrm{d}\mathbb{S}^{d-1}(\boldsymbol{\xi}) = \int_{\mathbb{B}^{d-1}} f(\varphi^\pm(\boldsymbol{x}))\, \frac{1}{\sqrt{1-\|\boldsymbol{x}\|^2}}\, \mathrm{d}\mathbb{R}^{d-1}(\boldsymbol{x}). \tag{2.6}$$

Proof. We are going to apply (2.4) for the maps $\varphi^{\pm}$. We have

$$\det(\varphi^{\pm}(\boldsymbol{x}), J_{\varphi^{\pm}}(\boldsymbol{x})) = \det\left(\varphi^{\pm}(\boldsymbol{x}), \quad \tfrac{\partial \varphi^{\pm}(\boldsymbol{x})}{\partial x_1}, \quad \cdots, \quad \tfrac{\partial \varphi_1^{\pm}}{\partial x_{d-1}}\right).$$

Computing the derivatives, we have

$$\det(\varphi^{\pm}(\boldsymbol{x}), J_{\varphi^{\pm}}(\boldsymbol{x})) = \det \begin{pmatrix} x_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_{d-1} & 0 & \cdots & 1 \\ \pm\sqrt{1-\|\boldsymbol{x}\|^2} & \frac{\mp x_1}{\sqrt{1-\|\boldsymbol{x}\|^2}} & \cdots & \frac{\mp x_{d-1}}{\sqrt{1-\|\boldsymbol{x}\|^2}} \end{pmatrix}.$$

For $i = 1, \ldots, d-1$, we add $\pm\frac{x_i}{\sqrt{1-\|\boldsymbol{x}\|^2}}$ times the i-th row to the last row of the matrix and obtain

$$\det(\varphi^{\pm}(\boldsymbol{x}), J_{\varphi^{\pm}}(\boldsymbol{x})) = \det \begin{pmatrix} x_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_{d-1} & 0 & \cdots & 1 \\ \pm\sqrt{1-\|\boldsymbol{x}\|^2} \pm \frac{\|\boldsymbol{x}\|^2}{\sqrt{1-\|\boldsymbol{x}\|^2}} & 0 & \cdots & 0 \end{pmatrix}.$$

The Laplace expansion rule of the determinant with respect to the last row gives

$$\begin{aligned} \det(\varphi^{\pm}(\boldsymbol{x}), J_{\varphi^{\pm}}(\boldsymbol{x})) &= (-1)^{d-1}\left(\pm\sqrt{1-\|\boldsymbol{x}\|^2} \pm \frac{\|\boldsymbol{x}\|^2}{\sqrt{1-\|\boldsymbol{x}\|^2}}\right) \\ &= \frac{\pm(-1)^{d-1}}{\sqrt{1-\|\boldsymbol{x}\|^2}}. \end{aligned}$$

The sign $\pm(-1)^{d-1}$ indicates whether $\varphi^{\pm}$ is orientation-preserving. However, since the integration depends on positive orientation, we have to change the sign in case that $\varphi^{\pm}$ switches the orientation. Hence, we have shown (2.6). ■

Additionally to the map (2.5) from the ball to the sphere, the volume form on $\mathbb{S}^{d-1}$ can also be expressed with the help of the volume form on $\mathbb{S}^{d-2}$ as follows. We decompose a vector $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_d)^\top \in \mathbb{S}^{d-1}$ as

$$\boldsymbol{\xi} = \xi_d\, \boldsymbol{\epsilon}^d + \sqrt{1-\xi_d^2}\, \boldsymbol{\xi}_{(d-1)}, \tag{2.7}$$

where $\xi_d \in [-1, 1]$ and $\boldsymbol{\xi}_{(d-1)} \in \mathbb{S}^{d-2} \times \{0\}$. The surface measure $\mathrm{d}\mathbb{S}^{d-1}$ of the sphere $\mathbb{S}^{d-1}$ can be decomposed for $d \geq 3$ by the formula [AH12, (1.16)]

$$\mathrm{d}\mathbb{S}^{d-1}(\boldsymbol{\xi}) = \mathrm{d}\mathbb{S}^{d-2}(\boldsymbol{\xi}_{(d-1)})\,(1-\xi_d^2)^{\frac{d-3}{2}}\,\mathrm{d}\xi_d, \tag{2.8}$$

which can be shown analogously to Proposition 2.2. Accordingly, the integral is written as

$$\int_{\mathbb{S}^{d-1}} f(\boldsymbol{\xi})\,\mathrm{d}\mathbb{S}^{d-1}(\boldsymbol{\xi}) = \int_{-1}^{1}\int_{\mathbb{S}^{d-2}} f(\boldsymbol{\xi})\,\mathrm{d}\mathbb{S}^{d-2}(\boldsymbol{\xi}_{(d-1)})\,(1-\xi_d^2)^{\frac{d-3}{2}}\,\mathrm{d}\xi_d. \tag{2.9}$$

Equation (2.9) allows to proof inductively that the volume of the sphere $\mathbb{S}^{d-1}$ is given by [AH12, (1.19)]

$$\left|\mathbb{S}^{d-1}\right| = \int_{\mathbb{S}^{d-1}} \mathrm{d}\mathbb{S}^{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)} = \begin{cases} \frac{2\pi^{d/2}}{(\frac{d}{2}-1)!}, & d \text{ even} \\ \frac{2^{\frac{d+1}{2}}\pi^{\frac{d-1}{2}}}{(d-2)!!}, & d \text{ odd}. \end{cases} \tag{2.10}$$

We note that (2.10) also holds in the case $d = 1$, where we have $|\mathbb{S}^0| = 2$.

2.1.2 Spherical harmonics

The Hilbert space $L^2(\mathbb{S}^{d-1})$ is defined as the space of all measurable functions $f\colon \mathbb{S}^{d-1} \to \mathbb{C}$, whose norm

$$\|f\|_{L^2(\mathbb{S}^{d-1})} := \sqrt{\langle f, f\rangle_{L^2(\mathbb{S}^{d-1})}}$$

is finite, where

$$\langle f, g\rangle_{L^2(\mathbb{S}^{d-1})} := \int_{\mathbb{S}^{d-1}} f(\boldsymbol{\xi})\,\overline{g(\boldsymbol{\xi})}\,\mathrm{d}\boldsymbol{\xi}$$

denotes the inner product in $L^2(\mathbb{S}^{d-1})$.

In this section, we introduce the construction of spherical harmonics, which form an orthonormal basis of $L^2(\mathbb{S}^{d-1})$. We mostly follow Atkinson and Han [AH12, Section 2], see also [Mül98, DX13].

A function $f\colon \mathbb{R}^d \to \mathbb{C}$ is called homogeneous of degree n if

$$f(\lambda \boldsymbol{x}) = \lambda^n f(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^d,$$

for all $\lambda > 0$. The function f is called *harmonic* if the Laplacian $\Delta f = 0$. For $n \in \mathbb{N}_0$, we denote by $\mathscr{H}_{n,d}(\mathbb{R}^d)$ the space of harmonic polynomials on $\mathbb{R}^d$ that are *homogeneous* of degree n and we call its restriction to the sphere

$$\mathscr{Y}_{n,d}(\mathbb{S}^{d-1}) := \left\{ f\big|_{\mathbb{S}^{d-1}} \,;\, f \in \mathscr{H}_{n,d}(\mathbb{R}^d)\right\} \tag{2.11}$$

the space of spherical harmonics of degree n, which has the dimension

$$N_{n,d} = \dim\left(\mathscr{Y}_{n,d}(\mathbb{S}^{d-1})\right) = \frac{(2n+d-2)(n+d-3)!}{n!(d-2)!}. \tag{2.12}$$

Legendre polynomials. We collect some facts about Legendre polynomials as found in our paper [QHL18]. The *Legendre polynomial* $P_{n,d}$ of degree $n \in \mathbb{N}_0$ in dimension $d \geq 2$ is given by the Rodrigues formula [AH12, (2.70)]

$$P_{n,d}(t) := (-1)^n \frac{(d-3)!!}{(2n+d-3)!!} (1-t^2)^{\frac{3-d}{2}} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n (1-t^2)^{n+\frac{d-3}{2}}, \qquad t \in [-1,1]. \tag{2.13}$$

The classical Legendre polynomials are $P_n := P_{n,3}$. Here, we use this name also in the general situation $d > 3$ as in [Mül98, AH12]. For $d = 2$, the Legendre polynomial $P_{n,2}$ is known as the Chebyshev polynomial of the first kind. For $d = 4$, the Legendre polynomial $P_{n,4}$ is known as the Chebyshev polynomial of the second kind. The Legendre polynomials can be defined recursively by [AH12, (2.86)]

$$P_{n,d}(t) = \frac{2n+d-4}{n+d-3} t P_{n-1,d}(t) - \frac{n-1}{n+d-3} P_{n-2,d}(t), \qquad t \in [-1,1], \tag{2.14}$$

for $n \geq 2$, initialized by $P_{0,d}(t) = 1$ and $P_{1,d}(t) = t$. The Legendre polynomials $P_{n,d}$ satisfy for all $n \in \mathbb{N}_0$ [AH12, (2.116)]

$$|P_{n,d}(t)| \leq P_{n,d}(1) = 1, \qquad t \in [-1,1]. \tag{2.15}$$

The Legendre polynomials are orthogonal with respect to the weight function

$$w_d(t) := (1-t^2)^{\frac{d-3}{2}}, \qquad t \in (-1,1). \tag{2.16}$$

In particular, they satisfy the orthogonality relation

$$\int_{-1}^{1} P_{n,d}(t)\, P_{m,d}(t)\, (1-t^2)^{\frac{d-3}{2}}\, \mathrm{d}t = \delta_{n,m} \frac{\left|\mathbb{S}^{d-1}\right|}{N_{n,d}\left|\mathbb{S}^{d-2}\right|} \tag{2.17}$$

for all $m, n \in \mathbb{N}_0$. We also define the normalized Legendre polynomial $\widetilde{P}_{n,d}$ of degree $n \in \mathbb{N}_0$ by

$$\widetilde{P}_{n,d}(t) := \sqrt{\frac{N_{n,d}\left|\mathbb{S}^{d-2}\right|}{\left|\mathbb{S}^{d-1}\right|}}\, P_{n,d}(t) = \frac{\sqrt{(2n+d-2)\,(n+d-3)!}}{2^{(d-2)/2}\sqrt{n!}\,\Gamma(\frac{d-1}{2})}\, P_{n,d}(t). \tag{2.18}$$

The normalized Legendre polynomials $\{\widetilde{P}_{n,d}\,;\, n \in \mathbb{N}_0\}$ form an orthonormal basis of the weighted Lebesgue space $L^2((-1,1); w_d)$ induced by the inner product

$$\langle f, g\rangle_{L^2((-1,1);w_d)} := \int_{-1}^{1} f(t)\, \overline{g(t)}\, (1-t^2)^{\frac{d-3}{2}}\, \mathrm{d}t.$$

Up to its normalization, the Legendre polynomial $P_{n,d}$ is equal to the Gegenbauer or ultraspherical polynomial

$$C_n^{(\frac{d-2}{2})} = \binom{n+d-3}{n} P_{n,d} \tag{2.19}$$

for $d \geq 3$, see [AH12, (2.145)]. The *Gegenbauer polynomial* $C_n^{(\alpha)}$ of degree $n \in \mathbb{N}_0$ and order $\alpha > -1/2$ satisfies the explicit expression [AS72, 22.3]

$$C_n^{(\alpha)}(t) := \frac{1}{\Gamma(\alpha)} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^m\, \Gamma(n-m+\alpha)}{m!\,(n-2m)!} (2t)^{n-2m} \tag{2.20}$$

and the Rodrigues formula

$$C_n^{(\alpha)}(t) = \frac{(-1)^n\, \Gamma(\alpha+\frac{1}{2})\, \Gamma(n+2\alpha)}{2^n\, n!\, \Gamma(2\alpha)\, \Gamma(\alpha+n+\frac{1}{2})} (1-t^2)^{\frac{1}{2}-\alpha} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n (1-t^2)^{n+\alpha-\frac{1}{2}}. \tag{2.21}$$

Jacobi polynomials. The Legendre polynomials are special cases of the Jacobi polynomials. The *Jacobi polynomial* $P_n^{(\alpha,\beta)}$ of degree $n \in \mathbb{N}_0$ and orders $\alpha, \beta > -1$ is given by the Rodrigues formula [AS72, Section 22.11] for $t \in [-1,1]$,

$$P_n^{(\alpha,\beta)}(t) := \frac{(-1)^n}{2^n n!} (1-t)^{-\alpha} (1+t)^{-\beta} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \left((1-t)^{n+\alpha} (1+t)^{n+\beta}\right). \tag{2.22}$$

Comparing (2.13) with (2.22), we see the relation of the Legendre polynomial $P_{n,d}$ with the Jacobi polynomial $P_n^{(\frac{d-3}{2},\frac{d-3}{2})}$. We have for $d \geq 3$

$$P_{n,d}(t) = \frac{2^n\, n!\, (d-3)!!}{(n+d-3)!!}\, P_n^{(\frac{d-3}{2},\frac{d-3}{2})}(t).$$

The Jacobi polynomials $P_n^{(\alpha,\beta)}$ satisfy for $n, n' \in \mathbb{N}_0$ the orthogonality relation [AS72, Section 22.2]

$$\begin{aligned}\int_{-1}^{1} P_n^{(\alpha,\beta)}(t)\, P_{n'}^{(\alpha,\beta)}(t)\, (1-t)^{\alpha}(1+t)^{\beta}\, \mathrm{d}x& \\ = \delta_{n,n'}\, \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)n!}&.\end{aligned} \tag{2.23}$$

Zonal harmonics. Let $n \in \mathbb{N}_0$, and let $Y_{n,d}^k$, $k = 1, \ldots, N_{n,d}$, be an orthonormal basis of the space $\mathscr{Y}_{n,d}(\mathbb{S}^{d-1})$. We define the zonal harmonic $Z_{n,d} \colon \mathbb{S}^{d-1} \times \mathbb{S}^{d-1} \to \mathbb{R}$ of degree $n \in \mathbb{N}_0$ by

$$Z_{n,d}(\boldsymbol{\xi}, \boldsymbol{\eta}) := \sum_{k=1}^{N_{n,d}} Y_{n,d}^k(\boldsymbol{\xi})\, \overline{Y_{n,d}^k(\boldsymbol{\eta})}, \qquad \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{S}^{d-1}. \tag{2.24}$$

The *zonal harmonic* $Z_{n,d}$ is the reproducing kernel of the space $\mathscr{Y}_{n,d}(\mathbb{S}^{d-1})$, i. e., for any $Y_{n,d} \in \mathscr{Y}_{n,d}(\mathbb{S}^{d-1})$ we have

$$Y_{n,d}(\boldsymbol{\xi}) = \int_{\mathbb{S}^{d-1}} Y_{n,d}(\boldsymbol{\eta})\, Z_{n,d}(\boldsymbol{\xi}, \boldsymbol{\eta})\, \mathrm{d}\boldsymbol{\eta}, \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1}.$$

The addition theorem [AH12, (2.24)] states that for $n \in \mathbb{N}_0$

$$\sum_{k=1}^{N_{n,d}} Y_{n,d}^k(\boldsymbol{\xi})\, \overline{Y_{n,d}^k(\boldsymbol{\eta})} = \frac{N_{n,d}}{|\mathbb{S}^{d-1}|}\, P_{n,d}(\boldsymbol{\xi}^\top \boldsymbol{\eta}), \qquad \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{S}^{d-1}. \tag{2.25}$$

Spherical Fourier series. Every function $f \in L^2(\mathbb{S}^{d-1})$ satisfies the expansion in the *spherical Fourier series*

$$f = \sum_{n=0}^{\infty} \sum_{k=1}^{N_{n,d}} \hat{f}_{n,d}^k\, Y_{n,d}^k \tag{2.26}$$

with the *spherical Fourier coefficients*

$$\hat{f}^k_{n,d} := \int_{\mathbb{S}^{d-1}} f(\boldsymbol{\xi})\, \overline{Y^k_{n,d}(\boldsymbol{\xi})}\, \mathrm{d}\boldsymbol{\xi}, \qquad n \in \mathbb{N}_0,\ k = 1, \dots, N_{n,d}, \tag{2.27}$$

and it satisfies Parseval's equality

$$\|f\|^2_{L^2(\mathbb{S}^{d-1})} = \sum_{n=0}^{\infty} \sum_{k=1}^{N_{n,d}} \left| \hat{f}^k_{n,d} \right|^2. \tag{2.28}$$

Convolution on the sphere. The spherical convolution of a function $\psi\colon [-1,1] \to \mathbb{C}$ with a function $f\colon \mathbb{S}^{d-1} \to \mathbb{C}$ is defined by

$$[\psi \star f](\boldsymbol{\xi}) := \int_{\mathbb{S}^{d-1}} f(\boldsymbol{\eta})\, \psi(\boldsymbol{\xi}^\top \boldsymbol{\eta})\, \mathrm{d}\boldsymbol{\eta}, \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1}. \tag{2.29}$$

The *Funk–Hecke formula* [AH12, Theorem 2.22] states that for a spherical harmonic $Y_{n,d} \in \mathscr{Y}_{n,d}(\mathbb{S}^{d-1})$ and a function $\psi \in L^1([-1,1])$ for which $\int_{-1}^1 |\psi(t)|\, (1-t^2)^{\frac{d-3}{2}}\, \mathrm{d}t$ is finite, we have

$$[\psi \star Y_{n,d}](\boldsymbol{\xi}) = Y_{n,d}(\boldsymbol{\xi})\, \left|\mathbb{S}^{d-2}\right| \int_{-1}^{1} \psi(t)\, P_{n,d}(t)\, (1-t^2)^{\frac{d-3}{2}}\, \mathrm{d}t, \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1}. \tag{2.30}$$

The Funk–Hecke formula was introduced for $\mathbb{S}^2$ by Funk [Fun15b] and Hecke [Hec17].

2.1.3 An explicit construction of spherical harmonics

We give an explicit construction of an orthonormal basis of spherical harmonics $Y^k_{n,d}$ of $\mathscr{Y}_{n,d}(\mathbb{S}^{d-1})$. This construction is done recursively with respect to the dimension d and it is based on the decomposition (2.7) of the volume form on $\mathbb{S}^{d-1}$. We mostly follow [AH12, Section 2].

Spherical harmonics on $\mathbb{S}^1$

We start with the case $d = 2$. The one-dimensional unit sphere

$$\mathbb{S}^1 = \{(\cos\varphi, \sin\varphi)^\top \in \mathbb{R}^2\ ;\ \varphi \in [0, 2\pi)\}$$

is the unit circle in the plane $\mathbb{R}^2$. The two complex-valued functions $(x,y) \mapsto (x \pm y\mathrm{i})^n$ are harmonic polynomials that are homogeneous of degree n in the plane $\mathbb{R}^2$. Restricted to $\mathbb{S}^1$ and using polar coordinates $(x,y) = (\cos\varphi, \sin\varphi) \in \mathbb{S}^1$, the two functions can be expressed as the trigonometric polynomials

$$x + y\mathrm{i} = \mathrm{e}^{\pm\mathrm{i}\varphi}.$$

An orthonormal basis of the space $\mathscr{Y}_{n,2}(\mathbb{S}^1)$ for $n \in \mathbb{N}$ consists of the two trigonometric polynomials

$$Y_{n,2}^1(\cos\varphi, \sin\varphi) := \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\mathrm{i}n\varphi}, \qquad Y_{n,2}^2(\cos\varphi, \sin\varphi) := \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\mathrm{i}n\varphi}, \qquad \varphi \in [0, 2\pi). \tag{2.31}$$

For $n = 0$, an orthonormal basis of the space $\mathscr{Y}_{0,2}(\mathbb{S}^1)$ consists of the single constant function $\frac{1}{\sqrt{2\pi}}$.

Spherical harmonics on $\mathbb{S}^{d-1}$

The case for general $d \geq 3$ requires some more effort. We define the *associated Legendre function* $P_{n,d}^m$ of degree $n \in \mathbb{N}_0$ and order $m = 0, \ldots, n$ in dimension $d \geq 3$ by [AH12, Proposition 2.42]

$$P_{n,d}^m(t) := \frac{n!\,\Gamma(\frac{d-1}{2})}{2^m\,(n-m)!\,\Gamma(m + \frac{d-1}{2})}\,(1-t^2)^{m/2}\,P_{n-m,d+2m}(t), \qquad t \in [-1, 1], \tag{2.32}$$

which is equal to the Legendre polynomial $P_{n,d} = P_{n,d}^0$ for $m = 0$. The associated Legendre functions $P_{n,d}^m$ are occasionally called associated Legendre polynomials, however, they are polynomials only if the order m is even. The normalized associated Legendre functions [AH12, (2.158)] are given by

$$\widetilde{P}_{n,d}^m := \frac{\sqrt{(2n+d-2)\,(n-m)!\,(n+m+d-3)!}}{2^{(d-2)/2}\,n!\,\Gamma(\frac{d-1}{2})}\,P_{n,d}^m. \tag{2.33}$$

We see that the normalized Legendre polynomial can be written as $\widetilde{P}_{n,d} = \widetilde{P}_{n,d}^0$. The normalized associated Legendre functions $\widetilde{P}_{n,d}^m$ satisfy the orthogonality relation [AH12, (2.160)]

$$\int_{-1}^{1} \widetilde{P}_{n,d}^m(t)\,\widetilde{P}_{n',d}^m(t)\,(1-t^2)^{\frac{d-3}{2}}\,\mathrm{d}t = \delta_{n,n'} \tag{2.34}$$

for $n, n' \in \mathbb{N}_0$ and $m = 0, \ldots, \min\{n, n'\}$.

We define an orthonormal basis of $L^2(\mathbb{S}^{d-1})$ recursively with respect to the dimension $d \geq 3$. As the base case of this recursion with respect to d, the spherical harmonics for dimension $d = 2$ are given in (2.31). For $m \in \mathbb{N}_0$, let $\{Y_{m,d-1}^j\,;\, j = 1, \ldots, N_{m,d-1}\}$ be an orthonormal basis of $\mathscr{Y}_{m,d-1}(\mathbb{S}^{d-2})$. As in (2.7), we write

$$\boldsymbol{\xi} = \sqrt{1-\xi_d^2}\,\boldsymbol{\xi}_{(d-1)} + \xi_d \boldsymbol{\varepsilon}^d \in \mathbb{S}^{d-1}.$$

Then the polynomials

$$Y_{n,d}^{m,j}(\boldsymbol{\xi}) := \widetilde{P}_{n,d}^m(\xi_d)\,Y_{m,d-1}^j(\boldsymbol{\xi}_{(d-1)}), \qquad m = 0, \ldots, n,\ j = 1, \ldots, N_{m,d-1}, \tag{2.35}$$

form an orthonormal basis of $\mathscr{Y}_{n,d}(\mathbb{S}^{d-1})$, see [AH12, (2.164)]. We note that the orthonormality of the functions (2.35) is a direct consequence of the assumed orthonormality of the spherical harmonics $Y^j_{m,d-1}$ on $\mathbb{S}^{d-1}$, the orthonormality (2.34) of the normalized associated Legendre functions $\widetilde{P}^m_{n,d}$ and the substitution rule (2.8). The spherical harmonics $Y^{m,j}_{n,d}$ are complete in $L^2(\mathbb{S}^{d-1})$, i. e., the set

$$\left\{Y^{m,j}_{n,d}\ ;\ n \in \mathbb{N}_0,\ m = 0, \dots, n,\ j = 1, \dots, N_{m,d-1}\right\}$$

forms an orthonormal basis of $L^2(\mathbb{S}^{d-1})$.

In each step of the recursion with respect to d, we have to perform a reparameterization of the indices of the spherical harmonics. We want to obtain an orthonormal basis of the space of spherical harmonics $\mathscr{Y}_{n,d}(\mathbb{S}^{d-1})$ of the form $Y^k_{n,d}$, $k = 1, \dots, N_{n,d}$. However, we have given the basis $Y^{m,j}_{n,d}$ as defined in (2.35). To this end, we convert the two indices (m, j) to a single index k. We set

$$k = k(m, j) := j + \sum_{l=0}^{m-1} N_{l,d-1}, \qquad m = 0, \dots, n,\ j = 1, \dots, N_{m,d-1}.$$

We note that the number of indices does not change this way, i.e., we have

$$k(n, N_{n,d-1}) = \sum_{l=0}^{m} N_{m,d-1} = N_{m,d},$$

cf. [AH12, (2.14)]. Then

$$Y^k_{n,d} := Y^{m,j}_{n,d}, \qquad k = k(m, j) \in \{1, \dots, N_{n,d-1}\}$$

is the desired orthonormal basis of the space of spherical harmonics $\mathscr{Y}_{n,d}(\mathbb{S}^{d-1})$.

Spherical harmonics on $\mathbb{S}^2$

In this section, we take a special look at the spherical harmonics for $d = 3$. Compared to the general case $d \geq 3$, we introduce a simplified notation for $d = 3$, as we used in [QHL18], see also [Mic13]. The two-dimensional sphere $\mathbb{S}^2$ has a parameterization in terms of the spherical coordinates

$$\boldsymbol{\xi}(\varphi, \vartheta) = (\cos\varphi\, \sin\vartheta, \sin\varphi\, \sin\vartheta, \cos\vartheta)^\top, \quad \varphi \in [0, 2\pi),\ t \in [-1, 1]. \tag{2.36}$$

We define the normalized associated Legendre function of degree $n \in \mathbb{N}_0$ and order $k = 0, \dots, n$ by the Rodrigues formula

$$\widetilde{P}^k_n(t) := \sqrt{\frac{2n+1}{4\pi}\,\frac{(n-k)!}{(n+k)!}}\,\frac{(-1)^k}{2^n n!}\left(1-t^2\right)^{k/2}\frac{\mathrm{d}^{n+k}}{\mathrm{d}t^{n+k}}\left(t^2-1\right)^n, \qquad t \in [-1, 1], \tag{2.37}$$

and

$$\widetilde{P}_n^{-k} := (-1)^k \, \widetilde{P}_n^k. \tag{2.38}$$

Note that this definition is in line with the associated Legendre functions $P_{n,3}^k$ from (2.33). In particular, we have for $n \in \mathbb{N}_0$ and $k = -n, \ldots, n$

$$\widetilde{P}_n^k(t) = (-1)^{\frac{k-|k|}{2}} \, \widetilde{P}_{n,3}^{|k|}(t), \qquad t \in [-1,1].$$

An orthonormal basis in the Hilbert space $L^2(\mathbb{S}^2)$ of square-integrable functions on the sphere is formed by the spherical harmonics

$$Y_n^k(\boldsymbol{\xi}(\varphi,t)) := \widetilde{P}_n^k(t)\, \mathrm{e}^{\mathrm{i}k\varphi}, \qquad \boldsymbol{\xi}(\varphi,t) \in \mathbb{S}^2, \tag{2.39}$$

of degree $n \in \mathbb{N}_0$ and order $k \in \{-n, \ldots, n\}$. This definition of the spherical harmonic Y_n^k on $\mathbb{S}^2$ is closely related to the general definition of $Y_{n,3}^{m,j}$ in (2.35). In particular, we have

$$Y_n^k = (-1)^{\frac{k-|k|}{2}} \, Y_{n,3}^{|k|,j(k)}, \qquad n \in \mathbb{N}_0,\ k \in \{-n, \ldots, n\},$$

where $j(k) = 1$ if $k \geq 0$ and $j(k) = 2$ if $k < 0$.

Remark 2.3. The factor $(-1)^k$ in the definition (2.38) of the associated Legendre polynomials $\widetilde{P}_n^k$ for negative k is sometimes called the Condon–Shortley phase factor, which implies that the relation

$$Y_n^k(\boldsymbol{\xi}) = (-1)^k \, \overline{Y_n^{-k}}(\boldsymbol{\xi}), \qquad \boldsymbol{\xi} \in \mathbb{S}^2$$

holds. The usage of this factor is inconsistent in the literature. Some authors like [PST98] define the spherical harmonics without this factor. □

2.1.4 Derivatives

In this section, which is based on [Que18], we introduce the surface gradient and Laplacian on the sphere as well as the space $C^s(\mathbb{S}^{d-1})$ of functions with continuous derivatives. For brevity, we denote by

$$\partial_i := \frac{\partial}{\partial x_i}$$

the partial derivative with respect to the i-th coordinate of $\boldsymbol{x} = (x_1, \ldots, x_d)^\top \in \mathbb{R}^d$. We extend a function $f \colon \mathbb{S}^{d-1} \to \mathbb{C}$ to the ambient space $\mathbb{R}^d \setminus \{\mathbf{0}\}$ by

$$f^\bullet(\boldsymbol{x}) := f\left(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\right), \qquad \boldsymbol{x} \in \mathbb{R}^d \setminus \{\mathbf{0}\}, \tag{2.40}$$

cf. [Mül98, (§14.32)]. The surface gradient $\boldsymbol{\nabla}^\bullet$ on the sphere is the projection of the gradient

$$\boldsymbol{\nabla} := \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}\right)^\top$$

in $\mathbb{R}^d$ onto the tangent space of the unit sphere $\mathbb{S}^{d-1}$. For $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$, we have

$$\boldsymbol{\nabla}^{\bullet} f(\boldsymbol{\xi}) := \boldsymbol{\nabla} f^{\bullet}(\boldsymbol{\xi}).$$

The restriction of the Laplacian

$$\Delta := \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$$

to the sphere $\mathbb{S}^{d-1}$ is known as the Laplace–Beltrami operator [Mül98, (§14.20)]

$$\Delta^{\bullet} f(\boldsymbol{\xi}) := \Delta f^{\bullet}(\boldsymbol{\xi}). \tag{2.41}$$

The spherical harmonics $Y_{n,d}$ are eigenfunctions of the Laplace–Beltrami operator $\Delta^{\bullet}$, see [DX13, (1.4.9)]. In particular, we have for any $Y_{n,d} \in \mathscr{Y}_{n,d}(\mathbb{S}^{d-1})$

$$\Delta^{\bullet} Y_{n,d} = -n\,(n+d-2)\,Y_{n,d}, \qquad n \in \mathbb{N}_0. \tag{2.42}$$

Spaces of functions having continuous derivatives. For a multi-index

$$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)^{\top} \in \mathbb{N}_0^d,$$

we define its one-norm

$$\|\boldsymbol{\alpha}\|_1 := \sum_{i=1}^{d} |\alpha_i|\,.$$

Let $s \in \mathbb{N}_0$. We denote by $C^s(\mathbb{S}^{d-1})$ the space of functions $f\colon \mathbb{S}^{d-1} \to \mathbb{C}$ whose extension $f^{\bullet}$ has continuous derivatives up to the order s with the norm

$$\|f\|_{C^s(\mathbb{S}^{d-1})} := \max_{\|\boldsymbol{\alpha}\|_1 \le s} \sup_{\boldsymbol{\xi} \in \mathbb{S}^{d-1}} \left|\partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} f^{\bullet}(\boldsymbol{\xi})\right|.$$

Furthermore, the space $C^{\infty}(\mathbb{S}^{d-1})$ consists of all functions $f\colon \mathbb{S}^{d-1} \to \mathbb{C}$ for which $f^{\bullet}$ has continuous derivatives of arbitrary order. The space $C(\mathbb{S}^{d-1}) := C^0(\mathbb{S}^{d-1})$ is the space of continuous functions with the uniform norm

$$\|f\|_{C(\mathbb{S}^{d-1})} := \sup_{\boldsymbol{\xi} \in \mathbb{S}^{d-1}} |f(\boldsymbol{\xi})|\,.$$

The definition implies for $f \in C^{s+1}(\mathbb{S}^{d-1})$

$$\|f\|_{C^s(\mathbb{S}^{d-1})} \le \|f\|_{C^{s+1}(\mathbb{S}^{d-1})}\,. \tag{2.43}$$

We define the space $C^s(\mathbb{S}^{d-1} \to \mathbb{R}^d)$ of vector fields

$$\boldsymbol{f}\colon \mathbb{S}^{d-1} \to \mathbb{R}^d, \quad \boldsymbol{f}(\boldsymbol{\xi}) = [f_i(\boldsymbol{\xi})]_{i=1}^{d},$$

with component functions $f_i \in C^s(\mathbb{S}^{d-1})$. The norm in $C^s(\mathbb{S}^{d-1} \to \mathbb{R}^d)$ is the Euclidean norm over the C^s-norms of its component functions, i. e., we set

$$\|\boldsymbol{f}\|_{C^s(\mathbb{S}^{d-1}\to\mathbb{R}^d)} := \sqrt{\sum_{i=1}^{d} \|f_i\|^2_{C^s(\mathbb{S}^{d-1})}}. \tag{2.44}$$

We see that for $f \in C^{s+1}(\mathbb{S}^{d-1})$

$$\|\boldsymbol{\nabla}^\bullet f\|^2_{C^s(\mathbb{S}^{d-1}\to\mathbb{R}^d)} = \sum_{i=1}^{d} \|\partial_i f^\bullet\|^2_{C^s(\mathbb{S}^{d-1})} \le \sum_{i=1}^{d} \|f\|^2_{C^{s+1}(\mathbb{S}^{d-1})} = d\ \|f\|^2_{C^{s+1}(\mathbb{S}^{d-1})}. \tag{2.45}$$

2.1.5 Spherical Sobolev spaces

We introduce spherical Sobolev spaces, which give a rigorous definition of the smoothness of functions that are defined on the sphere $\mathbb{S}^{d-1}$. This section uses material from [Que18], see also [AH12, Section 3.8].

A function $f\colon \mathbb{S}^{d-1} \to \mathbb{C}$ that has a finite representation

$$f = \sum_{n=0}^{N} \sum_{k=0}^{N_{n,d}} \hat{f}^k_{n,d}\, Y^k_{n,d}$$

with respect to spherical harmonics $Y^k_{n,d}$ is called a spherical polynomial of degree $N \in \mathbb{N}_0$ if $\hat{f}^k_{N,d} \neq 0$ for some k. For a spherical polynomial $f\colon \mathbb{S}^{d-1} \to \mathbb{C}$ with $d \ge 3$ and some $s \in \mathbb{R}$, we introduce the Sobolev norm

$$\|f\|^2_{H^s(\mathbb{S}^{d-1})} := \sum_{n=0}^{\infty} \sum_{k=1}^{N_{n,d}} \left(n + \tfrac{d-2}{2}\right)^{2s} \left|\left\langle f, Y^k_{n,d}\right\rangle_{L^2(\mathbb{S}^{d-1})}\right|^2, \tag{2.46}$$

cf. [AH12, (3.98)]. The Sobolev space $H^s(\mathbb{S}^{d-1})$ of order s is defined as the completion of the space of all spherical polynomials with respect to the Sobolev norm $\|\cdot\|_{H^s(\mathbb{S}^{d-1})}$. We note that (2.46) is a norm only for $d \ge 3$, whereas it is a semi-norm if $d = 2$.

By construction, the linear span of all spherical harmonics $Y^k_{n,d}$, which is equal to the set of all spherical polynomials, is dense in the Sobolev space $H^s(\mathbb{S}^{d-1})$. The Sobolev spaces are nested: the continuous embedding $H^s(\mathbb{S}^{d-1}) \hookrightarrow H^t(\mathbb{S}^{d-1})$ holds whenever $s > t$. The space $H^0(\mathbb{S}^{d-1})$ can be identified with $L^2(\mathbb{S}^{d-1})$, then (2.46) with $s = 0$ is equal to Parseval's equality (2.28). If s is a non-negative integer, then $H^s(\mathbb{S}^{d-1})$ can be imagined as the space of functions defined on $\mathbb{S}^{d-1}$ whose (distributional) derivatives up to order s are in $L^2(\mathbb{S}^{d-1})$. If $s > \frac{d-1}{2}$, the continuous embedding

$$H^s(\mathbb{S}^{d-1}) \hookrightarrow C(\mathbb{S}^{d-1}) \tag{2.47}$$

holds [AH12, (3.102)]. This basically means that every function in the Sobolev space $H^s(\mathbb{S}^{d-1})$ for $s > \frac{d-1}{2}$ is continuous.

We have seen in (2.42) that the spherical harmonics $Y_{n,d}^k$ are eigenfunctions of the Laplace–Beltrami operator $\Delta^\bullet$ with respective eigenvalues $-n\,(n+d-2)$. We obtain for $s \in \mathbb{N}_0$

$$\left(-\Delta^\bullet + \frac{(d-2)^2}{4}\right)^s Y_{n,d}^k = \left(n + \frac{d-2}{2}\right)^{2s} Y_{n,d}^k. \tag{2.48}$$

Formally, we see (2.48) as a definition of fractional powers $s \in \mathbb{R}$ of the Laplace–Beltrami operator acting on spherical harmonics $Y_{n,d}^k$. With this definition, we can extend the Laplace–Beltrami operator as a pseudo-differential operator on the Sobolev space $H^s(\mathbb{S}^{d-1})$, cf. [FGS98, Section 5.1]. We set for $s \in \mathbb{R}$ and $f \in H^s(\mathbb{S}^{d-1})$

$$\left(-\Delta^\bullet + \frac{(d-2)^2}{4}\right)^{s/2} f := \sum_{n=0}^{\infty} \sum_{k=1}^{N_{n,d}} \left(n + \frac{d-2}{2}\right)^s \hat{f}_{n,d}^k\, Y_{n,d}^k. \tag{2.49}$$

The so-defined operator $\left(-\Delta^\bullet + \frac{(d-2)^2}{4}\right)^{s/2}$ is a continuous linear operator from $H^s(\mathbb{S}^{d-1})$ to $L^2(\mathbb{S}^{d-1})$. It coincides with the classical definition in (2.41) if s is positive and even. The Sobolev norm (2.46) can also be stated with the help of the Laplace–Beltrami operator $\Delta^\bullet$ by

$$\|f\|_{H^s(\mathbb{S}^{d-1})} = \left\| \left(-\Delta^\bullet + \frac{(d-2)^2}{4}\right)^{s/2} f \right\|_{L^2(\mathbb{S}^{d-1})}, \tag{2.50}$$

cf. [AH12, (3.98)].

There are other definitions of the Sobolev norm $\|\cdot\|_{H^s(\mathbb{S}^{d-1})}$ that yield equivalent norms and thus the same Sobolev space $H^s(\mathbb{S}^{d-1})$. The factor $(n + \frac{d-2}{2})$ in (2.46) is replaced by $(n+1)$ in [Str81], by $\max(n,1)$ in [CF97], or by $\sqrt{1 + n(n+d-2)}$ in [JSW99].

Sobolev spaces of vector-valued functions

In the spirit of (2.44), we extend the definition of the Sobolev norm (2.46) to vector fields. Let $s \in \mathbb{R}$. The Sobolev norm of the vector field $\boldsymbol{f} \colon \mathbb{S}^{d-1} \to \mathbb{R}^d$ is the Euclidean norm over the Sobolev norms of its component functions $\boldsymbol{f}(\boldsymbol{\xi}) = [f_1(\boldsymbol{\xi}), \dots, f_d(\boldsymbol{\xi})]^\top$, i. e.,

$$\|\boldsymbol{f}\|_{H^s(\mathbb{S}^{d-1} \to \mathbb{R}^d)}^2 := \sum_{i=1}^{d} \|f_i\|_{H^s(\mathbb{S}^{d-1})}^2. \tag{2.51}$$

Lemma 2.4. *Let $s \in \mathbb{N}_0$ and $f \in H^{s+1}(\mathbb{S}^{d-1})$. We have*

$$\|f\|_{H^{s+1}(\mathbb{S}^{d-1})}^2 = \|\boldsymbol{\nabla}^\bullet f\|_{H^s(\mathbb{S}^{d-1} \to \mathbb{R}^d)}^2 + \tfrac{(d-2)^2}{4} \|f\|_{H^s(\mathbb{S}^{d-1})}^2. \tag{2.52}$$

Proof. Since the negative Laplace–Beltrami operator $-\Delta^\bullet$ is self-adjoint, we can write the Sobolev norm (2.50) as

$$\|f\|_{H^s(\mathbb{S}^{d-1})}^2 = \left\langle \left(-\Delta^\bullet + \tfrac{(d-2)^2}{4}\right)^s f, f \right\rangle_{L^2(\mathbb{S}^{d-1})}.$$

We have for $s \in \mathbb{N}_0$

$$\begin{aligned}\|f\|^2_{H^{s+1}(\mathbb{S}^{d-1})} &= \int_{\mathbb{S}^{d-1}} \left(\left(-\Delta^\bullet + \tfrac{(d-2)^2}{4}\right)^{s+1} f(\boldsymbol{\xi})\right) f(\boldsymbol{\xi}) \,\mathrm{d}\mathbb{S}^{d-1}(\boldsymbol{\xi}) \\ &= \int_{\mathbb{S}^{d-1}} \left(\left(-\Delta^\bullet + \tfrac{(d-2)^2}{4}\right)\left(-\Delta^\bullet + \tfrac{(d-2)^2}{4}\right)^{s} f(\boldsymbol{\xi})\right) f(\boldsymbol{\xi}) \,\mathrm{d}\mathbb{S}^{d-1}(\boldsymbol{\xi}).\end{aligned}$$

Then the Green–Beltrami identity [Mül98, §14, Lemma 1]

$$\begin{aligned}-\int_{\mathbb{S}^{d-1}} f(\boldsymbol{\xi})\, \Delta^\bullet g(\boldsymbol{\xi}) \,\mathrm{d}\mathbb{S}^{d-1}(\boldsymbol{\xi}) &= \int_{\mathbb{S}^{d-1}} \langle \boldsymbol{\nabla}^\bullet f(\boldsymbol{\xi}),\, \boldsymbol{\nabla}^\bullet g(\boldsymbol{\xi})\rangle \,\mathrm{d}\mathbb{S}^{d-1}(\boldsymbol{\xi}), \\ & \qquad f \in C^2(\mathbb{S}^{d-1}),\ g \in C^1(\mathbb{S}^{d-1})\end{aligned} \tag{2.53}$$

implies that

$$\begin{aligned}\|f\|^2_{H^{s+1}(\mathbb{S}^{d-1})} &= \int_{\mathbb{S}^{d-1}} \left\langle \boldsymbol{\nabla}^\bullet f(\boldsymbol{\xi}),\, \boldsymbol{\nabla}^\bullet \left(-\Delta^\bullet + \tfrac{(d-1)^2}{4}\right)^{s} f(\boldsymbol{\xi})\right\rangle \,\mathrm{d}\mathbb{S}^{d-1}(\boldsymbol{\xi}) \\ &\quad + \frac{(d-2)^2}{4} \int_{\mathbb{S}^{d-1}} \left(\left(-\Delta^\bullet + \tfrac{(d-2)^2}{4}\right)^{s} f(\boldsymbol{\xi})\right) f(\boldsymbol{\xi}) \,\mathrm{d}\mathbb{S}^{d-1}(\boldsymbol{\xi}).\end{aligned}$$

Since the gradient $\boldsymbol{\nabla}^\bullet$ and the Laplacian $\Delta^\bullet$ commute by Schwarz's theorem, we obtain (2.52). ∎

Interpolation with respect to the scale of Sobolev spaces

The norm of a bounded linear operator $\mathcal{A}\colon X \to Y$ between two Banach spaces X and Y with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively, is defined as

$$\|\mathcal{A}\|_{X\to Y} := \sup_{x \in X\setminus\{0\}} \frac{\|\mathcal{A}x\|_Y}{\|x\|_X}.$$

The following proposition shows that the boundedness of linear operators in Sobolev spaces $H^s(\mathbb{S}^{d-1})$ can be interpolated with respect to the smoothness parameter s. This result is derived from a more general interpolation theorem that can be found in [Tri95, Section 1.18].

Proposition 2.5. *Let $0 \le s_0 \le s_1$, and let*

$$\mathcal{A}\colon H^{s_0}(\mathbb{S}^{d-1}) \to H^{s_0}(\mathbb{S}^{d-1})$$

be a bounded linear operator such that its restriction

$$\mathcal{A}\big|_{H^{s_1}(\mathbb{S}^{d-1})} \colon H^{s_1}(\mathbb{S}^{d-1}) \to H^{s_1}(\mathbb{S}^{d-1})$$

is also bounded. For $\theta \in [0,1]$, we set $s_\theta := (1-\theta)s_0 + \theta s_1$. Then the restriction of $\mathcal{A}$ to $H^{s_\theta}(\mathbb{S}^{d-1}) \to H^{s_\theta}(\mathbb{S}^{d-1})$ is bounded with

$$\|\mathcal{A}\|_{H^{s_\theta}(\mathbb{S}^{d-1})\to H^{s_\theta}(\mathbb{S}^{d-1})} \le \|\mathcal{A}\|^{1-\theta}_{H^{s_0}(\mathbb{S}^{d-1})\to H^{s_0}(\mathbb{S}^{d-1})}\, \|\mathcal{A}\|^{\theta}_{H^{s_1}(\mathbb{S}^{d-1})\to H^{s_1}(\mathbb{S}^{d-1})}\,.$$

Proof. For $n \in \mathbb{N}_0$, let $\{Y_{n,d}^k \; ; \; k = 1, \dots, N_{n,d}\}$ be an orthonormal basis of the space $\mathcal{Y}_{n,d}(\mathbb{S}^{d-1})$, and let $f \in L^2(\mathbb{S}^{d-1})$. We write f as the Fourier series

$$f = \sum_{n=0}^{\infty} \sum_{k=1}^{N_{n,d}} \left\langle f, Y_{n,d}^k \right\rangle_{L^2(\mathbb{S}^{d-1})} Y_{n,d}^k.$$

On the index set

$$I := \{(n,k) \; ; \; n \in \mathbb{N}_0, \; k = 1, \dots, N_{n,d}\},$$

we define for $s \geq 0$ the weight function

$$w_s(n,k) := (n + \tfrac{d-2}{2})^{2s}.$$

Then the Sobolev space $H^s(\mathbb{S}^{d-1})$ is isometrically isomorphic to the weighted L^2-space

$$L^2(I; w_s) = \left\{ \hat{f} \colon I \to \mathbb{C} \; ; \; \|\hat{f}\|_{L^2(I;w_s)}^2 = \sum_{(n,k) \in I} \left| \hat{f}(n,k) \right|^2 w_s(n,k) < \infty \right\}$$

that consists of the Fourier coefficients

$$\hat{f}(n,k) = \left\langle f, Y_{n,d}^k \right\rangle_{L^2(\mathbb{S}^{d-1})}, \qquad (n,k) \in I,$$

on the set I with the counting measure. By [Tri95, Theorem 1.18.5], the complex interpolation space between $L^2(I; w_{s_0}) \cong H^{s_0}(\mathbb{S}^{d-1})$ and $L^2(I; w_{s_1}) \cong H^{s_1}(\mathbb{S}^{d-1})$ is

$$\left[L^2(I; w_{s_0}), L^2(I; w_{s_1}) \right]_\theta = L^2(I; w),$$

where

$$w(n,k) = \left(w_{s_0}(n,k) \right)^{1-\theta} \left(w_{s_1}(n,k) \right)^\theta = \left(n + \tfrac{d-2}{2} \right)^{2((1-\theta)s + \theta t)} = w_{s_\theta}(n,k).$$

Hence, $L^2(I; w) \cong H^{s_\theta}(\mathbb{S}^{d-1})$. The assertion is a property of the interpolation space. ■

Multiplication and composition operators

The following two theorems show that the multiplication and composition with a smooth function are continuous operators in spherical Sobolev spaces $H^s(\mathbb{S}^{d-1})$.

Theorem 2.6. *Let $s \in \mathbb{N}_0$. The multiplication operator*

$$H^s(\mathbb{S}^{d-1}) \times C^s(\mathbb{S}^{d-1}) \to H^s(\mathbb{S}^{d-1}), \quad (f, v) \mapsto fv$$

is continuous. In particular, for all $f \in H^s(\mathbb{S}^{d-1})$ and $v \in C^s(\mathbb{S}^{d-1})$, we have

$$\|fv\|_{H^s(\mathbb{S}^{d-1})} \leq c_d^s \, \|f\|_{H^s(\mathbb{S}^{d-1})} \, \|v\|_{C^s(\mathbb{S}^{d-1})}, \tag{2.54}$$

where

$$c_d = \sqrt{2d+2}.$$

Proof. We use induction over $s \in \mathbb{N}_0$. For $s = 0$, we have

$$\|fv\|^2_{L^2(\mathbb{S}^{d-1})} = \int_{\mathbb{S}^{d-1}} |f(\boldsymbol{\xi})\, v(\boldsymbol{\xi})|^2 \,\mathrm{d}\boldsymbol{\xi} \le \|f\|^2_{L^2(\mathbb{S}^{d-1})} \|v\|^2_{C(\mathbb{S}^{d-1})}.$$

Let the claimed equation (2.54) hold for $s \in \mathbb{N}_0$, and let $f \in H^{s+1}(\mathbb{S}^{d-1})$ and $v \in C^{s+1}(\mathbb{S}^{d-1})$. Then the decomposition (2.52) of the Sobolev norm yields

$$\begin{aligned}\|fv\|^2_{H^{s+1}(\mathbb{S}^{d-1})} &= \|\boldsymbol{\nabla}^\bullet(fv)\|^2_{H^s(\mathbb{S}^{d-1}\to\mathbb{R}^d)} + \tfrac{(d-2)^2}{4}\|fv\|^2_{H^s(\mathbb{S}^{d-1})}\\ &= \|f\boldsymbol{\nabla}^\bullet v + v\boldsymbol{\nabla}^\bullet f\|^2_{H^s(\mathbb{S}^{d-1}\to\mathbb{R}^d)} + \tfrac{(d-2)^2}{4}\|fv\|^2_{H^s(\mathbb{S}^{d-1})}.\end{aligned}$$

By the triangle inequality and since $(a+b)^2 \le 2(a^2+b^2)$ for all $a, b \in \mathbb{R}$, we obtain

$$\|fv\|^2_{H^{s+1}(\mathbb{S}^{d-1})} \le 2\|f\boldsymbol{\nabla}^\bullet v\|^2_{H^s(\mathbb{S}^{d-1}\to\mathbb{R}^d)} + 2\|v\boldsymbol{\nabla}^\bullet f\|^2_{H^s(\mathbb{S}^{d-1})\to\mathbb{R}^d} + \tfrac{(d-2)^2}{4}\|fv\|^2_{H^s(\mathbb{S}^{d-1})}.$$

By the induction hypothesis, we have

$$\begin{aligned}c_d^{-2s}\|fv\|^2_{H^{s+1}(\mathbb{S}^{d-1})} &\le 2\|f\|^2_{H^s(\mathbb{S}^{d-1})}\|\boldsymbol{\nabla}^\bullet v\|^2_{C^s(\mathbb{S}^{d-1}\to\mathbb{R}^d)} + 2\|\boldsymbol{\nabla}^\bullet f\|^2_{H^s(\mathbb{S}^{d-1}\to\mathbb{R}^d)}\|v\|^2_{C^s(\mathbb{S}^{d-1})}\\ &\quad + \tfrac{(d-2)^2}{4}\|f\|^2_{H^s(\mathbb{S}^{d-1})}\|v\|^2_{C^s(\mathbb{S}^{d-1})}.\\ &= 2\|f\|^2_{H^s(\mathbb{S}^{d-1})}\|\boldsymbol{\nabla}^\bullet v\|^2_{C^s(\mathbb{S}^{d-1}\to\mathbb{R}^d)} + \|\boldsymbol{\nabla}^\bullet f\|^2_{H^s(\mathbb{S}^{d-1}\to\mathbb{R}^d)}\|v\|^2_{C^s(\mathbb{S}^{d-1})}\\ &\quad + \|f\|^2_{H^{s+1}(\mathbb{S}^{d-1})}\|v\|^2_{C^s(\mathbb{S}^{d-1})},\end{aligned}$$

where we made use of the decomposition (2.52) of the Sobolev norm. Furthermore, we insert the bound (2.45) of the Sobolev norm of the gradient and obtain

$$\begin{aligned}c_d^{-2s}\|fv\|^2_{H^{s+1}(\mathbb{S}^{d-1})} &\le 2d\|f\|^2_{H^s(\mathbb{S}^{d-1})}\|v\|^2_{C^{s+1}(\mathbb{S}^{d-1})} + \|f\|^2_{H^{s+1}(\mathbb{S}^{d-1})}\|v\|^2_{C^s(\mathbb{S}^{d-1})}\\ &\quad + \|f\|^2_{H^{s+1}(\mathbb{S}^{d-1})}\|v\|^2_{C^s(\mathbb{S}^{d-1})}.\end{aligned}$$

Because the involved norms are non-decreasing with respect to s, we see that

$$\|fv\|_{H^{s+1}(\mathbb{S}^{d-1})} \le c_d^s\sqrt{2d+2}\,\|f\|_{H^{s+1}(\mathbb{S}^{d-1})}\|v\|_{C^{s+1}(\mathbb{S}^{d-1})},$$

which shows (2.54). ■

Theorem 2.7. *Let $s \in \mathbb{N}_0$, and let $\boldsymbol{v}\colon \mathbb{S}^{d-1} \to \mathbb{S}^{d-1}$ be bijective with $\boldsymbol{v} \in C^s(\mathbb{S}^{d-1} \to \mathbb{S}^{d-1})$ and $\boldsymbol{v}^{-1} \in C^1(\mathbb{S}^{d-1} \to \mathbb{S}^{d-1})$. Then there exists a constant $b_{d,s}(\boldsymbol{v})$ such that for all $f \in H^s(\mathbb{S}^{d-1})$, we have*

$$\|f \circ \boldsymbol{v}\|_{H^s(\mathbb{S}^{d-1})} \le b_{d,s}(\boldsymbol{v})\,\|f\|_{H^s(\mathbb{S}^{d-1})}.$$

Proof. We have for $s = 0$

$$\|f \circ \boldsymbol{v}\|^2_{L^2(\mathbb{S}^{d-1})} = \int_{\mathbb{S}^{d-1}} |f(\boldsymbol{v}(\boldsymbol{\xi}))|^2 \,\mathrm{d}\mathbb{S}^{d-1}(\boldsymbol{\xi}).$$

The substitution $\boldsymbol{\eta} = \boldsymbol{v}(\boldsymbol{\xi})$ yields with the substitution rule (2.2)

$$\|f \circ \boldsymbol{v}\|_{L^2(\mathbb{S}^{d-1})}^2 = \int_{\mathbb{S}^{d-1}} |f(\boldsymbol{\eta})|^2 \left[(\boldsymbol{v}^{-1})^*(\mathrm{d}\mathbb{S}^{d-1})\right](\boldsymbol{\eta}).$$

Since $\boldsymbol{v}^{-1} \in C^1(\mathbb{S}^{d-1} \to \mathbb{S}^{d-1})$ and thus the Jacobian $J_{\boldsymbol{v}^{-1}}(\boldsymbol{\xi})$ depends continuously on $\boldsymbol{\xi}$, there exists a continuous function $\nu\colon \mathbb{S}^{d-1} \to \mathbb{R}$ such that the pullback (2.1) satisfies $(\boldsymbol{v}^{-1})^*(\mathrm{d}\mathbb{S}^{d-1}) = \nu\, \mathrm{d}\mathbb{S}^{d-1}$. Hence, we have

$$\|f \circ \boldsymbol{v}\|_{L^2(\mathbb{S}^{d-1})}^2 \le \|f\|_{L^2(\mathbb{S}^{d-1})}^2 \, \|\nu\|_{C(\mathbb{S}^{d-1})},$$

which shows the claim for $s = 0$.

We use induction on $s \in \mathbb{N}_0$. By the decomposition (2.52) of the Sobolev norm, we have

$$\|f \circ \boldsymbol{v}\|_{H^{s+1}(\mathbb{S}^{d-1})}^2 = \|\boldsymbol{\nabla}^\bullet(f \circ \boldsymbol{v})\|_{H^s(\mathbb{S}^{d-1}\to\mathbb{R}^d)}^2 + \tfrac{(d-2)^2}{4} \|f \circ \boldsymbol{v}\|_{H^s(\mathbb{S}^{d-1})}^2 . \tag{2.55}$$

By the induction hypothesis, the second summand of (2.55) is bounded by

$$\|f \circ \boldsymbol{v}\|_{H^s(\mathbb{S}^d)} \le b_{d,s}(\boldsymbol{v}) \, \|f\|_{H^s(\mathbb{S}^d)} . \tag{2.56}$$

Furthermore, by (2.51) and the chain rule, we have for the first summand of (2.55)

$$\begin{aligned}
\|\boldsymbol{\nabla}^\bullet(f \circ \boldsymbol{v})\|_{H^s(\mathbb{S}^{d-1}\to\mathbb{R}^d)}^2 &= \|\boldsymbol{\nabla}(f \circ v)^\bullet\|_{H^s(\mathbb{S}^{d-1}\to\mathbb{R}^d)}^2 \\
&= \sum_{i=1}^d \|\partial_i (f \circ \boldsymbol{v})^\bullet\|_{H^s(\mathbb{S}^{d-1})}^2 \\
&= \sum_{i=1}^d \left\| \sum_{j=1}^d ((\partial_j f^\bullet) \circ \boldsymbol{v}^\bullet)\, \partial_i v_j^\bullet \right\|_{H^s(\mathbb{S}^{d-1})}^2 .
\end{aligned}$$

Applying the triangle inequality for the sum over j and Jensen's inequality $(\sum_{j=1}^d x_j)^2 \le d \sum_{j=1}^d x_j^2$, we obtain

$$\|\boldsymbol{\nabla}^\bullet(f \circ \boldsymbol{v})\|_{H^s(\mathbb{S}^{d-1}\to\mathbb{R}^d)}^2 \le \sum_{i=1}^d d \sum_{j=1}^d \left\|((\partial_j f^\bullet) \circ \boldsymbol{v}^\bullet)\, \partial_i v_j^\bullet\right\|_{H^s(\mathbb{S}^{d-1})}^2 .$$

By Theorem 2.6, we have

$$\begin{aligned}
\|\boldsymbol{\nabla}^\bullet(f \circ \boldsymbol{v})\|_{H^s(\mathbb{S}^{d-1}\to\mathbb{R}^d)}^2 &\le d\, c_d^{2s} \sum_{j=1}^d \|(\partial_j f^\bullet) \circ \boldsymbol{v}^\bullet\|_{H^s(\mathbb{S}^{d-1})}^2 \sum_{i=1}^d \left\|\partial_i v_j^\bullet\right\|_{C^s(\mathbb{S}^{d-1})}^2 \\
&\le d^2\, c_d^{2s} \sum_{j=1}^d \|(\partial_j f^\bullet) \circ \boldsymbol{v}^\bullet\|_{H^s(\mathbb{S}^{d-1})}^2 \, \|v_j\|_{C^{s+1}(\mathbb{S}^{d-1})}^2 ,
\end{aligned}$$

where the last line follows from (2.45). By the induction hypothesis, we see that

$$\|\boldsymbol{\nabla}^\bullet(f\circ\boldsymbol{v})\|^2_{H^s(\mathbb{S}^{d-1}\to\mathbb{R}^d)} \le d^2\, c_d^{2s}\, b_{d,s}(\boldsymbol{v})^2 \sum_{j=1}^d \|\partial_j f^\bullet\|^2_{H^s(\mathbb{S}^{d-1})}\, \|v_j\|^2_{C^{s+1}(\mathbb{S}^{d-1})}\,.$$

By (2.51) and the fact that $\|v_j\|^2_{C^{s+1}(\mathbb{S}^{d-1})} \le \|\boldsymbol{v}\|^2_{C^{s+1}(\mathbb{S}^{d-1}\to\mathbb{R}^d)}$ for all $j=1,\ldots,d$, we obtain

$$\|\boldsymbol{\nabla}^\bullet(f\circ\boldsymbol{v})\|^2_{H^s(\mathbb{S}^{d-1}\to\mathbb{R}^d)} \le d^2\, c_d^{2s}\, b_{d,s}(\boldsymbol{v})^2\, \|\boldsymbol{\nabla}^\bullet f\|^2_{H^s(\mathbb{S}^{d-1}\to\mathbb{R}^d)}\, \|\boldsymbol{v}\|^2_{C^{s+1}(\mathbb{S}^{d-1}\to\mathbb{R}^d)}\,.$$

Inserting the last equation and (2.56) into (2.55), we obtain

$$\begin{aligned}
&\|f\circ\boldsymbol{v}\|^2_{H^{s+1}(\mathbb{S}^{d-1})}\\
&= \|\boldsymbol{\nabla}^\bullet(f\circ\boldsymbol{v})\|^2_{H^s(\mathbb{S}^{d-1}\to\mathbb{R}^d)} + \tfrac{(d-2)^2}{4}\,\|f\circ\boldsymbol{v}\|^2_{H^s(\mathbb{S}^{d-1})}\\
&\le b_{d,s}(\boldsymbol{v})^2\left(d^2\, c_d^{2s}\, \|\boldsymbol{\nabla}^\bullet f\|^2_{H^s(\mathbb{S}^{d-1}\to\mathbb{R}^d)}\, \|\boldsymbol{v}\|^2_{C^{s+1}(\mathbb{S}^{d-1}\to\mathbb{R}^d)} + \tfrac{(d-2)^2}{4}\,\|f\|^2_{H^s(\mathbb{S}^d)}\right)\\
&= b_{d,s}(\boldsymbol{v})^2\left(\left(d^2\, c_d^{2s}\, \|\boldsymbol{v}\|^2_{C^{s+1}(\mathbb{S}^{d-1}\to\mathbb{R}^d)} - 1\right)\|\boldsymbol{\nabla}^\bullet f\|^2_{H^s(\mathbb{S}^{d-1}\to\mathbb{R}^d)} + \|f\|^2_{H^{s+1}(\mathbb{S}^{d-1})}\right)\\
&\le b_{d,s}(\boldsymbol{v})^2 d^2\, c_d^{2s}\, \|\boldsymbol{v}\|^2_{C^{s+1}(\mathbb{S}^{d-1}\to\mathbb{R}^d)}\, \|f\|^2_{H^{s+1}(\mathbb{S}^{d-1})}\,,
\end{aligned}$$

where we have made use of (2.52). ■

Remark 2.8. The last theorem resembles a similar result found in [IKT13, Theorem 1.2]: Let M be a smooth, closed and oriented d-dimensional manifold and, for $s > \frac{d}{2}+1$, let $\varphi \in H^s(M\to M)$ be an orientation-preserving C^1-diffeomorphism. Then the composition map

$$H^s(M)\to H^s(M),\ f\mapsto f\circ\varphi$$

is continuous. However, Theorem 2.7 is not a special case of this result because Theorem 2.7 requires only $s\ge 0$ but imposes stronger assumptions on φ. □

2.1.6 Linearization of the product of spherical harmonics

Any spherical harmonic $Y_{n,d}^k$ is a polynomial of degree n. The product of two spherical harmonics, $Y_{n_1,d}^{k_1} Y_{n_2,d}^{k_2}$, is a polynomial of degree n_1+n_2, which can thus be written as the sum of spherical harmonics of degree up to n_1+n_2 as follows. Let us fix a certain basis of spherical harmonics $Y_{n,d}^k \in \mathscr{Y}_{n,d}(\mathbb{S}^{d-1})$ for $n\in\mathbb{N}_0$ and $k=1,\ldots,N_{n,d}$. We define the Gaunt coefficients

$$G^{n,k,d}_{n_1,k_1,n_2,k_2} := \int_{\mathbb{S}^{d-1}} Y^{k_1}_{n_1,d}(\boldsymbol{\xi})\, Y^{k_2}_{n_2,d}(\boldsymbol{\xi})\, \overline{Y^k_{n,d}(\boldsymbol{\xi})}\, \mathrm{d}\boldsymbol{\xi}. \tag{2.57}$$

Then the product of two spherical harmonics can be written as the sum

$$Y^{k_1}_{n_1,d}(\boldsymbol{\xi})\, Y^{k_2}_{n_2,d}(\boldsymbol{\xi}) = \sum_{\substack{n=|n_1-n_2|\\ n-n_1-n_2\text{ even}}}^{n_1+n_2} \sum_{k=1}^{N_{n,d}} G^{n,k,d}_{n_1,k_1,n_2,k_2}\, Y^k_{n,d}(\boldsymbol{\xi}), \qquad \boldsymbol{\xi}\in\mathbb{S}^{d-1}. \tag{2.58}$$

In dimension $d = 3$, we write the Gaunt coefficients as

$$G_{n_1,k_1,n_2,k_2}^{n,k} := \int_{\mathbb{S}^2} Y_{n_1}^{k_1}(\boldsymbol{\xi})\, Y_{n_2}^{k_2}(\boldsymbol{\xi})\, \overline{Y_n^k(\boldsymbol{\xi})}\, \mathrm{d}\boldsymbol{\xi}, \tag{2.59}$$

which are zero unless all the conditions

$$|k_1| \le n_1,\ |k_2| \le n_2,\ |k| \le n$$

and

$$k = k_1 + k_2,\ n = |n_1 - n_2|, |n_1 - n_2| + 2, \ldots, n_1 + n_2$$

are satisfied. We have by the definition of the spherical harmonics in (2.39)

$$\begin{aligned} G_{n_1,k_1,n_2,k_2}^{n,k} &= \int_{-1}^{1} \widetilde{P}_{n_1}^{k_1}(t)\, \widetilde{P}_{n_2}^{k_2}(t)\, \widetilde{P}_n^{k}(t)\, \mathrm{d}t \int_0^{2\pi} \mathrm{e}^{\mathrm{i}k_1\varphi + k_2\varphi - k\varphi}\, \mathrm{d}\varphi \\ &= 2\pi\delta_{k,k_1-k_2} \int_{-1}^{1} \widetilde{P}_{n_1}^{k_1}(t)\, \widetilde{P}_{n_2}^{k_2}(t)\, \widetilde{P}_n^{k}(t)\, \mathrm{d}t \end{aligned}$$

As in the general case (2.58), the product of two spherical harmonics may be written as the spherical Fourier series

$$Y_{n_1}^{k_1}(\boldsymbol{\xi}) Y_{n_2}^{k_2}(\boldsymbol{\xi}) = \sum_{\substack{n=\max(|n_1-n_2|,|k_1+k_2|)\\ n-n_1-n_2 \text{ even}}}^{n_1+n_2} G_{n_1,k_1,n_2,k_2}^{n,k_1+k_2} Y_n^{k_1+k_2}(\boldsymbol{\xi}), \qquad \boldsymbol{\xi} \in \mathbb{S}^2. \tag{2.60}$$

The first explicit description of the coefficients $G_{n_1,k_1,n_2,k_2}^{n,k}$ was obtained by Gaunt [Gau29] in 1929. The Gaunt coefficients are closely related to the Clebsch–Gordan coefficients $C_{n_1,k_1,n_2,k_2}^{n,k}$ via the equation [VMK88, Section 5.6.2]

$$G_{n_1,k_1,n_2,k_2}^{n,k} = \sqrt{\frac{(2n_1+1)(2n_2+1)}{4\pi(2n+1)}} C_{n_1,0,n_2,0}^{n,0} C_{n_1,k_1,n_2,k_2}^{n,k}.$$

An explicit representation of the Clebsch–Gordan coefficients is given in [VMK88, Section 8.2.2] and reads

$$\begin{aligned} C_{n_1,k_1,n_2,k_2}^{n,k} = \delta_{k,k_1+k_2} &\sqrt{\frac{(n_1+n_2-n)!\,(n_1-n_2+n)!\,(-n_1+n_2+n)!}{(n_1+n_2+n+1)!}} \\ &\cdot \sqrt{\frac{(n+k)!\,(n-k)!\,(2n+1)}{(n_1+k_1)!\,(n_1-k_1)!\,(n_2+k_2)!\,(n_2-k_2)!}} \\ &\cdot \sum_{\ell} \frac{(-1)^{n_2+k_2+\ell}\,(n+n_2+k_1-\ell)!\,(n_1-k_1+\ell)!}{k!\,(n-n_1+n_2-\ell)!\,(n+k-\ell)!\,(n_1-n_2-k+\ell)!}, \end{aligned}$$

where the sum is over all integers ℓ for which the argument of every factorial is nonnegative. In the case $k_1 = k_2 = k = 0$, we have [VMK88, (8.32)]

$$C^{n,0}_{n_1,0,n_2,0} = \frac{(-1)^{\frac{n_1+n_2-n}{2}}\sqrt{2n+1}\,\frac{n_1+n_2+n}{2}!}{\frac{n_1+n_2-n}{2}!\,\frac{n+n_1-n_2}{2}!\,\frac{n-n_1+n_2}{2}!}\sqrt{\frac{(n_1+n_2-n)!\,(n+n_1-n_2)!\,(n-n_1+n_2)!}{(n+n_1+n_2+1)!}}$$

if $n_1 + n_2 - n$ is even and zero otherwise. In the case $n_1 = n_n$ and $k_1 = k_2$, we have [VMK88, (8.23)]

$$C^{l,2k}_{n,k,n,k} = \frac{(-1)^{n-\frac{l}{2}}\sqrt{2l+1}(n+\frac{l}{2})!}{\frac{l+2k}{2}!\frac{l-2k}{2}!(n-\frac{l}{2})}\sqrt{\frac{(l+2k)!(l-2k)!(2n-l)!}{(2n+l+1)!}}$$

if $2n + l$ is even and otherwise 0. Another closely related notation is the Wigner $3j$ symbol

$$\begin{pmatrix} n_1 & n_2 & n \\ k_1 & k_2 & k \end{pmatrix} = (-1)^{n_1-j_2-k_3}\sqrt{2n+1}\,C^{n,-k}_{n_1,k_1,n_2,m_2}.$$

2.2 Harmonic analysis on the rotation group

We give a brief overview about the harmonic analysis on the rotation group SO(3). This introduction uses material from [HPQ18] and is based on [HHK98]. Rotational Fourier transforms date back to Wigner in 1931, see [Wig31]. The rotation group SO(3) consists of all orthogonal (3×3)–matrices with determinant one equipped with the matrix multiplication as group operation, i. e.,

$$\mathrm{SO}(3) := \{Q \in \mathbb{R}^{3\times 3}\,;\, Q^\top Q = I,\, \det(Q) = 1\}.$$

Every rotation $Q \in \mathrm{SO}(3)$ can be expressed in terms of its Euler angles α, β, γ by

$$Q(\alpha,\beta,\gamma) := R_3(\alpha)\,R_2(\beta)\,R_3(\gamma), \qquad \alpha,\gamma \in [0,2\pi),\ \beta \in [0,\pi], \tag{2.61}$$

where $R_i(\alpha)$ denotes the rotation of the angle α about the $\boldsymbol{\xi}_i$–axis, i. e.,

$$R_3(\alpha) := \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad R_2(\beta) := \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}. \tag{2.62}$$

Note that we use this zyz-convention of the Euler angles throughout this thesis. The integral of a function $g\colon \mathrm{SO}(3) \to \mathbb{C}$ on the rotation group is given by

$$\int_{\mathrm{SO}(3)} g(Q)\,\mathrm{d}Q := \int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} g(Q(\alpha,\beta,\gamma))\sin(\beta)\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma.$$

We define the rotational harmonics or Wigner D-functions $D_n^{k,j}$ of degree $n \in \mathbb{N}_0$ and orders $k, j \in \{-n,\dots,n\}$ by

$$D_n^{k,j}(Q(\alpha,\beta,\gamma)) := \mathrm{e}^{-\mathrm{i}k\alpha} d_n^{k,j}(\cos\beta)\mathrm{e}^{-\mathrm{i}j\gamma},$$

where the Wigner d-functions are given by [VMK88, page 77]

$$d_n^{k,j}(t) := \frac{(-1)^{n-j}}{2^n} \sqrt{\frac{(n+k)!(1-t)^{j-k}}{(n-j)!(n+j)!(n-k)!(1+t)^{j+k}}} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n-k} \frac{(1+t)^{n+j}}{(1-t)^{-n+j}}$$

for $t \in [-1,1]$. The Wigner d-functions satisfy the orthogonality relation

$$\int_{-1}^{1} d_n^{k,j}(t)\, d_{n'}^{k,j}(t)\, \mathrm{d}t = \frac{2\delta_{n,n'}}{2n+1}.$$

As we did on the sphere, we define the space of square-integrable functions $L^2(\mathrm{SO}(3))$ with the inner product

$$\langle f, g\rangle_{L^2(\mathrm{SO}(3))} := \int_{\mathrm{SO}(3)} f(Q)\, \overline{g(Q)}\, \mathrm{d}Q.$$

By the Peter–Weyl theorem, the rotational harmonics $D_n^{k,j}$ are complete in $L^2(\mathrm{SO}(3))$ and satisfy the orthogonality relation

$$\left\langle D_n^{k,j}, D_{n'}^{k',j'}\right\rangle_{L^2(\mathrm{SO}(3))} = \int_{\mathrm{SO}(3)} D_n^{k,j}(Q)\, \overline{D_{n'}^{k',j'}(Q)}\, \mathrm{d}Q = \frac{8\pi^2}{2n+1}\, \delta_{n,n'}\delta_{k,k'}\delta_{j,j'}. \tag{2.63}$$

We define the rotational Fourier coefficients of a function $g \in L^2(\mathrm{SO}(3))$ by

$$\hat{g}_n^{k,j} := \frac{2n+1}{8\pi^2} \left\langle g, D_n^{k,j}\right\rangle_{L^2(\mathrm{SO}(3))}, \qquad n \in \mathbb{N}_0,\ k,j = -n,\dots,n. \tag{2.64}$$

Then the rotational Fourier expansion of g holds,

$$g = \sum_{n=0}^{\infty} \sum_{k,j=-n}^{n} \hat{g}_n^{k,j}\, D_n^{k,j}.$$

The rotational Fourier transform is also known as the SO(3) Fourier transform (SOFT) or the Wigner D-transform.

The rotational harmonics $D_n^{k,j}$ are eigenfunctions of the Laplace–Beltrami operator on SO(3) with the corresponding eigenvalues $-n(n+1)$. The rotational harmonics $D_n^{j,k}$ are the matrix entries of the left regular representations of SO(3), see [Hie07, VMK88]. In particular, the rotation of a spherical harmonic Y_n^k, $n \in \mathbb{N}_0$, $k = -n,\dots,n$, which was introduced in (2.39), satisfies

$$Y_n^k(Q^{-1}\boldsymbol{\xi}) = \sum_{j=-n}^{n} D_n^{j,k}(Q)\, Y_n^j(\boldsymbol{\xi}), \qquad Q \in \mathrm{SO}(3),\ \boldsymbol{\xi} \in \mathbb{S}^2. \tag{2.65}$$

Sobolev spaces on SO(3) are defined in a similar manner as we did on the sphere $\mathbb{S}^2$ in Section 2.1.5, see [Hie07, Section 2.6]. Let $s \in \mathbb{R}$. We define the Sobolev space $H^s(\mathrm{SO}(3))$ as the completion of the space of smooth functions $g\colon \mathrm{SO}(3) \to \mathbb{C}$ with respect to the Sobolev norm

$$\|g\|^2_{H^s(\mathrm{SO}(3))} = \sum_{n=0}^{\infty} \sum_{k,j=-n}^{n} \frac{8\pi^2}{2n+1} \left(n+\tfrac{1}{2}\right)^{2s} \left|\hat{g}_n^{k,j}\right|^2. \tag{2.66}$$

For $s = 0$, we have $H^0(\mathrm{SO}(3)) = L^2(\mathrm{SO}(3))$.

2.3 Factorials, double factorials and the Gamma function

The factorial of a positive integer n is defined by $n! := n\,(n-1)\cdots 1$. Similarly, double factorials are defined by

$$n!! := \begin{cases} n(n-2)\cdots 2, & n \text{ even} \\ n(n-2)\cdots 1, & n \text{ odd} \end{cases}$$

and $0!! = 1$. For negative integers, the recursion $n!! = n\,(n-2)!!$ yields the definition $(-2n)!! = 0$ and $(-2n-1)!! = \frac{(-1)^n}{(2n-1)!!}$ for $n \in \mathbb{N}_0$. In particular, we have $(-1)!! = 1$. Double factorials are related to the factorial via

$$(2n)!! = (2n)\,(2n-2)\cdots 2 = 2^n\,n!$$

and

$$(2n-1)!! = \frac{(2n)!}{(2n)!!} = \frac{(2n)!}{2^n\,n!}.$$

The Gamma function serves as generalization of the factorial to non-integers and is defined for $x > 0$ via the integral

$$\Gamma(x) := \int_0^\infty y^{x-1}\,\mathrm{e}^{-y}\,\mathrm{d}y.$$

It satisfies the relation

$$\Gamma(x+1) = x\,\Gamma(x) \tag{2.67}$$

as well as $\Gamma(n) = (n-1)!$ if n is a positive integer. Special values include $\Gamma(1) = 1$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Double factorials can also be expressed with the Gamma function. We have

$$(2n)!! = 2^n\,n! = 2^n\,\Gamma(n+1)$$

and, by (2.67),

$$\begin{aligned} (2n-1)!! &= (2n-1)\,(2n-3)\cdots 1 \\ &= 2^n \left(n - \frac{1}{2}\right)\left(n - \frac{3}{2}\right)\cdots\left(\frac{1}{2}\right) \\ &= 2^n\,\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \\ &= \frac{2^n}{\sqrt{\pi}}\,\Gamma\left(n + \frac{1}{2}\right). \end{aligned}$$

The following simple relation exists for the quotient of double factorials.

Proposition 2.9. *Let $n, k \in \mathbb{N}_0$ with $k \le 2n$. Then*

$$\frac{\Gamma(\frac{k}{2})}{2^m\,\Gamma(n+\frac{k}{2})} = \frac{(k-2)!!}{(2n+k-2)!!}. \tag{2.68}$$

Proof. By the functional equation (2.67) of the Gamma function, we have

$$\begin{aligned}
\Gamma\left(n+\frac{k}{2}\right) &= \left(n+\frac{k}{2}-1\right)\cdot\Gamma\left(n+\frac{k}{2}-1\right)\\
&= \left(n+\frac{k}{2}-1\right)\left(n+\frac{k}{2}-2\right)\cdots\left(\frac{k}{2}\right)\cdot\Gamma\left(\frac{k}{2}\right)\\
&= \frac{(2n+k-2)\,(2n+k-4)\cdots k}{2^n}\cdot\Gamma\left(\frac{k}{2}\right)\\
&= \frac{(2n+k-2)!!}{2^n\,(k-2)!!}\cdot\Gamma\left(\frac{k}{2}\right).
\end{aligned}$$

∎

The binomial coefficient of $z \in \mathbb{C}$ and $k \in \mathbb{N}_0$ is defined by

$$\binom{z}{k} := \frac{z\,(z-1)\cdots(z-k+1)}{k!}. \tag{2.69}$$

The relation with the Gamma function is

$$\binom{z}{k} = \frac{\Gamma(z+1)}{\Gamma(z-k+1)\,\Gamma(k+1)}. \tag{2.70}$$

Asymptotic approximation

We are going to use the notation of asymptotic equivalence. For two sequences $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$, we write

$$a_n \simeq b_n \qquad \text{for } n \to \infty$$

if there exists a sequence $(c_n)_{n\in\mathbb{N}_0}$ such that $a_n = c_n b_n$ and $\lim_{n\to\infty} c_n = 1$. Stirling's approximation says that

$$n! \simeq \sqrt{2\pi}\, n^{n+1/2}\,\mathrm{e}^{-n}. \tag{2.71}$$

The following error bound was proven in [Rob55]. For $n = 1, 2, \ldots$

$$n! = \sqrt{2\pi} n^{n+1/2} \mathrm{e}^{-n} \mathrm{e}^{r(n)},$$

where $1/(12n+1) < r(n) < 1/(12n)$. The statement for the Gamma function is

$$\Gamma(x) \simeq \sqrt{2\pi}\, x^{x-\frac{1}{2}}\,\mathrm{e}^{-x}, \qquad x \to \infty. \tag{2.72}$$

There are also asymptotic approximations of double factorials, cf. [Bau07]. Based on the Wallis product, we have the formula

$$(2n-1)!! \simeq \frac{(2n)!!}{\sqrt{\pi\,(n+\frac{1}{4})}}. \tag{2.73}$$

Furthermore, we have for $n \to \infty$

$$(2n)!! \simeq \left(\frac{2n}{\mathrm{e}}\right)^n \sqrt{2\pi n} \tag{2.74}$$

and

$$(2n-1)!! \simeq \sqrt{2}\left(\frac{2n}{\mathrm{e}}\right)^n . \tag{2.75}$$

Moreover, we derive the following asymptotic approximation for the binomial coefficient $\binom{n+\alpha}{n}$.

Proposition 2.10. *Let $\alpha \in \mathbb{C}$ and $n \in \mathbb{N}_0$. Then we have*

$$\binom{n+\alpha}{n} \simeq \frac{1}{\Gamma(\alpha+1)} n^\alpha, \qquad n \to \infty. \tag{2.76}$$

Proof. We have by (2.70)

$$\binom{n+\alpha}{n} = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\,\Gamma(n+1)}.$$

Inserting the asymptotic expansion (2.72) of the Gamma function, we obtain

$$\begin{aligned}
\binom{n+\alpha}{n} &\simeq \frac{1}{\Gamma(\alpha+1)} \frac{(n+\alpha+1)^{n+\alpha+\frac{1}{2}}\,\mathrm{e}^{-n-\alpha-1}}{(n+1)^{n+\frac{1}{2}}\,\mathrm{e}^{-n-1}} \\
&= \frac{1}{\Gamma(\alpha+1)} \left(\frac{n+\alpha+1}{n+1}\right)^{n+\frac{1}{2}} (n+\alpha+1)^\alpha\,\mathrm{e}^{-\alpha} \\
&= \frac{1}{\Gamma(\alpha+1)} \left(1+\frac{\alpha}{n+1}\right)^{n+\frac{1}{2}} (n+\alpha+1)^\alpha\,\mathrm{e}^{-\alpha}.
\end{aligned}$$

Inserting the definition of the exponential $\mathrm{e}^x = \lim_{n\to\infty}(1+\frac{x}{n})^n$, we obtain

$$\binom{n+\alpha}{n} \simeq \frac{1}{\Gamma(\alpha+1)} n^\alpha, \qquad n \to \infty.$$

■

3 Circular means on the sphere

This chapter is devoted to the mean operator $\mathcal{M}$ that integrates a function f defined on the sphere $\mathbb{S}^{d-1}$ along all $(d-2)$-dimensional subspheres of $\mathbb{S}^{d-1}$. The reconstruction of a function f from its mean values $\mathcal{M}f$ is an overdetermined problem, which says that it suffices to have the mean values $\mathcal{M}f$ only for a certain family of subspheres of $\mathbb{S}^{d-1}$ in order to uniquely determine the function f. This raises the question, which families of subspheres are sufficient. We provide a characterization of such families of subspheres with the help of a partial differential equation in the first subsection.

Afterwards, we analyze restrictions of the mean operator $\mathcal{M}$ to special families of subspheres of $\mathbb{S}^{d-1}$. We start with the well-known cases of the Funk–Radon transform, which takes the family of great circles of $\mathbb{S}^2$, in Section 3.2 and the spherical section transform, which corresponds to the subspheres of fixed radius, in Section 3.3. Later on, we investigate the generalized Funk–Radon transform in Section 3.4, where we take the j-th order directional derivative of the function f perpendicular to the great circle along which we integrate f. The subspheres with a fixed set of centers are subject to Section 3.5, which includes the vertical slice transform if the centers are on the equator. The sections of the sphere with hyperplanes through a fixed point are covered in Section 3.6. For these different restrictions of the mean operator $\mathcal{M}$, we show singular value decompositions and we investigate their nullspace, which tells us whether the respective operator is injective, and also their range.

3.1 The mean operator on the sphere

In this section, we start with the definition the mean operator $\mathcal{M}$. We show some basic properties and its singular value decomposition. We investigate the continuity of the mean operator in spherical Sobolev spaces in Section 3.1.2. Furthermore, we characterize the range of the mean operator as the set of solutions of a partial differential equation, which resembles John's equation, in Section 3.1.3.

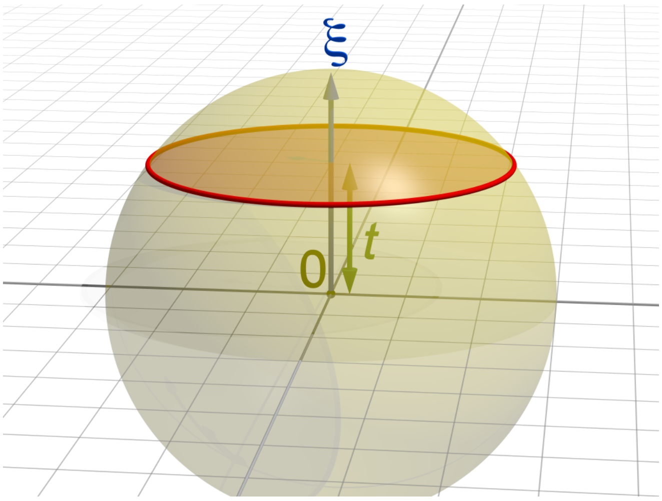

Figure 3.1: The red circle $C(\boldsymbol{\xi}, t)$ is obtained as the intersection of the two-sphere $\mathbb{S}^2$ with a plane. Here, $\boldsymbol{\xi} = \boldsymbol{\epsilon}^3$ is the north pole and $t = 0.6$.

3.1.1 Definition and basic properties

Any $(d-2)$-dimensional subsphere of $\mathbb{S}^{d-1}$ can be described as the intersection of the sphere $\mathbb{S}^{d-1}$ with the hyperplane

$$\left\{\boldsymbol{x} \in \mathbb{R}^d \,;\, \boldsymbol{\xi}^\top \boldsymbol{x} = t\right\},$$

which has the normal vector $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$ and the distance $t \in [-1, 1]$ to the origin. We define the $(d-2)$-dimensional subsphere

$$C(\boldsymbol{\xi}, t) := \left\{\boldsymbol{\eta} \in \mathbb{S}^{d-1} \,;\, \boldsymbol{\xi}^\top \boldsymbol{\eta} = t\right\}, \qquad (\boldsymbol{\xi}, t) \in \mathbb{S}^{d-1} \times [-1, 1], \tag{3.1}$$

which is illustrated in Figure 3.1. We call the normal vectors $\boldsymbol{\xi}$ and $-\boldsymbol{\xi}$ the poles of the subsphere $C(\boldsymbol{\xi}, t)$.

Definition 3.1. The mean operator

$$\mathcal{M} \colon C(\mathbb{S}^{d-1}) \to C(\mathbb{S}^{d-1} \times [-1, 1])$$

applied to a function $f \in C(\mathbb{S}^{d-1})$ is defined as

$$\mathcal{M}f(\boldsymbol{\xi}, t) := \begin{cases} \dfrac{1}{|\mathbb{S}^{d-2}|\,(1-t^2)^{\frac{d-2}{2}}} \displaystyle\int_{C(\boldsymbol{\xi},t)} f(\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta}, & \boldsymbol{\xi} \in \mathbb{S}^{d-1},\ t \in (-1, 1) \\ f(\pm\boldsymbol{\xi}), & \boldsymbol{\xi} \in \mathbb{S}^{d-1},\ t = \pm 1, \end{cases} \tag{3.2}$$

where $\mathrm{d}\boldsymbol{\eta}$ denotes the standard volume measure on the $(d-2)$-dimensional subsphere $C(\boldsymbol{\xi}, t)$ of $\mathbb{S}^{d-1}$.

The mean operator $\mathcal{M}f(\boldsymbol{\xi}, t)$ computes the mean value of the function f along the subsphere $C(\boldsymbol{\xi}, t)$. Spherical means have been studied since the second half of the twentieth century in approximation theory, see Rudin [Rud50]. To some authors, the operator $\mathcal{M}$ is also known as the spherical transform, see [Sal16].

Since the subsphere $C(\boldsymbol{\xi}, t)$ is a $(d-2)$-dimensional sphere of radius $\sqrt{1-t^2}$, its volume is given by

$$\int_{C(\boldsymbol{\xi},t)} \mathrm{d}\boldsymbol{\eta} = \left|\mathbb{S}^{d-2}\right| (1-t^2)^{\frac{d-2}{2}}.$$

Hence, the mean operator $\mathcal{M}$ maps the constant function $f \equiv 1$ to the constant function $\mathcal{M}f \equiv 1$.

Because $C(\boldsymbol{\xi}, t) = C(-\boldsymbol{\xi}, -t)$, we obtain the symmetry relation

$$\mathcal{M}f(\boldsymbol{\xi}, t) = \mathcal{M}f(-\boldsymbol{\xi}, -t), \qquad (\boldsymbol{\xi}, t) \in \mathbb{S}^{d-1} \times [-1, 1]. \tag{3.3}$$

Remark 3.2. For any $\boldsymbol{\eta} \in C(\boldsymbol{\xi}, t)$, we have

$$\|\boldsymbol{\xi} - \boldsymbol{\eta}\|^2 = \|\boldsymbol{\xi}\|^2 + \|\boldsymbol{\eta}\|^2 - 2\boldsymbol{\xi}^\top \boldsymbol{\eta} = 2 - 2t,$$

since $\|\boldsymbol{\xi}\| = \|\boldsymbol{\eta}\| = 1$. Hence, the subsphere $C(\boldsymbol{\xi}, t)$ can be obtained as the intersection of the unit sphere $\mathbb{S}^{d-1}$ with the sphere of radius $\sqrt{2-2t}$ around the pole $\boldsymbol{\xi}$, i. e.,

$$C(\boldsymbol{\xi}, t) = \{\boldsymbol{\eta} \in \mathbb{S}^{d-1}\, ;\, \|\boldsymbol{\xi} - \boldsymbol{\eta}\|^2 = 2 - 2t\}. \qquad \square$$

The integral of a spherical function $f\colon \mathbb{S}^{d-1} \to \mathbb{C}$ can be rewritten with the mean operator $\mathcal{M}$ as follows.

Corollary 3.3. Let $f\colon \mathbb{S}^{d-1} \to \mathbb{C}$ and $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$. We have

$$\int_{\mathbb{S}^{d-1}} f(\boldsymbol{\eta}) \,\mathrm{d}\mathbb{S}^{d-1}(\boldsymbol{\eta}) = \left|\mathbb{S}^{d-2}\right| \int_{-1}^{1} \mathcal{M}f(\boldsymbol{\xi}, t)\, (1-t^2)^{\frac{d-3}{2}} \,\mathrm{d}t. \tag{3.4}$$

Proof. We perform the proof for the case that $\boldsymbol{\xi} = \boldsymbol{\epsilon}^d$ is the north pole, otherwise we could choose a coordinate system where $\boldsymbol{\xi}$ is the north pole. We have by (2.8)

$$\begin{aligned}\int_{\mathbb{S}^{d-1}} f(\boldsymbol{\eta}) \,\mathrm{d}\mathbb{S}^{d-1}(\boldsymbol{\eta}) &= \int_{-1}^{1} \int_{\mathbb{S}^{d-2}} f\left(\sqrt{1-t^2}\, \boldsymbol{\eta}_{(d-1)} + t\boldsymbol{\epsilon}^d\right) \mathrm{d}\mathbb{S}^{d-2}(\boldsymbol{\eta}_{(d-1)})\, (1-t^2)^{\frac{d-3}{2}} \,\mathrm{d}t \\ &= \int_{-1}^{1} \mathcal{M}f(\boldsymbol{\epsilon}^d, t) \left|\mathbb{S}^{d-2}\right| (1-t^2)^{\frac{d-3}{2}} \,\mathrm{d}t. \qquad \blacksquare\end{aligned}$$

The spherical Fourier transform (2.26) turns out to be a powerful tool for analyzing the spherical mean operator $\mathcal{M}$. The following generalization of the Funk–Hecke formula (2.30) was proven by Berens, Butzer and Pawelke [BBP68, Section 4.2].

Theorem 3.4. *Let $Y_{n,d} \in \mathscr{Y}_{n,d}(\mathbb{S}^{d-1})$ be a spherical harmonic of degree $n \in \mathbb{N}_0$, and let $(\boldsymbol{\xi}, t) \in \mathbb{S}^{d-1} \times [-1, 1]$. Then we have*

$$\mathcal{M}Y_{n,d}(\boldsymbol{\xi}, t) = Y_{n,d}(\boldsymbol{\xi})\, P_{n,d}(t), \tag{3.5}$$

where $P_{n,d}$ is the Legendre polynomial of degree n in dimension d, see (2.13).

Proof. Let $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$ and $Y_{n,d} \in \mathscr{Y}_{n,d}(\mathbb{S}^{d-1})$. If $t = 1$, the assertion follows directly from the definition of the mean operator $\mathcal{M}$ in (3.2) and the fact that $P_{n,d}(1) = 1$, see (2.15). Then both sides of (3.5) equal $Y_{n,d}(\boldsymbol{\xi})$. In the case $t = -1$, we have on the left-hand side of (3.5)

$$\mathcal{M}Y_{n,d}(\boldsymbol{\xi}, -1) = Y_{n,d}(-\boldsymbol{\xi}) = (-1)^n \, Y_{n,d}(\boldsymbol{\xi})$$

and on the right-hand side

$$Y_{n,d}(\boldsymbol{\xi}) \, P_{n,d}(-1) = (-1)^n \, Y_{n,d}(\boldsymbol{\xi}).$$

Now let $t \in (-1, 1)$. We set the characteristic function

$$\psi(s) := \begin{cases} 1, & s \geq t, \\ 0, & s < t. \end{cases}$$

We compute by (3.4) the spherical convolution

$$\int_{\mathbb{S}^{d-1}} \psi(\boldsymbol{\xi}^\top \boldsymbol{\eta}) \, Y_{n,d}(\boldsymbol{\eta}) \, \mathrm{d}\mathbb{S}^{d-1}(\boldsymbol{\eta}) = \left|\mathbb{S}^{d-2}\right| \int_{-1}^{1} \mathcal{M}f(\boldsymbol{\xi}, s) \, \psi(s) \, (1 - s^2)^{\frac{d-3}{2}} \, \mathrm{d}s.$$

Inserting the definition of $\psi(s)$, we see that

$$\int_{\mathbb{S}^{d-1}} \psi(\boldsymbol{\xi}^\top \boldsymbol{\eta}) \, Y_{n,d}(\boldsymbol{\eta}) \, \mathrm{d}\mathbb{S}^{d-1}(\boldsymbol{\eta}) = \left|\mathbb{S}^{d-2}\right| \int_{t}^{1} \mathcal{M}f(\boldsymbol{\xi}, s) \, (1 - s^2)^{\frac{d-3}{2}} \, \mathrm{d}s. \tag{3.6}$$

On the other hand, we obtain by the Funk–Hecke formula (2.30)

$$\begin{aligned} \int_{\mathbb{S}^{d-1}} \psi(\boldsymbol{\xi}^\top \boldsymbol{\eta}) \, Y_{n,d}(\boldsymbol{\eta}) \, \mathrm{d}\mathbb{S}^{d-1}(\boldsymbol{\eta}) &= \left|\mathbb{S}^{d-2}\right| Y_{n,d}(\boldsymbol{\xi}) \int_{-1}^{1} \psi(s) \, P_{n,d}(s) \, (1 - s^2)^{\frac{d-3}{2}} \, \mathrm{d}t \\ &= \left|\mathbb{S}^{d-2}\right| Y_{n,d}(\boldsymbol{\xi}) \int_{t}^{1} P_{n,d}(s) \, (1 - s^2)^{\frac{d-3}{2}} \, \mathrm{d}s. \end{aligned} \tag{3.7}$$

Comparing (3.6) and (3.7), we obtain

$$\int_{t}^{1} \mathcal{M}f(\boldsymbol{\xi}, s) \, (1 - s^2)^{\frac{d-3}{2}} \, \mathrm{d}s = Y_{n,d}(\boldsymbol{\xi}) \int_{t}^{1} P_{n,d}(s) \, (1 - s^2)^{\frac{d-3}{2}} \, \mathrm{d}s.$$

Differentiation with respect to t yields

$$\mathcal{M}f(\boldsymbol{\xi}, t) \, (1 - t^2)^{\frac{d-3}{2}} = Y_{n,d}(\boldsymbol{\xi}) \, (1 - t^2)^{\frac{d-3}{2}} \, P_{n,d}(t),$$

which proves the assertion (3.5). ■

3.1.2 The mean operator in Sobolev spaces

We analyze the range of the mean operator $\mathcal{M}$ with respect to Sobolev spaces. To this end, we define the Sobolev space $H^{s,t}_{\mathrm{mix}}(\mathbb{S}^{d-1} \times (-1,1); w_d)$ of mixed orders $s,t \in \mathbb{R}$ on $\mathbb{S}^{d-1} \times (-1,1)$ with the weight $w_d(t) = (1-t^2)^{\frac{d-3}{2}}$ as the completion of the space $C^\infty(\mathbb{S}^{d-1} \times (-1,1))$ with respect to the Sobolev norm

$$\|g\|^2_{H^{s,t}_{\mathrm{mix}}(\mathbb{S}^{d-1}\times(-1,1);w_d)} := \sum_{n=0}^{\infty} \sum_{k=1}^{N_{n,d}} \sum_{l=0}^{\infty} \left|\hat{g}^k_{n,l,d}\right|^2 \left(n + \tfrac{d-2}{2}\right)^{2s} \left(l + \tfrac{d-2}{2}\right)^{2t}, \tag{3.8}$$

where the Fourier coefficients of $g \in C^\infty(\mathbb{S}^{d-1} \times (-1,1))$ are defined by

$$\hat{g}^k_{n,l,d} := \int_{-1}^{1} \int_{\mathbb{S}^{d-1}} g(\boldsymbol{\xi}, t)\, \overline{Y^k_{n,d}(\boldsymbol{\xi})}\, \widetilde{P}_{l,d}(t)\, (1-t^2)^{\frac{d-3}{2}}\, \mathrm{d}\boldsymbol{\xi}\, \mathrm{d}t$$

for $n, l \in \mathbb{N}_0$ and $k = 1, \dots, N_{n,d}$. In particular, any function $g \in H^{s,t}_{\mathrm{mix}}(\mathbb{S}^{d-1} \times (-1,1); w_d)$ can be written as the Fourier series

$$g(\boldsymbol{\xi}, t) = \sum_{n=0}^{\infty} \sum_{k=1}^{N_{n,d}} \sum_{l=0}^{\infty} \hat{g}^k_{n,l,d}\, Y^k_{n,d}(\boldsymbol{\xi})\, \widetilde{P}_{l,d}(t), \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1},\ t \in (-1,1). \tag{3.9}$$

We note that the Fourier series (3.9) converges in $H^{s+\frac{d-2}{2},0}_{\mathrm{mix}}(\mathbb{S}^{d-1} \times (-1,1); w_d)$ with respect to the Sobolev norm (3.8), but not necessarily pointwise for $(\boldsymbol{\xi}, t) \in \mathbb{S}^{d-1} \times (-1,1)$. The notion of Sobolev spaces H^s_{mix} of mixed smoothness is used in a similar manner on the d-dimensional torus, cf. [KSU15, Section 2.1].

For $s = t = 0$, we obtain the weighted Lebesgue space

$$L^2(\mathbb{S}^{d-1} \times (-1,1); w_d) := H^{0,0}_{\mathrm{mix}}(\mathbb{S}^{d-1} \times (-1,1); w_d).$$

The functions $(\boldsymbol{\xi}, t) \mapsto Y^k_{n,d}(\boldsymbol{\xi})\, \widetilde{P}_{l,d}(t)$ form an orthonormal basis of $L^2(\mathbb{S}^{d-1} \times (-1,1); w_d)$, see also (2.17). Similar to (2.50) for the Sobolev space $H^s(\mathbb{S}^{d-1})$, we have for $t = 0$ and $s \in \mathbb{R}$ an equivalent expression of the Sobolev norm

$$\|g\|_{H^{s,0}_{\mathrm{mix}}(\mathbb{S}^{d-1}\times(-1,1);w_d)} = \left\| \left(-\Delta^\bullet_{\boldsymbol{\xi}} + \frac{(d-2)^2}{4}\right)^{s/2} g \right\|_{L^2(\mathbb{S}^{d-1}\times(-1,1);w_d)}$$

in terms of the Laplace–Beltrami operator $\Delta^\bullet$, which was given in (2.48).

Theorem 3.5. *Let $s \in \mathbb{R}$. The mean operator $\mathcal{M}$ on the sphere $\mathbb{S}^{d-1}$ can be extended to a continuous linear operator*

$$\mathcal{M}\colon H^s(\mathbb{S}^{d-1}) \to H^{s+\frac{d-2}{2},0}_{\mathrm{mix}}(\mathbb{S}^{d-1} \times (-1,1); w_d).$$

Proof. Let $f \in C^\infty(\mathbb{S}^{d-1})$. We write f as the spherical Fourier series (2.26). By Theorem 3.4 together with (2.18), we obtain

$$\mathcal{M}f(\boldsymbol{\xi}, t) = \sum_{n=0}^{\infty} \sum_{k=1}^{N_{n,d}} \hat{f}^k_{n,d} \, Y^k_{n,d}(\boldsymbol{\xi}) \sqrt{\frac{|\mathbb{S}^{d-1}|}{N_{n,d} \, |\mathbb{S}^{d-2}|}} \, \widetilde{P}_{n,d}(t). \tag{3.10}$$

Then we have

$$\|\mathcal{M}f\|^2_{H_{\mathrm{mix}}^{s+\frac{d-2}{2},0}(\mathbb{S}^{d-1}\times(-1,1);w_d)} = \sum_{n=0}^{\infty} \sum_{k=1}^{N_{n,d}} \left|\hat{f}^k_{n,d}\right|^2 \frac{|\mathbb{S}^{d-1}|}{N_{n,d} \, |\mathbb{S}^{d-2}|} \left(n + \tfrac{d-2}{2}\right)^{2s+d-2}.$$

We have by (2.12)

$$\begin{aligned} N_{n,d} &= \frac{(2n+d-2)\,(n+d-3)!}{n!(d-2)!} \\ &= \frac{(2n+d-2)\,(n+d-3)(n+d-2)\cdots(n+1)}{(d-2)!}. \end{aligned}$$

Expanding the product, we obtain asymptotically for $n \to \infty$

$$N_{n,d} \simeq \frac{2}{(d-2)!} \, n^{d-2}. \tag{3.11}$$

Hence, the term $\frac{1}{N_{n,d}} \left(n + \frac{d-2}{2}\right)^{d-2}$ converges for $n \to \infty$ to a nonzero real number. Then, there exists a constant $c_{s,d} > 0$, which is independent of f such that

$$\begin{aligned} \|\mathcal{M}f\|^2_{H_{\mathrm{mix}}^{s+\frac{d-2}{2},0}(\mathbb{S}^{d-1}\times(-1,1);w_d)} &\le c_{s,d} \sum_{n=0}^{\infty} \sum_{k=1}^{N_{n,d}} \left|\hat{f}^k_{n,d}\right|^2 \left(n + \tfrac{d-2}{2}\right)^{2s} \\ &= c_{s,d} \, \|f\|_{H^s(\mathbb{S}^{d-1})}, \end{aligned}$$

Since $C^\infty(\mathbb{S}^{d-1})$ is dense in the Sobolev space $H^s(\mathbb{S}^{d-1})$, the mean operator $\mathcal{M}$ extends to a continuous linear operator from $H^s(\mathbb{S}^{d-1})$ to $H_{\mathrm{mix}}^{s+\frac{d-2}{2},0}(\mathbb{S}^{d-1} \times (-1,1); w_d)$. ■

Singular value decomposition. Theorem 3.4 forms a singular value decomposition of the mean operator $\mathcal{M}\colon H^s(\mathbb{S}^{d-1}) \to H_{\mathrm{mix}}^{s+\frac{d-2}{2},0}(\mathbb{S}^{d-1} \times (-1,1); w_d)$. We first recall some general theory about the singular value decomposition of compact operators, see [EHN96, Section 2.2]. Let $\mathcal{K}\colon X \to Y$ be a compact linear operator between two separable Hilbert spaces X and Y. A complete singular system

$$\{(u_n, v_n, \hat{\mathcal{K}}_n)\,;\, n \in \mathbb{N}_0\} \tag{3.12}$$

consists of an orthonormal basis $\{u_n\}_{n=0}^\infty$ of X, a set of orthonormal functions $\{v_n\}_{n=0}^\infty$ in Y and singular values $\{\hat{\mathcal{K}}_n\}_{n=0}^\infty \subset \mathbb{C}$, which satisfy $\hat{\mathcal{K}}_n \to 0$ for $n \to \infty$ such that the operator $\mathcal{K}$ can be diagonalized as

$$\mathcal{K}f = \sum_{n=0}^{\infty} \hat{\mathcal{K}}_n \, \langle f, u_n \rangle \, v_n, \qquad f \in X.$$

If all singular values $\hat{\mathcal{K}}_n$ are nonzero, the operator $\mathcal{K}$ is injective and for $g = \mathcal{K}f$, we have the inversion formula

$$f = \sum_{n=0}^{\infty} \frac{\langle g, v_n \rangle}{\hat{\mathcal{K}}_n} u_n.$$

The instability of an inverse problem can be characterized by the decay of the singular values $\hat{\mathcal{K}}_n$. The inverse problem of solving $\mathcal{K}f = g$ for f is called mildly ill-posed of degree $\alpha > 0$ if $\hat{\mathcal{K}}_n \in \mathcal{O}(n^{-\alpha})$ for $n \to \infty$. The faster the singular values $\hat{\mathcal{K}}_n$ decay to zero for $n \to \infty$, the more ill-posed the inverse problem becomes.

For the case of the mean operator $\mathcal{M}$ on the sphere, Theorem 3.5 implies in particular for $s = 0$ that $\mathcal{M}$ is a compact operator from $L^2(\mathbb{S}^{d-1})$ to $L^2(\mathbb{S}^{d-1} \times [-1,1]; w_d)$. Hence, we see that (3.10) forms the singular value decomposition of $\mathcal{M}$ as follows.

Corollary 3.6. The mean operator $\mathcal{M}\colon L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1} \times [-1,1]; w_d)$ on the sphere $\mathbb{S}^{d-1}$ has the singular value decomposition consisting of the complete singular system

$$\left\{ \left(Y_{n,d}^k, Y_{n,d}^k\, \widetilde{P}_{n,d}, \sqrt{\frac{|\mathbb{S}^{d-1}|}{N_{n,d}\, |\mathbb{S}^{d-2}|}} \right) ; n \in \mathbb{N}_0,\ k = 1, \dots, N_{n,d} \right\}.$$

3.1.3 A range characterization

We have seen that there are some redundancies in the range of the spherical mean operator $\mathcal{M}$, like the symmetry condition (3.3). A more sophisticated tool for analyzing the range of such Radon-type transforms is John's equation, which was first shown in 1938 by John [Joh38] for the ray transform, which computes the integrals along all lines in the Euclidean space $\mathbb{R}^d$. John's equation is an ultrahyperbolic partial differential equation which holds for all functions in the range of the ray transform. An adaption of John's equation for the Funk transform on the rotation group SO(3) was shown in [NS99]. We can obtain a similar partial differential equation for the mean operator $\mathcal{M}$ on the sphere. We point out that both the ray transform in $\mathbb{R}^d$ and the Funk transform on the rotation group SO(3) are about integrals along one-dimesnsional submanifolds, whereas the mean operator $\mathcal{M}$ takes the integrals along submanifolds whose dimension is one smaller than the dimension of the sphere $\mathbb{S}^{d-1}$.

Theorem 3.7. *Let $f \in C^2(\mathbb{S}^{d-1})$ and $d \geq 2$. We denote by $\Delta_{\boldsymbol{\xi}}^\bullet$ the Laplace–Beltrami operator (2.41) that acts with respect to $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$. Then, for $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$ and $t \in (-1,1)$, the mean operator $\mathcal{M}f(\boldsymbol{\xi},t) = g(\boldsymbol{\xi},t)$ satisfies the differential equation*

$$\Delta_{\boldsymbol{\xi}}^\bullet g(\boldsymbol{\xi},t) = (1-t^2)^{\frac{3-d}{2}} \frac{\partial}{\partial t} \left((1-t^2)^{\frac{d-1}{2}} \frac{\partial}{\partial t} g(\boldsymbol{\xi},t) \right), \tag{3.13}$$

which can also be written in the form

$$\Delta_{\boldsymbol{\xi}}^\bullet g(\boldsymbol{\xi},t) = \left((1-t^2) \frac{\partial^2}{\partial t^2} - (d-1)\, t \frac{\partial}{\partial t} \right) g(\boldsymbol{\xi},t).$$

Proof. We write the function $f \in C^2(\mathbb{S}^{d-1}) \subset L^2(\mathbb{S}^{d-1})$ as spherical Fourier series (2.26) and obtain by Theorem 3.4

$$\mathcal{M}f(\boldsymbol{\xi}, t) = \sum_{n=0}^{\infty} \sum_{k=1}^{N_{n,d}} \hat{f}_{n,d}^k \, Y_{n,d}^k(\boldsymbol{\xi}) \, P_{n,d}(t), \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1},\ t \in (-1,1).$$

Let $n \in \mathbb{N}_0$ and $k \in \{1, \ldots, N_{n,d}\}$. We are going to use the fact that any spherical harmonic $Y_{n,d}^k$ is an eigenfunction of the Laplace–Beltrami operator $\Delta^\bullet$. In particular, we have by (2.42)

$$\Delta^\bullet Y_{n,d}^k = -n\,(n+d-2)\, Y_{n,d}^k.$$

Hence, we have

$$\Delta_{\boldsymbol{\xi}}^\bullet \mathcal{M}f(\boldsymbol{\xi}, t) = \sum_{n=0}^{\infty} \sum_{k=1}^{N_{n,d}} -n\,(n+d-2)\, \hat{f}_{n,d}^k \, Y_{n,d}^k(\boldsymbol{\xi}) \, P_{n,d}(t). \tag{3.14}$$

We note that the right-hand side of (3.14) converges in $L^2(\mathbb{S}^{d-1})$ for all $t \in (-1,1)$ because, by the bound (2.15) of the Legendre polynomials $P_{n,d}$, we have

$$\sum_{n=0}^{\infty} \sum_{k=1}^{N_{n,d}} \left| -n\,(n+d-2)\, \hat{f}_{n,d}^k \, Y_{n,d}^k(\boldsymbol{\xi}) \, P_{n,d}(t) \right| \le \sum_{n=0}^{\infty} \sum_{k=1}^{N_{n,d}} \left| n\,(n+d-2)\, \hat{f}_{n,d}^k \, Y_{n,d}^k(\boldsymbol{\xi}) \right|$$

and the latter sum converges with respect to $L^2(\mathbb{S}^{d-1})$ since we assumed that $f \in C^2(\mathbb{S}^{d-1}) \subset H^2(\mathbb{S}^{d-1})$. The Legendre polynomials $P_{n,d}$ satisfy the differential equation [AH12, (2.82)]

$$(1-t^2)^{\frac{3-d}{2}} \frac{\mathrm{d}}{\mathrm{d}t} \left((1-t^2)^{\frac{d-1}{2}} \frac{\mathrm{d}}{\mathrm{d}t} P_{n,d}(t) \right) = -n\,(n+d-2)\, P_{n,d}(t). \tag{3.15}$$

Combining (3.14) and (3.15), we obtain

$$\begin{aligned}
\Delta_{\boldsymbol{\xi}}^\bullet \mathcal{M}f(\boldsymbol{\xi}, t) &= (1-t^2)^{\frac{3-d}{2}} \sum_{n=0}^{\infty} \sum_{k=1}^{N_{n,d}} \frac{\partial}{\partial t} \left((1-t^2)^{\frac{d-1}{2}} \frac{\partial}{\partial t} \hat{f}_{n,d}^k \, Y_{n,d}^k(\boldsymbol{\xi}) \, P_{n,d}(t) \right) \\
&= (1-t^2)^{\frac{3-d}{2}} \frac{\partial}{\partial t} \left((1-t^2)^{\frac{d-1}{2}} \frac{\partial}{\partial t} \sum_{n=0}^{\infty} \sum_{k=1}^{N_{n,d}} \hat{f}_{n,d}^k \, Y_{n,d}^k(\boldsymbol{\xi}) \, P_{n,d}(t) \right) \\
&= (1-t^2)^{\frac{3-d}{2}} \frac{\partial}{\partial t} \left((1-t^2)^{\frac{d-1}{2}} \frac{\partial}{\partial t} \mathcal{M}f(\boldsymbol{\xi}, t) \right),
\end{aligned}$$

which proves the assertion. ■

The Theorems 3.7 and 3.5 give necessary conditions for a function to be in the range of the mean operator $\mathcal{M}$. It turns out that the John-type equation (3.13) forms also a sufficient condition. In order to give such condition, we provide an interpretation of the differential equation (3.13) if the function g is not differentiable in the classical sense. We already gave a definition of the Laplace–Beltrami operator $\Delta^\bullet$ on Sobolev spaces in (2.49). Furthermore, we use the same approach to extend the definition of the differential operator $(1-t^2)^{\frac{3-d}{2}} \frac{\partial}{\partial t}(1-t^2)^{\frac{d-1}{2}} \frac{\partial}{\partial t}$ to a pseudo-differential operator. Following (3.15), we set for $g \in H^{0,2}_{\mathrm{mix}}(\mathbb{S}^{d-1} \times (-1,1); w_d)$

$$(1-t^2)^{\frac{3-d}{2}} \frac{\partial}{\partial t}(1-t^2)^{\frac{d-1}{2}} \frac{\partial}{\partial t} g(\boldsymbol{\xi}, t) = \sum_{n,l=0}^{\infty} \sum_{k=1}^{N_{n,d}} -n\,(n+d-2)\, \hat{g}^k_{n,l,d}\, Y^k_{n,d}(\boldsymbol{\xi}) \widetilde{P}_{l,d}(t).$$

Theorem 3.8. *Let $s \in \mathbb{R}$ and let $g \in H^{s+\frac{d-2}{2},0}_{\mathrm{mix}}(\mathbb{S}^{d-1} \times (-1,1); w_d)$ fulfill the John-type equation* (3.13)*, i. e.,*

$$\Delta^\bullet_{\boldsymbol{\xi}} g(\boldsymbol{\xi}, t) = (1-t^2)^{\frac{3-d}{2}} \frac{\partial}{\partial t} \left((1-t^2)^{\frac{d-1}{2}} \frac{\partial}{\partial t} g(\boldsymbol{\xi}, t) \right), \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1},\ t \in (-1,1).$$

Then there exists a function $f \in H^s(\mathbb{S}^{d-1})$ such that $\mathcal{M} f = g$.

Proof. Let us write the function $g \in H^{s+\frac{d-2}{2},0}_{\mathrm{mix}}(\mathbb{S}^{d-1} \times (-1,1); w_d)$ as the Fourier series (3.9), i. e.,

$$g(\boldsymbol{\xi}, t) = \sum_{n=0}^{\infty} \sum_{k=1}^{N_{n,d}} \sum_{l=0}^{\infty} \hat{g}^k_{n,l,d}\, Y^k_{n,d}(\boldsymbol{\xi})\, \widetilde{P}_{l,d}(t), \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1},\ t \in (-1,1).$$

Since g fulfills the John-type differential equation (3.13), we obtain with (3.14) and (3.15) that

$$-n\,(n+d-2)\, \hat{g}^k_{n,l,d} = -l\,(l+d-2)\, \hat{g}^k_{n,l,d}$$

holds for all $n, l \in \mathbb{N}_0$ and $k = 1, \ldots, N_{n,d}$. Hence, we see that the Fourier coefficients $\hat{g}^k_{n,l,d}$ must vanish unless $-n\,(n+d-2) = -l\,(l+d-2)$. Solving for n yields the two solutions $n = l$ and $n = -l-d+2$, where the latter solution can only be reached for $d = 2$ and $n = l = 0$ because $n, l \geq 0$ and $d \geq 2$. Hence, we see that $\hat{g}^k_{n,l,d}$ vanishes whenever $n \neq l$ and thus we have

$$g(\boldsymbol{\xi}, t) = \sum_{n=0}^{\infty} \sum_{k=1}^{N_{n,d}} \hat{g}^k_{n,n,d}\, Y^k_{n,d}(\boldsymbol{\xi})\, \widetilde{P}_{n,d}(t).$$

By the definition (2.18) of the normalized Legendre polynomial $\widetilde{P}_{n,d}$, we have

$$g(\boldsymbol{\xi}, t) = \sum_{n=0}^{\infty} \sum_{k=1}^{N_{n,d}} \hat{g}^k_{n,n,d}\, Y^k_{n,d}(\boldsymbol{\xi}) \sqrt{\frac{N_{n,d}\, |\mathbb{S}^{d-2}|}{|\mathbb{S}^{d-1}|}} P_{n,d}(t). \tag{3.16}$$

The assumption $g \in H^{s+\frac{d-2}{2},0}_{\mathrm{mix}}(\mathbb{S}^{d-1} \times (-1,1); w_d)$ implies the convergence of the series

$$\sum_{n=0}^{\infty} \sum_{k=1}^{N_{n,d}} \left|\hat{g}^k_{n,n,d}\right|^2 \left(n + \tfrac{d-2}{2}\right)^{2s+d-2}.$$

Since we have $N_{n,d} \simeq \frac{2}{(d-2)!}\, n^{d-2}$ for $n \to \infty$ by (3.11), we see that also the series

$$\sum_{n=0}^{\infty} \sum_{k=1}^{N_{n,d}} \frac{N_{n,d}\,|\mathbb{S}^{d-2}|}{|\mathbb{S}^{d-1}|} \left(n + \tfrac{d-2}{2}\right)^{2s} \left|\hat{g}^k_{n,n,d}\right|^2$$

converges. Hence, we can define the function $f \in H^s(\mathbb{S}^{d-1})$ by its Fourier series

$$f := \sum_{n=0}^{\infty} \sum_{k=1}^{N_{n,d}} \hat{g}^k_{n,n,d} \sqrt{\frac{N_{n,d}\,|\mathbb{S}^{d-2}|}{|\mathbb{S}^{d-1}|}}\, Y^k_{n,d}.$$

With Theorem 3.4, we obtain

$$\mathcal{M}f(\boldsymbol{\xi}, t) = \sum_{n=0}^{\infty} \sum_{k=1}^{N_{n,d}} \hat{g}^k_{n,n,d} \sqrt{\frac{N_{n,d}\,|\mathbb{S}^{d-2}|}{|\mathbb{S}^{d-1}|}}\, Y^k_{n,d}(\boldsymbol{\xi})\, P_{n,d}(t), \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1},\ t \in (-1,1).$$

Comparing the last equation with (3.5), we see that $g = \mathcal{M}f$. ■

Remark 3.9. In the proof of Theorem 3.8, we have seen that the Fourier coefficients $\hat{g}^k_{n,l,d}$ of the function $g = \mathcal{M}f$ vanish if $n \neq l$. For such a function g, it follows from the definition of the Sobolev norm (3.8) that

$$\|g\|_{H^{s,t}_{\mathrm{mix}}(\mathbb{S}^{d-1}\times(-1,1);w_d)} = \|g\|_{H^{s+\alpha,t-\alpha}_{\mathrm{mix}}(\mathbb{S}^{d-1}\times(-1,1);w_d)}$$

for any $\alpha \in \mathbb{R}$. Hence, in the statement of Theorem 3.8, we could alternatively impose the condition that $g \in H^{s,t}_{\mathrm{mix}}(\mathbb{S}^{d-1} \times (-1,1); w_d)$ for some $s, t \geq 0$ with $s + t = \frac{d-2}{2}$. □

Injectivity sets of the mean operator

Let $s \geq 0$. A set $D \subset \mathbb{S}^{d-1} \times [-1,1]$ is called an injectivity set of the mean operator $\mathcal{M}$ if the equation $\mathcal{M}f\big|_D = 0$ implies that $f = 0$ for all functions $f \in H^s(\mathbb{S}^{d-1})$. In Theorem 3.7, we have seen that $\mathcal{M}f$ is always a solution of the John-type equation (3.13) for all $f \in H^s(\mathbb{S}^{d-1})$. In Theorem 3.8, we have seen that all solutions of (3.13) can be expressed as $\mathcal{M}f$ with some function f. In what follows, we show the equivalence between injectivity sets D of the mean operator $\mathcal{M}$ and the unique solvability of the differential equation (3.13) with boundary values on D.

However, if we only assume that $\mathcal{M}f$ is in the Sobolev space $H^{s,t}_{\mathrm{mix}}(\mathbb{S}^{d-1} \times [-1,1]; w_d)$, it is not necessarily possible to have point evaluations of $\mathcal{M}f$ and thus it is not clear how to understand the restriction $\mathcal{M}f\big|_D$ in this case. In order to avoid these difficulties, we use Sobolev embeddings that guarantee the continuity of the functions f and $\mathcal{M}f$. We have already seen the Sobolev embedding (2.47) for $H^s(\mathbb{S}^{d-1})$. The following lemma forms an analogue on $H^{s,0}_{\mathrm{mix}}(\mathbb{S}^{d-1} \times [-1,1]; w_d)$.

Lemma 3.10. *Let $s > d - \frac{3}{2}$, and let $g \in H^{s,0}_{\mathrm{mix}}(\mathbb{S}^{d-1} \times [-1,1]; w_d)$ satisfy $\hat{g}^k_{n,l,d} = 0$ whenever $n \neq l$. Then the function g has a continuous representative.*

Proof. We start with the Fourier series expansion (3.9) of the function g,

$$g = \sum_{n=0}^{\infty} \sum_{k=1}^{N_{n,d}} \hat{g}_{n,n,d}^k \, Y_{n,d}^k \, \widetilde{P}_{n,d}.$$

We are going to show that this Fourier series of g converges pointwise absolutely. Let $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$ and $t \in [-1,1]$. We have

$$\begin{aligned} |g(\boldsymbol{\xi},t)| &= \left| \sum_{n=0}^{\infty} \sum_{k=1}^{N_{n,d}} \hat{g}_{n,n,d}^k \; Y_{n,d}^k(\boldsymbol{\xi}) \, \widetilde{P}_{n,d}(t) \right| \\ &\le \sum_{n=0}^{\infty} \sum_{k=1}^{N_{n,d}} \left(n + \tfrac{d-2}{2}\right)^s \left|\hat{g}_{n,n,d}^k\right| \, \left(n + \tfrac{d-2}{2}\right)^{-s} \left|Y_{n,d}^k(\boldsymbol{\xi})\right| \, \left|\widetilde{P}_{n,d}(t)\right|. \end{aligned}$$

With the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |g(\boldsymbol{\xi},t)| \le & \left(\sum_{n=0}^{\infty} \sum_{k=1}^{N_{n,d}} \left(n + \tfrac{d-2}{2}\right)^{2s} \left|\hat{g}_{n,n,d}^k\right|^2 \right)^{1/2} \\ & \cdot \left(\sum_{n=0}^{\infty} \sum_{k=1}^{N_{n,d}} \left(n + \tfrac{d-2}{2}\right)^{-2s} \left|Y_{n,d}^k(\boldsymbol{\xi})\right|^2 \left|\widetilde{P}_{n,d}(t)\right|^2 \right)^{1/2}. \end{aligned}$$

By the definition of the Sobolev norm (3.8), the addition formula (2.25) for the spherical harmonics $Y_{n,d}^k$, and (2.18) together with the bound (2.15) of the Legendre polynomials, we obtain

$$|g(\boldsymbol{\xi},t)| \le \|g\|_{H_{\mathrm{mix}}^{s,0}(\mathbb{S}^{d-1}\times[-1,1];w_d)} \left(\sum_{n=0}^{\infty} \left(n + \tfrac{d-2}{2}\right)^{-2s} \frac{N_{n,d}}{|\mathbb{S}^{d-1}|} \frac{N_{n,d}\left|\mathbb{S}^{d-2}\right|}{|\mathbb{S}^{d-1}|} \right)^{1/2}.$$

Since, by (3.11), we have $N_{n,d} \simeq \frac{2}{(d-2)!} n^{d-2}$ for $n \to \infty$, we see that the last sum converges if and only if $-2s + 2(d-2) < -1$, which is equivalent to $s > d - \frac{3}{2}$. ∎

Theorem 3.11. *Let $D \subset \mathbb{S}^{d-1} \times [-1,1]$, $g_0 \colon D \to \mathbb{C}$, and let $s > \frac{d-1}{2}$. Then the following two statements are equivalent:*

i) The problem

$$\mathcal{M}\big|_D f = g_0 \tag{3.17}$$

has a unique solution $f \in H^s(\mathbb{S}^{d-1})$.

ii) The John-type differential equation (3.13), i. e.,

$$\Delta_{\boldsymbol{\xi}}^{\bullet} g(\boldsymbol{\xi},t) = (1-t^2)^{\frac{3-d}{2}} \frac{\partial}{\partial t} \left((1-t^2)^{\frac{d-1}{2}} \frac{\partial}{\partial t} g(\boldsymbol{\xi},t) \right),$$

with boundary condition $g\big|_D = g_0$ has a unique solution

$$g \in H_{\mathrm{mix}}^{s+\frac{d-2}{2},0}(\mathbb{S}^{d-1} \times [-1,1]; w_d).$$

Proof. We start by showing that (i) implies (ii). Let $f \in H^s(\mathbb{S}^{d-1})$ be the unique solution of (3.17). We set $g = \mathcal{M}f$. By Theorem 3.5, we have $g \in H_{\mathrm{mix}}^{s+\frac{d-2}{2},0}(\mathbb{S}^{d-1} \times [-1,1]; w_d)$. Then g is a solution of the John-type equation (3.13) by Theorem 3.7 and it satisfies $g\big|_D = g_0$ by construction. In order to show the uniqueness, we assume that $\tilde{g} \in H_{\mathrm{mix}}^{s+\frac{d-2}{2},0}(\mathbb{S}^{d-1} \times [-1,1]; w_d)$ is another solution of (3.13) with $\tilde{g}\big|_D = g_0$. By Theorem 3.8, there exists a function $\tilde{f} \in H^s(\mathbb{S}^{d-1})$ such that $\mathcal{M}\tilde{f} = \tilde{g}$. By the assumed uniqueness of problem (i), we see that $f = \tilde{f}$ and hence $g = \tilde{g}$.

Now we show the reverse implication (ii) $\Rightarrow$ (i). Let $g \in H_{\mathrm{mix}}^{s+\frac{d-2}{2},0}(\mathbb{S}^{d-1} \times [-1,1]; w_d)$ be the unique solution of the John-type equation (3.13) satisfying $g\big|_D = g_0$. We note that, since $s > \frac{d-1}{2}$, Lemma 3.10 shows that g is continuous and hence the pointwise evaluations of g and $g\big|_D$ are well-defined. By Theorem 3.8, there exists a function $f \in H^s(\mathbb{S}^{d-1})$ such that $\mathcal{M}f = g$. Hence, f is a solution of (i). To show the uniqueness, we assume that a function $\tilde{f}$ solves (i). Then we see that $\mathcal{M}\tilde{f}$ is also a solution of (3.13) by Theorem 3.7 with $\mathcal{M}\tilde{f}\big|_D = g_0$. The uniqueness of (ii) implies that $\mathcal{M}\tilde{f}$ must coincide with $g = \mathcal{M}f$. Hence, we have $f = \tilde{f}$. ■

Remark. The condition $s > \frac{d-1}{2}$ in Theorem 3.11 implies that f is continuous by the Sobolev embedding (2.47). This implies that the function $\mathcal{M}f$ is continuous.

3.2 The Funk–Radon transform

The most well-known restriction of the mean operator $\mathcal{M}$ is the *Funk–Radon transform*

$$\mathcal{F}f(\boldsymbol{\xi}) := \mathcal{M}f(\boldsymbol{\xi}, 0) = \frac{1}{|\mathbb{S}^{d-2}|} \int_{\boldsymbol{\xi}^\top \boldsymbol{\eta} = 0} f(\boldsymbol{\eta}) \,\mathrm{d}\boldsymbol{\eta}, \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1}. \tag{3.18}$$

The Funk–Radon transform (FRT) $\mathcal{F}$ computes the mean values along all maximal subspheres of $\mathbb{S}^{d-1}$. A maximal subsphere of the sphere $\mathbb{S}^{d-1}$ is the intersection of $\mathbb{S}^{d-1}$ with a hyperplane that contains the origin. Maximal subspheres are totally geodesic submanifolds of $\mathbb{S}^{d-1}$, i.e., any geodesic on the maximal subsphere is also a geodesic on the whole sphere. A maximal subsphere of the two-dimensional sphere $\mathbb{S}^2$ is known as a great circle. A depiction of great circles on $\mathbb{S}^2$ is found in Figure 3.2.

Great circle integrals first appeared in 1905 in the work of Minkowski [Min05], which we cite after its German translation [Min11]. The Funk–Radon transform was introduced by Funk in 1911 and originally called the "Kreisintegralfunktion" (circle integral function) of f. We cite Funk's 1911 dissertation [Fun11] from the identical parts in his research article [Fun13]. The Funk–Radon transform is also known by the names *Funk transform*, *spherical Radon transform* [GW92], *Minkowski–Funk transform* [Rub00] or *totally geodesic Radon transform on the sphere* [Hel90].

We observe that the Funk–Radon transform of an odd function $f(\boldsymbol{\eta}) = -f(-\boldsymbol{\eta})$ vanishes everywhere because if a maximal subsphere contains a point $\boldsymbol{\eta} \in \mathbb{S}^{d-1}$, it must

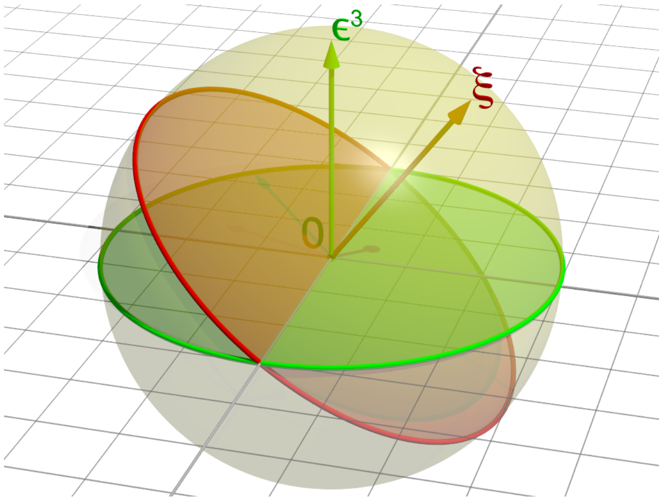

Figure 3.2: Two great circles of the sphere $\mathbb{S}^2$, along which the Funk–Radon transform takes the mean values. The green circle, which is the equator of $\mathbb{S}^2$, is perpendicular to the north pole $\boldsymbol{\epsilon}^3$ and the red circle is perpendicular to some other vector $\boldsymbol{\xi} \in \mathbb{S}^2$.

also contain its antipodal point $-\boldsymbol{\eta}$. So we can only expect to recover even functions $f(\boldsymbol{\xi}) = f(-\boldsymbol{\xi})$ from their respective Funk–Radon transform $\mathcal{F}f$. Furthermore, Equation (3.3) implies that the Funk–Radon transform $\mathcal{F}f$ of any function f is even, i. e.,

$$\mathcal{F}f(\boldsymbol{\xi}) = \mathcal{F}f(-\boldsymbol{\xi}), \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1}.$$

3.2.1 Eigenvalue decomposition

As a special case of (3.5) for the mean operator $\mathcal{M}$ we obtain the eigenvalue decomposition of the Funk–Radon transform $\mathcal{F}$.

Theorem 3.12. *Let $Y_{n,d} \in \mathscr{Y}_{n,d}(\mathbb{S}^{d-1})$ be a spherical harmonic of degree $n \in \mathbb{N}_0$. Then, we have*

$$\mathcal{F}Y_{n,d} = P_{n,d}(0)\, Y_{n,d}, \tag{3.19}$$

where

$$P_{n,d}(0) = \begin{cases} \dfrac{(-1)^{n/2}\,(n-1)!!\,(d-3)!!}{(n+d-3)!!}, & n \text{ even} \\ 0, & n \text{ odd}. \end{cases} \tag{3.20}$$

Proof. Equation (3.19) is a special case of (3.5) with $t = 0$. By the recurrence relation (2.14) of the Legendre polynomial $P_{n,d}$, we obtain for $n \in \mathbb{N}_0$ and $d \geq 3$

$$P_{n,d}(0) = -\frac{n-1}{n+d-3} P_{n-2,d}(0) = \begin{cases} (-1)^{n/2} \frac{(n-1)!!\,(d-3)!!}{(n+d-3)!!}, & n \text{ even} \\ 0, & n \text{ odd}. \end{cases}$$

Since the spherical harmonics $Y_{n,d}^k$ are complete in $L^2(\mathbb{S}^{d-1})$ shows that this is indeed an eigenvalue decomposition. ■

The eigenvalue decomposition (3.19) was shown for $\mathbb{S}^2$ by Minkowski [Min05] in 1905. The general situation of arbitrary dimension d was investigated by Goodey and Groemer [GG90] in 1990. The eigenvalue decomposition also shows that the Funk–Radon transform is injective for even functions $f(\boldsymbol{\xi}) = f(-\boldsymbol{\xi})$. The following range description with respect to spherical Sobolev spaces was shown by Strichartz [Str81, Paragraph 4] in 1981.

Theorem 3.13. *Let $H^s_{\mathrm{even}}(\mathbb{S}^{d-1})$ denote the Sobolev space of even functions $f(\boldsymbol{\xi}) = f(-\boldsymbol{\xi})$. The Funk–Radon transform is a bounded and bijective operator*

$$\mathcal{F}\colon H^s_{\mathrm{even}}(\mathbb{S}^{d-1}) \to H^{s+\frac{d-2}{2}}_{\mathrm{even}}(\mathbb{S}^{d-1}).$$

In particular, there exist constants $c_1, c_2 > 0$ such that for all $f \in H^s_{\mathrm{even}}(\mathbb{S}^{d-1})$

$$c_1 \left\|\mathcal{F}f\right\|_{H^{s+\frac{d-2}{2}}_{\mathrm{even}}(\mathbb{S}^{d-1})} \le \left\|f\right\|_{H^s_{\mathrm{even}}(\mathbb{S}^{d-1})} \le c_2 \left\|\mathcal{F}f\right\|_{H^{s+\frac{d-2}{2}}_{\mathrm{even}}(\mathbb{S}^{d-1})}.$$

Proof. Even though this is a well-known result, we give the proof in which we examine the asymptotic decay of the eigenvalues $P_{n,d}(0)$ in more detail. We have for $n \in \mathbb{N}_0$

$$|P_{2n,d}(0)| = \frac{(2n-1)!!\,(d-3)!!}{(2n+d-3)!!}.$$

Replacing the double factorials by the Gamma function as in (2.68), we have

$$|P_{2n,d}(0)| = \frac{(2n-1)!!\,\Gamma(\frac{d-1}{2})}{2^n\,\Gamma(2n+\frac{d-1}{2})}.$$

With the asymptotic approximations (2.75) and (2.72) of the double factorial and the Gamma function, respectively, we obtain for $n \to \infty$

$$\begin{aligned}
|P_{2n,d}(0)| &\simeq \frac{\sqrt{2}\,(2n)^n\,\mathrm{e}^{-n}\,\Gamma(\frac{d-1}{2})}{2^n\sqrt{2\pi}\,(n+\frac{d-1}{2})^{n+\frac{d-2}{2}}\,\mathrm{e}^{-n-\frac{d-1}{2}}} \\
&= \frac{n^n\,\mathrm{e}^{\frac{d-1}{2}}\,\Gamma(\frac{d-1}{2})}{\sqrt{\pi}\,(n+\frac{d-1}{2})^{n+\frac{d-2}{2}}} \\
&= \mathrm{e}^{\frac{d-1}{2}}\,\Gamma(\tfrac{d-1}{2})\,\pi^{-\frac12}\left(\frac{n}{n+\frac{d-1}{2}}\right)^n\left(n+\frac{d-1}{2}\right)^{\frac{2-d}{2}}.
\end{aligned}$$

Because $\mathrm{e}^x = \lim_{n\to\infty}(1+\frac{x}{n})^n$ for $x \in \mathbb{C}$, we obtain

$$|P_{2n,d}(0)| \simeq \mathrm{e}^{\frac{d-1}{2}}\,\Gamma(\tfrac{d-1}{2})\,\pi^{-\frac12}\,\mathrm{e}^{\frac{-d+1}{2}}\,(n+d-1)^{\frac{2-d}{2}}.$$

Hence, we have for $n \to \infty$

$$|P_{2n,d}(0)| \simeq \Gamma(\tfrac{d-1}{2})\, \pi^{-\frac{1}{2}}\, n^{\frac{2-d}{2}}. \tag{3.21}$$

By the definition of the Sobolev norm (2.46), we have

$$\|f\|^2_{H^s_{\mathrm{even}}(\mathbb{S}^{d-1})} = \sum_{\substack{n=0\\ n \text{ even}}}^{\infty} \sum_{k=1}^{N_{n,d}} \left(n + \tfrac{d-2}{2}\right)^{2s} \left| \left\langle f, Y^k_{n,d} \right\rangle_{L^2(\mathbb{S}^{d-1})} \right|^2,$$

and, by the eigenvalue decomposition (3.19),

$$\|\mathcal{F}f\|^2_{H^{s+\frac{d-2}{2}}_{\mathrm{even}}(\mathbb{S}^{d-1})} = \sum_{\substack{n=0\\ n \text{ even}}}^{\infty} \sum_{k=1}^{N_{n,d}} |P_{n,d}(0)|^2 \left(n + \tfrac{d-2}{2}\right)^{2s+d-2} \left| \left\langle f, Y^k_{n,d} \right\rangle_{L^2(\mathbb{S}^{d-1})} \right|^2.$$

Hence, we see that (3.13) holds if and only if

$$c_1\, |P_{n,d}(0)| \le (n + \tfrac{d-2}{2})^{\frac{2-d}{2}} \le c_2\, |P_{n,d}(0)|$$

for all even $n \in \mathbb{N}_0$. The latter follows from (3.21) and the fact that $P_{n,d}(0) \neq 0$ for all even n. On the other hand, we have $P_{n,d}(0) = 0$ for all odd $n \in \mathbb{N}_0$. ■

3.2.2 Inversion

Since the Funk–Radon transform $\mathcal{F}\colon H^s_{\mathrm{even}}(\mathbb{S}^{d-1}) \to H^{s+\frac{d-2}{2}}_{\mathrm{even}}(\mathbb{S}^{d-1})$ is bijective as we have seen in Theorem 3.13, we can ask for its inversion. Given $g \in H^{s+\frac{d-2}{2}}_{\mathrm{even}}(\mathbb{S}^{d-1})$, we want to solve the problem

$$\mathcal{F}f = g.$$

In what follows, we give an overview about different inversion methods from the literature. Funk [Fun13] proved an inversion formula of the Funk–Radon transform on the two-sphere $\mathbb{S}^2$ based on the solution of an Abel integral equation, see also [NW00, Section 2.5.1]. We have

$$f(\boldsymbol{\xi}) = \mathcal{F}f(\boldsymbol{\xi}) + \int_0^{\pi/2} \frac{1}{\cos\vartheta} \frac{\partial}{\partial\vartheta} \mathcal{M}[\mathcal{F}f](\boldsymbol{\xi}, \cos\vartheta)\, \mathrm{d}\vartheta, \qquad \boldsymbol{\xi} \in \mathbb{S}^2. \tag{3.22}$$

An inversion formula that makes use of complex analysis is due to Bailey et al. [BEGM03].

The Funk–Radon transform on the two-dimensional sphere $\mathbb{S}^2$ can be reduced to the Radon transform (3.67) on the plane as shown in [GRS94], see also [NW00, Section 2.5.2]. Since the function $f\colon \mathbb{S}^2 \to \mathbb{C}$ is even, it suffices to consider f in the northern hemisphere. This reduction is done via a central (gnomonic) projection from the origin to the plane tangential to sphere at the north pole $\boldsymbol{\epsilon}^3$. It was shown that

$$\mathcal{F}f(\boldsymbol{\xi}) = \mathcal{R}\phi((\xi_1, \xi_2), -\xi_3), \qquad \boldsymbol{\xi} \in \mathbb{S}^2,$$

where

$$\phi(\boldsymbol{y}) = 2(1+\|\boldsymbol{y}\|^2)^{-\frac{3}{2}}\, f\left(\frac{\boldsymbol{y}}{\sqrt{1+\|\boldsymbol{y}\|^2}}, \frac{1}{\sqrt{1+\|\boldsymbol{y}\|^2}}\right), \qquad \boldsymbol{y} \in \mathbb{R}^2.$$

Inversion formulas of the Funk–Radon transform in higher dimensions on $\mathbb{S}^{d-1}$ were found by Helgason [Hel59] for d even and Semjanistyi [Sem61] for d odd. We state the following inversion formula, which is due to Helgason [Hel90, Theorem 3.2]. An even function $f\colon \mathbb{S}^{d-1} \to \mathbb{C}$ can be reconstructed from its Funk–Radon transform $\mathcal{F}f$ by

$$f(\boldsymbol{\xi}) = \frac{2^{d-2}}{(d-3)!}\left[\left(\frac{\mathrm{d}}{\mathrm{d}(u^2)}\right)^{d-2}\int_0^u \mathcal{M}[\mathcal{F}f](\boldsymbol{\xi},v)\, v^{d-2}\,(u^2-v^2)^{\frac{d-4}{2}}\,\mathrm{d}v\right]_{u=1}, \qquad \boldsymbol{\xi}\in\mathbb{S}^{d-1}. \tag{3.23}$$

In the case of the two-sphere $\mathbb{S}^2$, Helgason's formula (3.23) can be expressed in the more simple form [Hel11, Theorem 4.1, Chapter II]

$$f(\boldsymbol{\eta}) = \frac{\mathrm{d}}{\mathrm{d}u}\int_0^u \mathcal{M}[\mathcal{F}f](\boldsymbol{\eta},v)\,\frac{v}{\sqrt{u^2-v^2}}\,\mathrm{d}v\bigg|_{u=1}, \qquad \boldsymbol{\eta}\in\mathbb{S}^2.$$

Another inversion formula of the Funk–Radon transform on $\mathbb{S}^{d-1}$ by Helgason [Hel11, Theorem 1.17, Chapter III] works only if d is even, where we have

$$f = \frac{(2\mathrm{i})^{2-d}\,\pi}{\Gamma(\frac{d-1}{2})^2}\prod_{j=0}^{\frac{d-4}{2}}\left(\boldsymbol{\Delta}^{\bullet} - (d-3-2j)(1+2j)\right)\mathcal{F}^2 f.$$

We note that Helgason's work does not use the normalization factor $\left|\mathbb{S}^{d-2}\right|^{-1}$ of the Funk–Radon transform as we do in (3.18).

Explicit inversion formulas of the Funk–Radon transform such as (3.22) or (3.23) are important for many theoretical considerations. In practical applications, however, they often lack numerical stability, which makes other methods more suitable in this case. The eigenvalue decomposition from Theorem 3.12 paves a simple and straightforward way to invert the Funk–Radon transform $\mathcal{F}$ as follows.

Proposition 3.14. *Let $s \ge 0$ and $g \in H_{\mathrm{even}}^{s+\frac{d-2}{2}}(\mathbb{S}^{d-1})$. Then the problem $\mathcal{F}f = g$ is solved by*

$$f(\boldsymbol{\xi}) = \sum_{\substack{n=0\\ n \text{ even}}}^{\infty}\sum_{k=1}^{N_{n,d}} \frac{1}{P_{n,d}(0)}\,\hat{g}_{n,d}^k\, Y_{n,d}^k(\boldsymbol{\xi}), \qquad \boldsymbol{\xi}\in\mathbb{S}^{d-1}, \tag{3.24}$$

where the spherical Fourier coefficients $\hat{g}_{n,d}^k$ of the function g are given in (2.27).

Remark 3.15. The inversion formula (3.24) of the Funk–Radon transform allows for a numerical implementation when the sum over n is truncated to $n \le N \in \mathbb{N}_0$, as we pointed out in [HQ15]. Let $g\colon \mathbb{S}^{d-1} \to \mathbb{C}$ be given. At first, we compute an approximation of the spherical Fourier coefficients of g,

$$\hat{g}_{n,d}^k = \int_{\mathbb{S}^{d-1}} g(\boldsymbol{\xi})\, \overline{Y_{n,d}^k(\boldsymbol{\xi})}\, \mathrm{d}\boldsymbol{\xi}, \qquad n = 0, \ldots, N, ,\ k = 1, \ldots, N_{n,d} \tag{3.25}$$

via a quadrature formula. Then, we can approximate a reconstruction of the function f via

$$f(\boldsymbol{\xi}) = \sum_{\substack{n=0\\ n \text{ even}}}^{N} \sum_{k=1}^{N_{n,d}} \frac{1}{P_{n,d}(0)}\, \hat{g}_{n,d}^k\, Y_{n,d}^k(\boldsymbol{\xi}), \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1}. \tag{3.26}$$

For the two-dimensional case of $\mathbb{S}^2$, there are spherical Fourier algorithms available for the fast evaluation of (3.25) and (3.26), see [KP03, KP08, Sch13] as well as [PPST18, Section 9.6] and the references therein. Some of these fast spherical Fourier algorithms are implemented in [KKP]. □

Remark 3.16. In most applications, one usually measures only a noisy version of the data, $g + \delta$, where δ denotes a small noise. However, it does not make much sense to make assumptions on the smoothness of the noise like $\delta \in H^{s+\frac{d-2}{2}}(\mathbb{S}^{d-1})$. This makes the reconstruction of the function f an ill-posed problem.

For the reconstruction of a function f given noisy samples of the Funk–Radon transform $\mathcal{F}f$, it is necessary to apply a regularization method. There is a great variety of regularization methods in the literature, cf. [EHN96]. Louis et al. [LRSS11] have applied the mollifier method for the numerical inversion of the Funk–Radon transform; locally supported mollifiers were discussed in [RS13].

In our article [HQ15], we have combined the mollifier method with the singular value decomposition (3.24) leading to a fast numerical algorithm that is based on the spherical Fourier transform. With this approach, the algorithm from Remark 3.15 is modified such that the summands in (3.26) are multiplied with certain mollification coefficients. In numerical tests, this algorithm shows a reconstruction error that is comparable to the error of the direct implementation of the mollifier method, while it is considerably faster than the direct implementation. □

3.2.3 Applications

The Funk–Radon transform and its inverse have several theoretical and practical applications. Integrals along great circles of the sphere $\mathbb{S}^2$ were first used by Minkowski [Min05]. He utilized the singular value decomposition (3.19) to solve the geometrical problem whether the bodies of constant width and the bodies of constant circumference are the same. We give a brief description of the problem and the solution technique in the following remark.

Remark 3.17. Let $K \subset \mathbb{R}^3$ be a convex body, i. e. a convex, compact set with nonempty interior. We define the support function of K by

$$h_K(\boldsymbol{\xi}) := \sup\{\boldsymbol{\xi}^\top \boldsymbol{x}\ ;\ \boldsymbol{x} \in K\}, \qquad \boldsymbol{\xi} \in \mathbb{S}^2.$$

The width function w_K of the body K in the direction $\boldsymbol{\xi} \in \mathbb{S}^2$ is given by

$$w_K(\boldsymbol{\xi}) := h_K(\boldsymbol{\xi}) + h_K(-\boldsymbol{\xi}).$$

The width $w_K(\boldsymbol{\xi})$ can be imagined as the smallest distance between two parallel planes perpendicular to $\boldsymbol{\xi}$ such that K fits between these planes. The circumference $u_K(\boldsymbol{\xi})$ is defined as the circumference of the intersection of K with the plane $\{\boldsymbol{x} \in \mathbb{R}^3 : \boldsymbol{x}^\top \boldsymbol{\xi} = 0\}$. Minkowski showed the relation with the Funk–Radon transform

$$u_K(\boldsymbol{\xi}) = 2\pi \mathcal{F} h_K(\boldsymbol{\xi}).$$

A body K is of constant width or circumference, if w_K or u_K is constant, respectively. We develop h_K and u_K in spherical Fourier series (2.26). Then w_K is constant if and only if $\hat{h}_K(n) = 0$ for all $n \in \{2, 4, 6, \dots\}$. This is equivalent to $\hat{u}_K(n) = \hat{F}(n)\, \hat{h}_K(n) = 0$ for all $n \geq 1$, which says that u_K is constant. So a convex body is of constant width if and only if it is of constant circumference. □

In the mathematical field of geometric tomography, the Funk–Radon transform is useful for the description of star bodies, see [Gar06, Chapter 4].

There are also practical applications of the Funk–Radon transform in different imaging modalities. One example is the Q–ball imaging [Tuc04], which is a technique in magnetic resonance imaging. The inversion of the Funk–Radon transform is used in synthetic aperture radar (SAR) [YY11] and for the inversion of the conical Radon transform [Ter15], which occurs in Compton imaging.

It is also used for the inversion of another Radon-type transform arising in photoacoustic tomography [HMS16]. The Radon-type transform

$$\mathcal{R}_P f(\boldsymbol{\theta}, t) := \int_{\boldsymbol{\theta}^\top \boldsymbol{\alpha} = 0} \int_{\mathbb{S}^2} f(\boldsymbol{\alpha} + t\boldsymbol{\beta})\, \mathrm{d}\boldsymbol{\beta}\, \mathrm{d}\boldsymbol{\alpha}, \qquad (\boldsymbol{\theta}, t) \in \mathbb{S}^2 \times (0, \infty)$$

first integrates a function on $\mathbb{R}^3$ along the sphere $\boldsymbol{\alpha} + t\, \mathbb{S}^2$ and then takes the integral over the midpoints $\boldsymbol{\alpha}$ of the spheres along the circle $\{\boldsymbol{\alpha} \in \mathbb{S}^2 \,;\, \boldsymbol{\theta}^\top \boldsymbol{\alpha} = 0\}$.

3.3 The spherical section transform

We come to a restriction of the mean operator $\mathcal{M}$, which also serves as a generalization of the Funk–Radon transform $\mathcal{F}$. Let $z \in [-1, 1]$ be arbitrary but fixed. We define the *spherical section transform* $\mathcal{T}_z \colon C(\mathbb{S}^{d-1}) \to C(\mathbb{S}^{d-1})$ by

$$\mathcal{T}_z f(\boldsymbol{\xi}) := \mathcal{M} f(\boldsymbol{\xi}, z), \quad \boldsymbol{\xi} \in \mathbb{S}^{d-1}.$$

The operator $\mathcal{T}_z$ takes the mean values along all $(d-2)$-dimensional subspheres that have fixed radius $\sqrt{1 - z^2}$. Such subspheres are illustrated in Figure 3.3. In the case $z = 0$, we have the Funk–Radon transform $\mathcal{T}_0 = \mathcal{F}$. The operator $\mathcal{T}_z$ was introduced

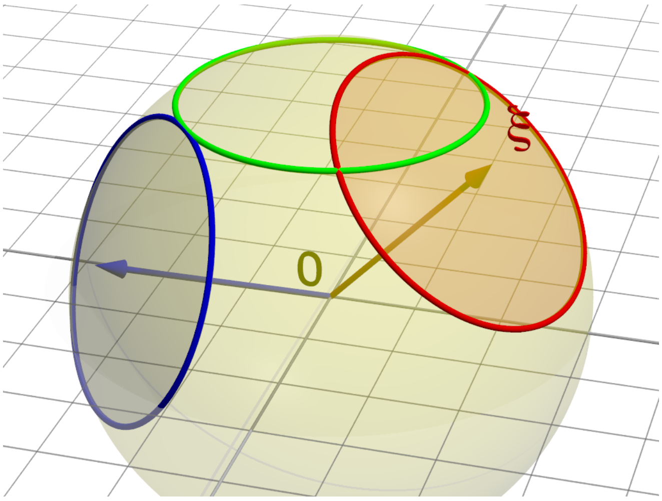

Figure 3.3: This picture shows three circles on the two-sphere $\mathbb{S}^2$ with the same radius. The spherical section transform $\mathcal{T}_z$ computes the mean values along such circles.

by Rudin [Rud50] in 1950 and is known as the *spherical section transform* [Rub00], the *translation operator* [DX13] or the *shift operator* [Rus93].

As a special case of Theorem 3.4 for the mean operator $\mathcal{M}$, we obtain the eigenvalue decomposition of the spherical section transform $\mathcal{T}_z$.

Proposition 3.18. *Let $z \in [-1,1]$, and let $Y_{n,d} \in \mathscr{Y}_{n,d}(\mathbb{S}^{d-1})$ be a spherical harmonic of degree $n \in \mathbb{N}_0$. Then, we have*

$$\mathcal{T}_z Y_{n,d} = P_{n,d}(z)\, Y_{n,d}, \qquad n \in \mathbb{N}_0, \tag{3.27}$$

where the eigenvalues $P_{n,d}(z)$ are the Legendre polynomials (2.13) *evaluated at z.*

As a consequence of the eigenvalue decomposition (3.27), we obtain the following injectivity result of the spherical section transform $\mathcal{T}_z$.

Proposition 3.19. *Let $z \in [-1,1]$. The spherical section transform $\mathcal{T}_z \colon L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1})$ is injective if and only if*

$$P_{n,d}(z) \neq 0 \qquad \text{for all } n \in \mathbb{N}_0.$$

Hence, the operator $\mathcal{T}_z$ is injective for all except countably many $z \in [-1,1]$. In particular, the set of all z for which the spherical section transform $\mathcal{T}_z$ is not injective, has the Lebesgue measure zero, but it is dense everywhere in $[-1,1]$.

The injectivity result of the spherical section transform $\mathcal{T}_z$ in Proposition 3.19 is due to Schneider [Sch69] in 1969. It was inspired by the so-called “Freak Theorem” of Ungar [Ung54] from 1954, who had given a similar injectivity condition for the integrals over the spherical caps $\{\boldsymbol{\xi} \in \mathbb{S}^2\,;\, \xi_3 \geq z\}$ where $z \in [-1,1]$.

The result of Strichartz about Sobolev estimates for the Funk–Radon transform $\mathcal{F}$ in Theorem 3.13 extends to the following bound of the spherical section transform $\mathcal{T}_z$ in Sobolev spaces.

Theorem 3.20. *Let $z \in (-1,1)$ and $s \in \mathbb{R}$. The spherical section transform $\mathcal{T}_z$ is a continuous operator*

$$\mathcal{T}_z \colon H^s(\mathbb{S}^{d-1}) \to H^{s+\frac{d-2}{2}}(\mathbb{S}^{d-1}).$$

Proof. Let $\lambda > 0$ and $n \in \mathbb{N}_0$. We use Darboux's extension of Laplace's formula for the Gegenbauer polynomial [Sze75, (8.21.14)] with $p = 1$, i. e.,

$$C_n^{(\lambda)}(\cos\theta) = 2\binom{n+\lambda-1}{n}\frac{\cos((n+\lambda)\theta - \lambda\pi/2)}{(2\sin\theta)^\lambda} + \mathcal{O}(n^{\lambda-2}).$$

Let $z \in (-1,1)$ be fixed. Hence, we have for $n \to \infty$

$$\left|C_n^{(\lambda)}(z)\right| \in \mathcal{O}\binom{n+\lambda-1}{n}.$$

By the relation (2.19) between the Legendre polynomial $P_{n,d}$ and the Gegenbauer polynomial $C_n^{(\lambda)}$ and inserting $\lambda = \frac{d-2}{2}$, we have for $n \to \infty$

$$|P_{n,d}(z)| \in \mathcal{O}\left(\frac{\binom{n+\frac{d-2}{2}-1}{n}}{\binom{n+d-3}{n}}\right)$$

Inserting the asymptotic expansion (2.76) of the binomial coefficient, we obtain

$$|P_{n,d}(z)| \in \mathcal{O}\left(\frac{n^{\frac{d-2}{2}-1}}{n^{d-3}}\right) = \mathcal{O}\left(n^{\frac{2-d}{2}}\right). \tag{3.28}$$

The assertion follows analogously to the proof of Theorem 3.13. ■

We note that Theorem 3.20 does not hold for the extremal cases $z = \pm 1$. However, we see that $\mathcal{T}_1$ is the identity operator and $\mathcal{T}_{-1}$ is the reflection operator in the origin, both of which are continuous operators $H^s(\mathbb{S}^{d-1}) \to H^s(\mathbb{S}^{d-1})$.

Remark 3.21. Contrary to Theorem 3.13 for the Funk–Radon transform $\mathcal{F}$, there is no corresponding lower bound to (3.28) of the form $|P_{n,d}(z)| \ge c\,n^{\frac{2-d}{2}}$. So, even if we choose a $z \in (-1,1)$ for which the spherical section transform $\mathcal{T}_z$ is bijective, the inverse $\mathcal{T}_z^{-1}$ is not always bounded from $H^{\frac{d-2}{2}}(\mathbb{S}^{d-1})$ to $L^2(\mathbb{S}^{d-1})$, depending on z. Theorem 3.20 is a special case of [Rub00, Theorem 2.9] by Rubin, who also did a more detailed analysis of the injectivity of the spherical section transform, especially on $\mathbb{S}^3$. He showed that $\mathcal{T}_{\cos\theta}$ is injective for $\theta \in \{\frac{\pi}{3}, \frac{\pi}{4}\}$ on $\mathbb{S}^2$. However, it is unknown whether there exists any $\beta \in \mathbb{Q}$ besides the well-known cases $\beta \in \{0, \frac{1}{2}, 1\}$ for which $\mathcal{T}_{\cos(\beta\pi)}$ is not injective. □

Application. The spherical section transform $\mathcal{T}_z$ appears in the study of the conical Radon transform, which is used in Compton camera imaging, see [Moo17]. The conical Radon transform $\mathcal{C}$ maps a function $f\colon \mathbb{R}^3 \to \mathbb{R}$ to its integral along the circular cone with center $\boldsymbol{u} \in \mathbb{R}^3$, direction $\boldsymbol{\xi} \in \mathbb{S}^2$ and opening angle $\psi \in (0,\pi)$, i. e.,

$$\mathcal{C}f(\boldsymbol{u},\boldsymbol{\xi},\psi) := \int_0^\infty \int_{C(\boldsymbol{\xi},\cos\psi)} f(\boldsymbol{u}+r\boldsymbol{\eta})\, r \,\mathrm{d}\boldsymbol{\eta}\,\mathrm{d}r,$$

where the subsphere $C(\boldsymbol{\xi},\cos\psi) \subset \mathbb{S}^2$ is defined in (3.1). When we fix the center $\boldsymbol{u}$, the inner integral is the spherical section transform $\mathcal{T}_{\cos\psi}$ of the function $\boldsymbol{\eta} \mapsto f(\boldsymbol{u}+r\boldsymbol{\eta})\, r$. An overview of the conical Radon transform and its application in Compton camera imaging is provided in [TKK18].

3.4 The generalized Funk–Radon transform $\mathcal{S}^{(j)}$

This section is devoted to the operator $\mathcal{S}^{(j)}$ that generalizes the Funk–Radon transform to derivatives. Most of the material can be found in the first part of our article [QHL18].

Let $j \in \mathbb{N}_0$. We define the *generalized Funk–Radon transform* $\mathcal{S}^{(j)}$ for $f \in C^\infty(\mathbb{S}^{d-1})$ by

$$\mathcal{S}^{(j)}f(\boldsymbol{\xi}) := \int_{\mathbb{S}^{d-1}} \delta^{(j)}(\boldsymbol{\xi}^\top\boldsymbol{\eta})\, f(\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta}, \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1}. \tag{3.29}$$

Here, $\delta^{(j)}$ denotes the j-th derivative of the Dirac delta distribution, which is defined by its application to a test function $\psi \in C^\infty[-1,1]$

$$\int_{-1}^1 \delta^{(j)}(t)\,\psi(t)\,\mathrm{d}t = (-1)^j \int_{-1}^1 \delta(t)\,\psi^{(j)}(t)\,\mathrm{d}t = (-1)^j\psi^{(j)}(0). \tag{3.30}$$

Equation (3.29) can be interpreted as the spherical convolution (2.29) with $\delta^{(j)}$. As for now, we define $\mathcal{S}^{(j)}$ only for smooth functions. Like we did for the Funk–Radon transform $\mathcal{F}$ in Theorem 3.13, we will later extend $\mathcal{S}^{(j)}$ to appropriate Sobolev spaces in Section 3.4.2.

The generalized Funk–Radon transform $\mathcal{S}^{(j)}$ was introduced by Louis [Lou16] in 2016. The study of $\mathcal{S}^{(j)}$ is motivated by the cone-beam tomography, to which it is related via Grangeat's formula, see Section 4.1.

In the following, we explain the above definition of the generalized Funk–Radon transform $\mathcal{S}^{(j)}$ in (3.29) and present equivalent formulas.

Corollary 3.22. Let $f \in C^\infty(\mathbb{S}^{d-1})$. The generalized Funk–Radon transform (3.29) can be written as

$$\mathcal{S}^{(j)}f(\boldsymbol{\xi}) = \int_{\boldsymbol{\xi}^\top\boldsymbol{\eta}=0} \left(-\frac{\partial}{\partial\boldsymbol{\xi}}\right)^j f(\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta}, \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1}, \tag{3.31}$$

where $\frac{\partial}{\partial\boldsymbol{\xi}}$ denotes the directional derivative with respect to $\boldsymbol{\xi}$.

Proof. We state the proof for $\boldsymbol{\xi} = \boldsymbol{\epsilon}^d$ being the north pole of $\mathbb{S}^{d-1}$, the general case follows by rotational symmetry. Using the decomposition $\boldsymbol{\eta} = \sqrt{1-\eta_d^2}\,\boldsymbol{\eta}_{(d-1)} + \eta_d \boldsymbol{\epsilon}^d$ as in (2.8), we have

$$\mathcal{S}^{(j)} f(\boldsymbol{\epsilon}^d) = \int_{\mathbb{S}^{d-1}} \int_{-1}^{1} \delta^{(j)}(\eta_d)\, f\left(\sqrt{1-\eta_d^2}\,\boldsymbol{\eta}_{(d-1)} + \eta_d \boldsymbol{\epsilon}^d\right) \mathrm{d}\eta_d \,\mathrm{d}\boldsymbol{\eta}_{(d-1)}.$$

With the definition of the delta distribution $\delta^{(j)}$ in (3.30), we obtain

$$\mathcal{S}^{(j)} f(\boldsymbol{\epsilon}^d) = \int_{\mathbb{S}^{d-1}} (-1)^j \left(\frac{\partial}{\partial \boldsymbol{\epsilon}^d}\right)^j f(\boldsymbol{\eta}_{(d-1)})\,\mathrm{d}\boldsymbol{\eta}_{(d-1)},$$

which shows the assertion. ■

The last corollary shows that, in the case $j = 0$, the operator $\mathcal{S}^{(0)}$ matches the Funk–Radon transform $\mathcal{F}$, see also [Rip11, Lemma 2.2]. Furthermore, we note that the generalized Funk–Radon transform $\mathcal{S}^{(j)}$ is not a restriction of the mean operator $\mathcal{M}$ considered that $j \geq 1$.

Corollary 3.23. Let $f \in C^\infty(\mathbb{S}^{d-1})$ and $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$. We have

$$\mathcal{S}^{(j)} f(\boldsymbol{\xi}) = (-1)^j \left|\mathbb{S}^{d-2}\right| \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^j (1-t^2)^{\frac{d-3}{2}} \mathcal{M}f(\boldsymbol{\xi}, t)\bigg|_{t=0}. \tag{3.32}$$

Furthermore, let $Y_{n,d}^k \in \mathscr{Y}_{n,d}(\mathbb{S}^{d-1})$ be a spherical harmonic of degree $n \in \mathbb{N}_0$ with $k = 1, \ldots, N_{n,d}$. Then, we have

$$\mathcal{S}^{(j)} Y_{n,d}^k(\boldsymbol{\xi}) = (-1)^j \left|\mathbb{S}^{d-2}\right| Y_{n,d}^k(\boldsymbol{\xi}) \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^j (1-t^2)^{\frac{d-3}{2}} P_{n,d}(t)\bigg|_{t=0}. \tag{3.33}$$

Proof. We apply equation (3.4) to the definition (3.29) of the generalized Funk–Radon transform $\mathcal{S}^{(j)}$ and obtain

$$\mathcal{S}^{(j)} f(\boldsymbol{\xi}) = \left|\mathbb{S}^{d-2}\right| \int_{-1}^{1} \delta^{(j)}(t)\,(1-t^2)^{\frac{d-3}{2}} \mathcal{M}f(\boldsymbol{\xi}, t)\,\mathrm{d}t.$$

Inserting the definition (3.30) of the delta distribution $\delta^{(j)}$ implies (3.32). Inserting the spherical harmonic $f = Y_{n,d}^k$ into (3.32) and considering the singular value decomposition (3.5) of the mean operator $\mathcal{M}$ shows (3.33). ■

We note that equation (3.33) can also be obtained by formally applying the Funk–Hecke formula (2.30) with $\psi = \delta^{(j)}$. However, the Funk–Hecke formula originally requires that $\psi \in L^1(-1,1)$.

3.4.1 Singular value decomposition

Like for the classical Funk–Radon transform $\mathcal{F}$, we obtain an eigenvalue decomposition of the generalized Funk–Radon transform $\mathcal{S}^{(j)}$ in terms of spherical harmonics.

Theorem 3.24. *Let $j \in \mathbb{N}_0$. The generalized Funk–Radon transform $\mathcal{S}^{(j)} \colon C^\infty(\mathbb{S}^{d-1}) \to C^\infty(\mathbb{S}^{d-1})$ satisfies the eigenvalue decomposition*

$$\mathcal{S}^{(j)} Y_{n,d}^k = \hat{\mathcal{S}}^{(j)}(n)\, Y_{n,d}^k, \qquad n \in \mathbb{N}_0,\ k = 1, \dots, N_{n,d},$$

with the eigenvalues for $n+j$ even and ($n \ge j-d+3$ or d even)

$$\hat{\mathcal{S}}^{(j)}(n) := \left|\mathbb{S}^{d-2}\right| (-1)^{\frac{n+j}{2}} \frac{(n+j-1)!!\,(d-3)!!}{(n-j+d-3)!!} \tag{3.34}$$

$$= \pi^{\frac{d-2}{2}} (-1)^{\frac{n+j}{2}}\, 2^{j+1} \frac{\Gamma\left(\frac{n+j+1}{2}\right)}{\Gamma\left(\frac{n-j+d-1}{2}\right)} \tag{3.35}$$

and otherwise

$$\hat{\mathcal{S}}^{(j)}(n) := 0.$$

Proof. Let $n \in \mathbb{N}_0$ and $k \in \{1, \dots, N_{n,d}\}$. By (3.33), we have

$$\mathcal{S}^{(j)} Y_{n,d}(\boldsymbol{\xi}) = \left|\mathbb{S}^{d-2}\right| (-1)^j\, Y_{n,d}^k(\boldsymbol{\xi}) \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^j P_{n,d}(t)\,(1-t^2)^{\frac{d-3}{2}} \bigg|_{t=0}. \tag{3.36}$$

By Rodrigues' formula (2.13), we have

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^j P_{n,d}(t)\,(1-t^2)^{\frac{d-3}{2}} \bigg|_{t=0} = (-1)^n \frac{(d-3)!!}{(2n+d-3)!!} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n+j} (1-t^2)^{n+\frac{d-3}{2}} \bigg|_{t=0}. \tag{3.37}$$

We apply the generalized binomial theorem, which states for $a, b, z \in \mathbb{C}$

$$(a+b)^z = \sum_{k=0}^{\infty} \binom{z}{k} a^{z-k}\, b^k, \tag{3.38}$$

where the binomial coefficient with complex argument is defined in (2.69). Let $t \in \mathbb{R}$. The generalized binomial theorem (3.38) implies

$$(1-t^2)^{n+\frac{d-3}{2}} = \sum_{k=0}^{\infty} \binom{n+\frac{d-3}{2}}{k} (-1)^k\, t^{2k}. \tag{3.39}$$

Evaluating the $(n+j)$-th derivative of (3.39) at $t=0$ and taking into account that $\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^\ell t^{2k} \Big|_{t=0} = (2k)!\,\delta_{\ell,2k}$, we obtain if $n+j$ is even

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n+j} (1-t^2)^{n+\frac{d-3}{2}} \bigg|_{t=0} &= \sum_{k=0}^{\infty} \binom{n+\frac{d-3}{2}}{k} (-1)^k \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n+j} t^{2k} \bigg|_{t=0} \\ &= \binom{n+\frac{d-3}{2}}{\frac{n+j}{2}} (-1)^{\frac{n+j}{2}}\,(n+j)!\end{aligned} \tag{3.40}$$

and zero otherwise. By its definition in (2.69), the binomial coefficient $\binom{z}{k}$ is zero if and only if both z is a nonnegative integer and $z < k$. Hence, the binomial coefficient $\binom{n+\frac{d-3}{2}}{\frac{n+j}{2}}$ from (3.40) is nonzero if and only if $\frac{d-3}{2}$ is not an integer or $n + \frac{d-3}{2} \geq \frac{n+j}{2}$. This condition can be simplified to that d is even or $n \geq j - d + 3$. The binomial coefficient $\binom{n+\frac{d-3}{2}}{\frac{n+j}{2}}$ is nonzero if and only if $\frac{d-3}{2}$ is not an integer or $n + \frac{d-3}{2} \geq \frac{n+j}{2}$, which is equivalent to d even or $n \geq j - d + 3$. Then we have

$$\begin{aligned} \binom{n+\frac{d-3}{2}}{\frac{n+j}{2}} &= \frac{\left(\frac{2n+d-3}{2}\right)\left(\frac{2n+d-3}{2}-1\right)\cdots\left(\frac{2n+d-3}{2}-\frac{n+j}{2}+1\right)}{\left(\frac{n+j}{2}\right)!} \\ &= \frac{(2n+d-3)!!}{(n-j+d-3)!!\,(n+j)!!}. \end{aligned} \tag{3.41}$$

Combining (3.37), (3.40) and (3.41), we obtain

$$\begin{aligned} &\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^j P_{n,d}(t)\,(1-t^2)^{\frac{d-3}{2}}\Bigg|_{t=0} \\ &= (-1)^n \frac{(d-3)!!}{(2n+d-3)!!} \frac{(2n+d-3)!!}{(n+j)!!(n-j+d-3)!!} (-1)^{\frac{n+j}{2}} (n+j)! \\ &= (-1)^{\frac{n-j}{2}} \frac{(n+j-1)!!\,(d-3)!!}{(n-j+d-3)!!} \end{aligned}$$

if $n+j$ is even and ($n \geq j-d+3$ or d even), and zero otherwise. Plugging into (3.36) shows (3.34). Inserting the volume (2.10) of $\mathbb{S}^{d-2}$ into (3.34), we have for $n+j$ even and ($n \geq j-d+3$ or d even)

$$\begin{aligned} \hat{\mathcal{S}}^{(j)}(n) &= \left|\mathbb{S}^{d-2}\right| (-1)^{\frac{n-j}{2}} \frac{(n+j-1)!!\,(d-3)!!}{(n-j+d-3)!!} \\ &= \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)} (-1)^{\frac{n-j}{2}} \frac{(n+j-1)(n+j-3)\cdots 1}{(n-j+d-3)(n-j+d-5)\cdots(d-1)} \\ &= \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)} (-1)^{\frac{n-j}{2}} \frac{2^{\frac{n+j}{2}}\left(\frac{n+j-1}{2}\right)\left(\frac{n+j-1}{2}-1\right)\cdots\left(\frac{1}{2}\right)}{2^{\frac{n-j}{2}}\left(\frac{n-j+d-1}{2}-1\right)\left(\frac{n-j+d-1}{2}-2\right)\cdots\left(\frac{d-1}{2}\right)}. \end{aligned}$$

With the functional equation (2.67) of the Gamma function, we obtain

$$\begin{aligned} \hat{\mathcal{S}}^{(j)}(n) &= \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)} (-1)^{\frac{n-j}{2}} 2^j \frac{\Gamma\left(\frac{n+j+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{n-j+d-1}{2}\right)} \\ &= \pi^{\frac{d-2}{2}} (-1)^{\frac{n-j}{2}} 2^{j+1} \frac{\Gamma\left(\frac{n+j+1}{2}\right)}{\Gamma\left(\frac{n-j+d-1}{2}\right)}, \end{aligned}$$

which shows (3.35). ■

Theorem 3.24 contains for $j = 0$ the eigenvalue decomposition (3.19) of the Funk–Radon transform $\mathcal{F}$.

3.4.2 $\mathcal{S}^{(j)}$ in Sobolev spaces

In this section, we extend the generalized Funk–Radon transform $\mathcal{S}^{(j)}$ to a continuous operator between Sobolev spaces. As a first step, we derive an asymptotic approximation of the eigenvalues $\hat{\mathcal{S}}^{(j)}(n)$ from Theorem 3.24 as n goes to infinity.

Lemma 3.25. *Let $j \in \mathbb{N}_0$. We have for $n \to \infty$ with $n+j$ even and $n \geq j$*

$$\left|\hat{\mathcal{S}}^{(j)}(n)\right| \simeq n^{j-\frac{d-2}{2}}\, \pi^{\frac{d-1}{2}}\, 2^{\frac{d}{2}}.$$

Proof. Let $n+j$ be even and $n \geq j$. We have by (3.35)

$$\left|\hat{\mathcal{S}}^{(j)}(n)\right| = \pi^{\frac{d-2}{2}}\, 2^{j+1}\, \frac{\Gamma\left(\frac{n+j+1}{2}\right)}{\Gamma\left(\frac{n-j+d-1}{2}\right)}.$$

We apply Stirling's approximation (2.72) of the Gamma function to the eigenvalues (3.35) and obtain for $n \to \infty$

$$\begin{aligned}
\left|\hat{\mathcal{S}}^{(j)}(n)\right| &\simeq \pi^{\frac{d-2}{2}}\, 2^{j+1}\, \frac{\left(\frac{n+j+1}{2}\right)^{\frac{n+j}{2}}\, \mathrm{e}^{-\frac{n+j+1}{2}}}{\left(\frac{n-j+d-1}{2}\right)^{\frac{n-j+d-2}{2}}\, \mathrm{e}^{-\frac{n-j+d-1}{2}}} \\
&= \pi^{\frac{d-2}{2}}\, 2^{\frac{d}{2}}\, \mathrm{e}^{\frac{d-2}{2}-j}\, \frac{(n+j+1)^{\frac{n+j}{2}}}{(n-j+d-1)^{\frac{n-j+d-2}{2}}} \\
&= 2^{\frac{d}{2}}\, \pi^{\frac{d-2}{2}}\, \mathrm{e}^{\frac{d-2}{2}-j} \left(1 + \frac{2j-d+2}{n-j+d-1}\right)^{\frac{n}{2}} (n+j+1)^{\frac{j}{2}}\, (n-j+d-1)^{\frac{j+2-d}{2}}.
\end{aligned}$$

Considering that $\mathrm{e}^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$, we obtain

$$\left|\hat{\mathcal{S}}^{(j)}(n)\right| \simeq 2^{\frac{d}{2}}\, \pi^{\frac{d-2}{2}}\, \mathrm{e}^{\frac{d-2}{2}-j}\, \mathrm{e}^{\frac{2j-d+2}{2}}\, n^{j+\frac{2-d}{2}} = 2^{\frac{d}{2}}\, \pi^{\frac{d-2}{2}}\, n^{j-\frac{d-2}{2}}. \qquad \blacksquare$$

Now, we are able to show the generalization of the Sobolev estimates from Theorem 3.13 as well as a complete characterization of the nullspace or kernel for the generalized Funk–Radon transform $\mathcal{S}^{(j)}$.

Theorem 3.26. *Let $s \in \mathbb{R}$ and $j \in \mathbb{N}_0$. The generalized Funk–Radon transform $\mathcal{S}^{(j)}$ extends to a continuous operator*

$$\mathcal{S}^{(j)} \colon H^s(\mathbb{S}^{d-1}) \to H^{s-j+\frac{d-2}{2}}(\mathbb{S}^{d-1}). \tag{3.42}$$

If $j < \frac{d-2}{2}$, then $\mathcal{S}^{(j)} \colon L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1})$ is compact. The nullspace of $\mathcal{S}^{(j)}$ is the closed linear span

$$\overline{\operatorname{span}}\left\{\mathscr{Y}_{n,d}(\mathbb{S}^{d-1})\ ;\ n+j \text{ odd or } (n \leq j-d+1 \text{ and } d \text{ odd})\right\}. \tag{3.43}$$

If d is odd and $j \geq d-1$, the nullspace of $\mathcal{S}^{(j)}$ comprises the sum of all polynomials of degree up to $j-d+1$ and all odd (even) functions whenever j is even (odd). Otherwise, the null-space of $\mathcal{S}^{(j)}$ comprises all odd (even) functions whenever j is even (odd).

Proof. It follows from Lemma 3.25 that the sequence

$$\mathbb{N}_0 \ni n \mapsto \left|\hat{\mathcal{S}}^{(j)}(n)\right| \left(n + \frac{d-2}{2}\right)^{-j+\frac{d-2}{2}}$$

has an upper bound that is independent of n. Hence, the continuity of (3.42) follows from definition (2.46) of the Sobolev space analogously to the proof of Theorem 3.13. The compactness of $\mathcal{S}^{(j)} \colon L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1})$ for $j < \frac{d-2}{2}$ follows because then the eigenvalues $\hat{\mathcal{S}}^{(j)}(n)$ converge to 0 for $n \to \infty$. The nullspace of $\mathcal{S}^{(j)}$ consists of the closed linear span of all spherical harmonics $Y_{n,d}^k$ where $n \in \mathbb{N}_0$ satisfies $\hat{\mathcal{S}}^{(j)}(n) = 0$, i. e.,

$$\overline{\operatorname{span}}\{Y_{n,d} \in \mathscr{Y}_{n,d}(\mathbb{S}^{d-1}) \; ; n \in \mathbb{N}_0, \, \hat{\mathcal{S}}^{(j)}(n) = 0\},$$

which is equal to (3.43). ∎

The order of smoothness $s - j + \frac{d-2}{2}$ of the Sobolev space in (3.42) is not unexpected, because $\mathcal{S}^{(j)}$ consists of j differentiations, which lower the order of smoothness by j, and the integration along a $(d-2)$-dimensional submanifold, which raises the order of smoothness by $\frac{d-2}{2}$.

3.4.3 Special cases of j

In this section, we take a look at the generalized Funk–Radon transform $\mathcal{S}^{(j)}$ for certain special choices of j, some of which are already well-known operators from literature. Even though $\mathcal{S}^{(j)}$ was initially defined in (3.29) only for $j \in \mathbb{N}_0$, we can extend it to negative integers j by the singular value decomposition (3.34).

Example 3.27. Inserting $j = -1$ in the equation (3.35) of the eigenvalues of $\mathcal{S}^{(-1)}$ yields for odd $n \in \mathbb{N}_0$

$$\hat{\mathcal{S}}^{(-1)}(n) = \left|\mathbb{S}^{d-2}\right| (-1)^{\frac{n-1}{2}} \frac{(n-2)!!\,(d-3)!!}{(n+d-2)!!} = 2\,(-1)^{\frac{n-1}{2}} \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{1}{2}\right)} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+d}{2}\right)}. \tag{3.44}$$

and $\hat{\mathcal{S}}^{(-1)}(n) = 0$ for even $n \in \mathbb{N}_0$. As shown in [Rub99], the values (3.44) are the eigenvalues of the modified *hemispherical transform*

$$\mathcal{S}^{(-1)} f(\boldsymbol{\xi}) = \frac{1}{2} \int_{\mathbb{S}^{d-1}} \operatorname{sgn}(\boldsymbol{\xi}^\top \boldsymbol{\eta})\, f(\boldsymbol{\eta})\, \mathrm{d}\boldsymbol{\eta}, \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1}, \tag{3.45}$$

where sgn denotes the sign function, see . The modified hemispherical transform $\mathcal{S}^{(-1)}$ is a continuous and bijective operator

$$\mathcal{S}^{(-1)} \colon L^2_{\mathrm{odd}}(\mathbb{S}^{d-1}) \to H^{\frac{d}{2}}_{\mathrm{odd}}(\mathbb{S}^{d-1}),$$

where $L^2_{\mathrm{odd}}(\mathbb{S}^{d-1})$ denotes the subspace of the Hilbert space $L^2(\mathbb{S}^{d-1})$ that contains only odd functions $f(\boldsymbol{\xi}) = -f(-\boldsymbol{\xi})$. Furthermore, we note that the sign function $\frac{1}{2}\operatorname{sgn}(t)$ can be interpreted as the antiderivative of the delta distribution $\delta(t)$.

Originally, the hemispherical transform was defined in a slightly different manner by Funk [Fun15a] (see also [Cam84]), namely as

$$\mathcal{H}f(\boldsymbol{\xi}) = \int_{\boldsymbol{\xi}^\top\boldsymbol{\eta}\geq 0} f(\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta}, \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1},$$

which computes the integrals of the function f on hemispheres $\{\boldsymbol{\eta} \in \mathbb{S}^{d-1} \,;\, \boldsymbol{\xi}^\top\boldsymbol{\eta} \geq 0\}$. The classical hemispherical transform $\mathcal{H}$ is the spherical convolution (2.29) with the characteristic function $\psi(t) = \mathbf{1}(t \geq 0)$ instead of $\frac{1}{2}\operatorname{sgn}(t)$ for $\mathcal{S}^{(-1)}$. It has applications for discrete choice models in statistics, see [GK13]. The two hemispherical transforms $\mathcal{H}$ and $\mathcal{S}^{(-1)}$ differ only for the constant part of a function. In particular, we have for a continuous function $f\colon \mathbb{S}^{d-1} \to \mathbb{C}$

$$\mathcal{H}f(\boldsymbol{\xi}) = \mathcal{S}^{(-1)}f(\boldsymbol{\xi}) + \frac{1}{2}\int_{\mathbb{S}^{d-1}} f(\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta}, \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1}. \qquad \square$$

Example 3.28. Inserting $j = -2$ in the equation (3.35) gives the eigenvalues

$$\hat{\mathcal{S}}^{(-2)}(n) = \begin{cases} 2\left|\mathbb{S}^{d-2}\right|(-1)^{\frac{n-2}{2}} \frac{(n-3)!!\,(d-2)!!}{(n+d-1)!!}, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

of the *spherical cosine transform*

$$\mathcal{S}^{(-2)}f(\boldsymbol{\xi}) = \frac{1}{2}\int_{\mathbb{S}^{d-1}} \left|\boldsymbol{\xi}^\top\boldsymbol{\eta}\right| f(\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta},$$

cf. [Gro96, Lemma 3.4.5]. The spherical cosine transform $\mathcal{S}^{(-2)}$ is the spherical convolution (2.29) with the absolute value function $\psi(t) = \frac{1}{2}|t|$ for $t \in [-1, 1]$. The second (distributional) derivative of $\frac{1}{2}|t|$ is the delta distribution $\delta(t)$. By Theorem 3.26, the spherical cosine transform is a continuous and bijective operator

$$\mathcal{S}^{(-2)}\colon L^2_{\mathrm{even}}(\mathbb{S}^{d-1}) \to H^{\frac{d+2}{2}}_{\mathrm{even}}(\mathbb{S}^{d-1}),$$

see also [Pet61]. The inversion of the spherical cosine transform is important for the analysis of spatial fiber systems in biology or metallography [KP05] and was subject of the papers [Rub02, LRSS11, RS13].

The name "spherical cosine transform" arises from the fact that for two points $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{S}^{d-1}$ and $t = \boldsymbol{\xi}^\top\boldsymbol{\eta}$, we have $\cos(d(\boldsymbol{\xi}, \boldsymbol{\eta})) = t$, where $d(\boldsymbol{\xi}, \boldsymbol{\eta})$ denotes the geodesic distance between $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ on the sphere $\mathbb{S}^{d-1}$.

Replacing the cosine by the sine, we see that $\sin(d(\boldsymbol{\xi}, \boldsymbol{\eta})) = \sqrt{1-t^2}$. This fact motivates the definition of the *spherical sine transform* as the spherical convolution (2.29) with the function $\psi(t) = \sqrt{1-t^2}$. The spherical sine transform is a continuous operator in the spaces $L^2_{\mathrm{even}}(\mathbb{S}^{d-1}) \to H^d_{\mathrm{even}}(\mathbb{S}^{d-1})$, see [HS02]. $\square$

Example 3.29. Let $d \ge 2$ be even. In the case $j = \frac{d-2}{2}$, the generalized Funk–Radon transform $\mathcal{S}^{(\frac{d-2}{2})}$ has the eigenvalues

$$\hat{\mathcal{S}}^{(\frac{d-2}{2})}(n) = \left|\mathbb{S}^{d-2}\right| (-1)^{\frac{2n+d-2}{4}} (d-3)!! = 2\,(2\pi)^{\frac{d-2}{2}} (-1)^{\frac{2n+d-2}{4}}, \qquad n+j \text{ even}$$

and

$$\hat{\mathcal{S}}^{(\frac{d-2}{2})}(n) = 0, \qquad n+j \text{ odd}.$$

The eigenvalues $\hat{\mathcal{S}}^{(\frac{d-2}{2})}(n)$ are, except for their sign, independent of the degree n. Hence, if $\frac{d-2}{2}$ is even (odd), the operator $\mathcal{S}^{(\frac{d-2}{2})} \colon L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1})$ restricted to the even (odd) functions is a constant multiple of an isometry. □

The following theorem shows an inversion formula for the Funk–Radon transform in even dimensions d.

Theorem 3.30. *Let $d \ge 2$ be even. Then any even function $f \colon \mathbb{S}^{d-1} \to \mathbb{C}$ can be reconstructed from its Funk–Radon transform $g = \mathcal{S}^{(0)} f = \mathcal{F} f$ by*

$$f = \frac{1}{\left|\mathbb{S}^{d-2}\right|^2 \left((d-3)!!\right)^2} \mathcal{S}^{(d-2)} g. \tag{3.46}$$

Proof. Let $n \in \mathbb{N}_0$ be even. On the one hand, we have the eigenvalues

$$\hat{\mathcal{S}}^{(d-2)}(n) = \left|\mathbb{S}^{d-2}\right| (-1)^{\frac{n+d-2}{2}} \frac{(n+d-3)!!\,(d-3)!!}{(n-1)!!}.$$

On the other hand, the Funk–Radon transform $\mathcal{F} = \mathcal{S}^{(0)}$ has the eigenvalues

$$\hat{\mathcal{S}}^{(0)}(n) = \left|\mathbb{S}^{d-2}\right| (-1)^{\frac{n}{2}} \frac{(n-1)!!\,(d-3)!!}{(n+d-3)!!}.$$

Hence, the product of the two operators has the constant eigenvalues

$$\widehat{\mathcal{S}^{(d-2)}\mathcal{S}^{(0)}}(n) = \left|\mathbb{S}^{d-2}\right|^2 (-1)^{\frac{d-2}{2}} (d-3)!!^2. \qquad \blacksquare$$

We have already seen different inversion formulas of the Funk–Radon transform $\mathcal{F}$ in Section 3.2.2. The inversion formula (3.46) contains $d-2$ derivatives and the integration of $\mathcal{F} f$ along a $(d-2)$-dimensional subsphere, which are exactly the same numbers as in Helgason's inversion formula (3.23).

3.4.4 Similar transforms

In this section, we consider two integral transforms, which are equal to the generalized Funk–Radon transform $\mathcal{S}^{(j)}$ for certain but not all parameters j.

An integro-differential transform. Let $j \in \mathbb{N}_0$ and $\vartheta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. We define the integro-differential transform $\mathcal{R}_\vartheta^{(j)} \colon C^j(\mathbb{S}^{d-1}) \to C(\mathbb{S}^{d-1})$ by

$$\mathcal{R}_\vartheta^{(j)} f(\boldsymbol{\xi}) := \int_{\boldsymbol{\xi}^\top \boldsymbol{\omega}=0} \left(\frac{\partial}{\partial \vartheta}\right)^j f(\boldsymbol{\xi} \sin\vartheta + \boldsymbol{\omega}\cos\vartheta)\,\mathrm{d}\boldsymbol{\omega}, \qquad \boldsymbol{\xi} \in \mathbb{S}^d,$$

which was introduced in [MMÓ00]. The operator $\mathcal{R}_\vartheta^{(0)}$ has been investigated in [Sch69]. For $j > 0$, we first take the j-th derivative of f perpendicular to the circle of integration. It was shown in [MMÓ01] that the nullspace of the operator $\mathcal{R}_\vartheta^{(j)}$ is

$$\overline{\operatorname{span}} \left\{ \mathscr{Y}_{n,d}(\mathbb{S}^{d-1}) \;;\; \left(\frac{\mathrm{d}}{\mathrm{d}\vartheta}\right)^j P_{n,d}(\sin\vartheta) = 0 \right\}.$$

If $\vartheta = 0$, we write $\mathcal{R}^{(j)} = \mathcal{R}_0^{(j)}$. If $j \ge 1$, the null–space of $\mathcal{R}^{(j)} \colon C^j(\mathbb{S}^{d-1}) \to C(\mathbb{S}^{d-1})$ equals for j odd (even) the set $\{f \in C^m(\mathbb{S}^d) \;;\; f$ is even (f is the sum of an odd function and a constant)$\}$.

Theorem 3.31. *We have $\mathcal{S}^{(0)} = \mathcal{R}^{(0)}$ and $\mathcal{S}^{(1)} = -\mathcal{R}^{(1)}$.*

Proof. For $j = 0$, we see that $\mathcal{S}^{(0)} = \mathcal{R}^{(0)}$ is the Funk–Radon transform. By [MMÓ01], the operator $\mathcal{R}^{(j)}$ has as eigenfunctions the spherical harmonics $Y_{n,d}^k$ and the eigenvalues

$$\hat{\mathcal{R}}^{(j)}(n) = \left|\mathbb{S}^{d-2}\right| \frac{\mathrm{d}}{\mathrm{d}\vartheta} P_{n,d}(\sin\vartheta)\Big|_{\vartheta=0}.$$

Theorem 3.24 shows that $\mathcal{S}^{(j)}$ has the same eigenfunctions. Hence, the two operators coincide if their respective eigenvalues do. We have for $j = 1$ on the one hand

$$\hat{\mathcal{R}}^{(1)}(n) = \left|\mathbb{S}^{d-2}\right| P_{n,d}'(\sin\vartheta)\,\cos\vartheta\big|_{\vartheta=0} = \left|\mathbb{S}^{d-2}\right| P_{n,d}'(0)$$

and on the other hand

$$\hat{\mathcal{S}}^{(1)}(n) = -\left|\mathbb{S}^{d-2}\right| \frac{\mathrm{d}}{\mathrm{d}t} P_{n,d}(t)\,(1-t^2)^{\frac{d-3}{2}}\Big|_{t=0} = -\left|\mathbb{S}^{d-2}\right| P_{n,d}'(0). \qquad \blacksquare$$

Remark 3.32. Theorem 3.31 does not hold for all $j \in \mathbb{N}_0$. If $d = 3$ and $j = 2$, we have for $n \in \mathbb{N}_0$

$$\hat{\mathcal{R}}^{(2)}(n) = 2\pi\big(P_n''(\sin\vartheta)\,\cos^2\vartheta - P_n'(\sin\vartheta)\,\sin\vartheta\big)\Big|_{\vartheta=0} = 2\pi\,P_n''(0) = \hat{\mathcal{S}}^{(2)}(n).$$

However, for $d = 3$ and $j = 3$

$$\begin{aligned}\hat{\mathcal{R}}^{(3)}(n) &= 2\pi\big(P_n'''(\sin\vartheta)\,\cos^3\vartheta - 3P_n''(\sin\vartheta)\,\cos\vartheta\,\sin\vartheta - P_n'(\sin\vartheta)\,\cos\vartheta\big)\Big|_{\vartheta=0} \\ &= 2\pi\,(P_n'''(0) - P_n'(0))\end{aligned}$$

does not coincide with $-\hat{\mathcal{S}}^{(3)}(n) = 2\pi\,P_n'''(0)$. $\square$

The alpha-cosine transform. A recent topic of research is the alpha-cosine transform or Blaschke–Levy representation [Kol97, Rub99] for $\alpha \in \mathbb{Z}$,

$$\mathcal{H}^{(\alpha)} f(\boldsymbol{\xi}) = \int_{\mathbb{S}^{d-1}} \left|\boldsymbol{\xi}^\top \boldsymbol{\eta}\right|^\alpha f(\boldsymbol{\eta})\, \mathrm{d}\boldsymbol{\eta} \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1}, \tag{3.47}$$

with the singular values

$$\hat{\mathcal{H}}^{(\alpha)}(n) = \begin{cases} (-1)^{n/2} \dfrac{\Gamma\left(\frac{n-\alpha}{2}\right)}{\Gamma\left(\frac{n+d+\alpha}{2}\right)}, & n \text{ even} \\ 0, & n \text{ odd.} \end{cases}$$

Hence, for j even and $\alpha = -j-1$, the α-cosine transform $\mathcal{H}^{(-j-1)}$ is, up to a constant factor, equal to the generalized Funk–Radon transform $\mathcal{S}^{(j)}$.

Furthermore, the alpha-cosine transform with odd kernel is given by

$$\mathcal{H}^{(\alpha)}_{\mathrm{odd}} f(\boldsymbol{\xi}) = \int_{\mathbb{S}^{d-1}} \left|\boldsymbol{\xi}^\top \boldsymbol{\eta}\right|^\alpha \operatorname{sgn}(\boldsymbol{\xi}^\top \boldsymbol{\eta})\, f(\boldsymbol{\eta})\, \mathrm{d}\boldsymbol{\eta}, \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1},$$

which differs from (3.47) only in the factor $\operatorname{sgn}(\boldsymbol{\xi}^\top \boldsymbol{\eta})$. For j odd and $\alpha = -j-1$, it was shown in [Kaz18] that $\mathcal{H}^{(-j-1)}_{\mathrm{odd}}$ is, up to a constant factor, equal to the generalized Funk–Radon transform $\mathcal{S}^{(j)}$.

3.5 Sections with a fixed set of centers

In this section, we come to an important class of restrictions of the mean operator $\mathcal{M}$. We take a look at the manifold that consists of all subspheres of $\mathbb{S}^{d-1}$ whose poles are located on some set $S \subset \mathbb{S}^{d-1}$. More precisely, we consider the restriction of the mean operator $\mathcal{M}$ to manifolds of the form

$$T_S := S \times [-1,1] = \{(\boldsymbol{\xi}, t) \,;\, \boldsymbol{\xi} \in S, t \in [-1,1]\}.$$

In other words, we take the sections of the sphere with hyperplanes whose normal vectors are contained in the set S.

We are going to start with general sets $S \subset \mathbb{S}^{d-1}$ and come to the cases where S is a subsphere or, more specifically, the equator of $\mathbb{S}^{d-1}$ in the Sections 3.5.2 and 3.5.3, respectively.

3.5.1 Centers in arbitrary curves

We call a set $S \subset \mathbb{S}^{d-1}$ an injectivity set of the mean operator if the restriction $\mathcal{M}|_{T_S}$ is injective. Quinto and Zalcman noted the following result characterizing the injectivity sets of the mean operator restricted to T_S in the 1980s. However, it remained unpublished for some time and was published in [AQ96, Theorem 7.1] with only a sketch of the proof in 1996. A full proof was later given in [AVZ99]. In the following, we present a simplified proof that makes use of the zonal harmonics $Z_{n,d}$ given in (2.24).

Proposition 3.33. *Let $S \subset \mathbb{S}^{d-1}$. The spherical mean operator $\mathcal{M}|_{T_S}$ restricted to the manifold T_S is injective if and only if the set S is not a subset of the zero set of any nontrivial spherical harmonic $Y_{n,d} \in \mathscr{Y}_{n,d}(\mathbb{S}^{d-1})$ for any $n \in \mathbb{N}$.*

Proof. We first show the sufficiency. Let $Y_{n,d} \in \mathscr{Y}_{n,d}(\mathbb{S}^{d-1})$ be a spherical harmonic of degree $n \in \mathbb{N}$ with $Y_{n,d}(\boldsymbol{\xi}) = 0$ for all $\boldsymbol{\xi} \in S$. Then, we have by Theorem 3.4 for all $(\boldsymbol{\xi}, t) \in T_S$

$$\mathcal{M}Y_{n,d}(\boldsymbol{\xi}, t) = P_{n,d}(t)\, Y_{n,d}(\boldsymbol{\xi}) = 0.$$

For the necessity, we assume that the operator $\mathcal{M}|_{T_S}$ is not injective, i. e., there exists a nontrivial function $f \neq 0$ with $\mathcal{M}f(\boldsymbol{\xi}, t) = 0$ for all $\boldsymbol{\xi} \in S$. Let $Z_{n,d}$ denote the zonal harmonic of degree n as defined in (2.24). Since f is not the zero function, there exists an $n \in \mathbb{N}_0$ such that $Z_{n,d} \star f$ is a nontrivial spherical harmonic of degree n, where the spherical convolution $\star$ was defined in (2.29). On the other hand, we have by (3.4) for any $\boldsymbol{\xi} \in S$

$$\begin{aligned} Z_{n,d} \star f(\boldsymbol{\xi}) &= \int_{\mathbb{S}^{d-1}} f(\boldsymbol{\eta})\, Z_{n,d}(\boldsymbol{\xi}^\top \boldsymbol{\eta})\, \mathrm{d}\boldsymbol{\eta} \\ &= \frac{|\mathbb{S}^{d-2}|}{|\mathbb{S}^{d-1}|} \int_{-1}^{1} (1-t^2)^{\frac{d-3}{2}}\, Z_{n,d}(t)\, \mathcal{M}f(\boldsymbol{\xi}, t)\, \mathrm{d}t = 0, \end{aligned}$$

since $\mathcal{M}f(\boldsymbol{\xi}, t)$ vanishes for all $\boldsymbol{\xi} \in S$ and $t \in [-1, 1]$. Hence, S is a subset of the zero set of the nontrivial spherical harmonic $Z_{n,d} \star f \in \mathscr{Y}_{n,d}(\mathbb{S}^{d-1})$. ■

One should note that, in the last proposition, we consider the zero sets of all spherical harmonics $Y_{n,d} \in \mathscr{Y}_{n,d}(\mathbb{S}^{d-1})$, see (2.11), not only those from a specific basis $Y_{n,d}^k$.

Remark 3.34. In the case of the one-dimensional sphere $\mathbb{S}^1$, the zero sets of spherical harmonics have a rather simple form. Any spherical harmonic $Y_{n,2} \in \mathscr{Y}_{n,2}(\mathbb{S}^1)$ of degree $n \in \mathbb{N}$ can be written as the linear combination $Y_{n,2}(\cos\varphi, \sin\varphi) = a\mathrm{e}^{\mathrm{i}n\varphi} + b\mathrm{e}^{-\mathrm{i}n\varphi}$, where $\varphi \in [0, 2\pi)$, with some coefficients $a, b \in \mathbb{C}$, see (2.31). From $Y_{n,2} = 0$, we obtain that $\mathrm{e}^{2\mathrm{i}n\varphi} = -\frac{b}{a}$. We choose $\gamma \in \mathbb{R}$ such that $\mathrm{e}^{2\mathrm{i}n\gamma} = -\frac{b}{a}$. Then we have $\mathrm{e}^{2\mathrm{i}n\varphi} = \mathrm{e}^{2\mathrm{i}n\gamma}$ and hence we see that $\varphi = \gamma + \frac{k}{n}\pi$ for some $k \in \mathbb{Z}$. Consequently, the zero set of a spherical harmonic $Y_{n,2}$ is the set $\{(\cos\varphi, \sin\varphi) \in \mathbb{S}^1\ ;\ \varphi = \frac{k}{n}\pi + \gamma,\ k \in \mathbb{Z}\}$ for some $\gamma \in \mathbb{R}$. Geometrically, the zero set of a spherical harmonic $Y_{n,2}$ is characterized as a set of $2n$ equidistant points on the circle $\mathbb{S}^1$. □

Complete systems of radial functions

A closely related question to the injectivity sets of the mean operator is the completeness of systems of radial functions. For a function $f\colon [-1, 1] \to \mathbb{C}$ on the unit interval and a center point $\boldsymbol{a} \in \mathbb{S}^{d-1}$, we define the corresponding radial function $f_{\boldsymbol{a}}\colon \mathbb{S}^{d-1} \to \mathbb{C}$ on the sphere by

$$f_{\boldsymbol{a}}(\boldsymbol{\xi}) := f(\boldsymbol{a}^\top \boldsymbol{\xi}), \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1}.$$

We note that $f \in L^2([-1,1]; w_d)$ if and only if $f_{\boldsymbol{a}} \in L^2(\mathbb{S}^{d-1})$. This can be seen with the help of (3.4), which implies that

$$\int_{\mathbb{S}^{d-1}} f_{\boldsymbol{a}}(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} = \left|\mathbb{S}^{d-2}\right| \int_{-1}^{1} \mathcal{M} f_{\boldsymbol{a}}(\boldsymbol{a}, t)\,(1-t^2)^{\frac{d-3}{2}}\,\mathrm{d}t = \left|\mathbb{S}^{d-2}\right| \int_{-1}^{1} f(t)\,(1-t^2)^{\frac{d-3}{2}}\,\mathrm{d}t.$$

In the following theorem, we show that the problem of completeness of systems of radial functions is the same as the injectivity of restrictions of the mean operator. This is the analogue of a result from [AQ96] about radial functions on the Euclidean space $\mathbb{R}^d$.

Theorem 3.35. *Let $S \subset \mathbb{S}^{d-1}$. We denote by*

$$\mathscr{L}(S) := \overline{\{f_{\boldsymbol{a}} \; ; \; f \in L^2([-1,1]; w_d), \boldsymbol{a} \in S\}} \subset L^2(\mathbb{S}^{d-1})$$

the closure of all radial functions with center in S. Then the mean operator $\mathcal{M}|_{T_S}$ is injective for the set S if and only if the set $\mathscr{L}(S)$ of radial functions centered on S is dense in $L^2(\mathbb{S}^{d-1})$.

Proof. We note that $\mathscr{L}(S)$ is dense in the Hilbert space $L^2(\mathbb{S}^{d-1})$ if and only if its orthogonal complement

$$\mathscr{L}(S)^{\perp} = \{g \in L^2(\mathbb{S}^{d-1}) \; ; \; \langle f, g \rangle_{L^2(\mathbb{S}^{d-1})} = 0 \text{ for all } f \in \mathscr{L}(S)\}$$

consists of only the zero function. Let $g \in L^2(\mathbb{S}^{d-1})$. Then $g \in \mathscr{L}(S)^{\perp}$ if and only if

$$\int_{\mathbb{S}^{d-1}} g(\boldsymbol{\xi})\, f(\boldsymbol{a}^\top \boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} = 0$$

for all $f \in L^2([-1,1]; w_d)$ and $\boldsymbol{a} \in S$. By (3.4), we obtain

$$\int_{\mathbb{S}^{d-1}} \mathcal{M} g(\boldsymbol{a}, t)\, f(t)\,(1-t^2)^{\frac{d-3}{2}}\,\mathrm{d}t = 0$$

for all $f \in L^2([-1,1]; w_d)$ and $\boldsymbol{a} \in S$. Setting $f(t) = \overline{\mathcal{M} g(\boldsymbol{a}, t)}$, we see that

$$\int_{\mathbb{S}^{d-1}} \left|\mathcal{M} g(\boldsymbol{a}, t)\right|^2\,(1-t^2)^{\frac{d-3}{2}}\,\mathrm{d}t = 0$$

and hence

$$\mathcal{M} g(\boldsymbol{a}, \cdot) = 0$$

for all $\boldsymbol{a} \in S$. The inverse implication also holds. Hence, we have shown that $g \in \mathscr{L}(S)^{\perp}$ if and only if $\mathcal{M}|_{T_S} g = 0$. Consequently, $\mathscr{L}(S)^{\perp}$ contains only the zero function if and only if the restricted mean operator $\mathcal{M}_{T_S}$ is injective. ■

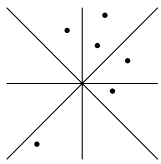

Figure 3.4: The union of the Coxeter system Σ_4 with a finite set.

A related problem in the Euclidean space

The mean operator $\mathcal{M}\big|_{T_S}$ is closely related with its analogue on the Euclidean space $\mathbb{R}^d$, which is known as the circular Radon transform or the spherical-mean Radon transform. We define the circular Radon transform $\mathcal{C}$ for $f\colon \mathbb{R}^d \to \mathbb{C}$

$$\mathcal{C}f(\boldsymbol{x}, r) := \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} f(\boldsymbol{x} + r\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}, \qquad \boldsymbol{x} \in S \subset \mathbb{R}^d,\ r > 0, \tag{3.48}$$

which is the mean value of the function f along the sphere with midpoint $\boldsymbol{x} \in \mathbb{R}^d$ and radius $r > 0$.

In 1988, Andersson [And88] considered the inversion of the circular Radon transform $\mathcal{C}$ for the case that the midpoints $\boldsymbol{x}$ are in the hyperplane $S = \{\boldsymbol{x} \in \mathbb{R}^d\ ;\ x_d = 0\}$. Inversion formulas where the set of midpoints S is a sphere were shown by Finch et al. [FPR04] for odd dimension d and Kunyansky [Kun07] for general $d \geq 2$. The case that S is an ellipsoid was covered by Haltmeier [Hal14].

Let us denote by $C_c(\mathbb{R}^d)$ the set of continuous functions $\mathbb{R}^d \to \mathbb{C}$ with compact support. The injectivity problem of $\mathcal{C}$ for general sets S in $\mathbb{R}^2$ and compactly supported functions was solved in 1996 by Agranovsky and Quinto [AQ96] as follows.

Proposition 3.36. *Let $n \in \mathbb{N}$. We denote by*

$$\Sigma_n := \{t\mathrm{e}^{\pi\mathrm{i}k/n}\ ;\ t \in \mathbb{R},\ k = 1, \dots, n\},$$

the Coxeter system of n lines in $\mathbb{R}^2$. The circular Radon transform $\mathcal{C}$ restricted to the set S is injective on $C_c(\mathbb{R}^2)$ if and only if S is not contained in any set of the form

$$\omega(\Sigma_N) \cup F,$$

where ω is a rigid motion in $\mathbb{R}^2$ and F is a finite set, see Figure 3.4.

It is conjectured that an generalized version of Proposition 3.36 holds in the d-dimensional space $\mathbb{R}^d$. However, only one direction of the equivalence has been proven. We state this result as follows, see the review article [AK05] and the references therein.

Proposition 3.37. *Let $S \subset \mathbb{R}^d$ be not contained in any set of the form*

$$\omega(\Sigma_n) \cup F,$$

where ω is a rigid motion in $\mathbb{R}^d$, Σ_n is the zero set of a homogeneous, harmonic polynomial of degree $n \in \mathbb{N}$ and F is an algebraic subset in $\mathbb{R}^d$ of codimension at least 2. Then the circular Radon transform $\mathcal{C}$ of the set S is injective on $C_c(\mathbb{R}^d)$. In particular, it is injective if S is the boundary of a bounded open set with nonempty interior.

The sets $\Sigma_n \subset \mathbb{R}^d$ are the zero sets of the spherical harmonics of degree n. Restricted to the unit sphere $\mathbb{S}^{d-1}$, the sets $\Sigma_n\big|_{\mathbb{S}^{d-1}}$, $n \in \mathbb{N}$, are exactly the sets for which the mean operator $\mathcal{M}\big|_{T_S}$ is not injective, as we have seen in Proposition 3.33.

3.5.2 Centers on a hyperplane section

After the theory about the general situation, we now come to restrictions of the mean operator $\mathcal{M}$ to the manifolds $S\times[-1,1]$, where the set of centers S is a $(d-2)$-dimensional subsphere of $\mathbb{S}^{d-1}$. In this case, we are able to obtain a more concrete criterion about the injectivity. For simplicity of calculations, we assume without loss of generality by rotational symmetry that this subsphere S is the intersection of the sphere $\mathbb{S}^{d-1}$ with a hyperplane parallel to the equator, i. e., we set

$$S_z := \{\boldsymbol{\xi} \in \mathbb{S}^{d-1} \; ; \xi_d = z\}$$

for some $z \in (-1,1)$. As in (2.7), we can write any point $\boldsymbol{\xi} \in S_z$ in the form

$$\boldsymbol{\xi} = \sqrt{1-z^2}\,\boldsymbol{\sigma} + z\boldsymbol{\epsilon}^d$$

for some $\boldsymbol{\sigma} \in \mathbb{S}^{d-2}\times\{0\}$. Here and in the following, we occasionally identify $\mathbb{S}^{d-2}\times\{0\} \subset \mathbb{R}^d$ with $\mathbb{S}^{d-2}$. For $f \in C(\mathbb{S}^{d-1})$ and $z \in (-1,1)$, we set

$$\mathcal{V}_z f(\boldsymbol{\sigma},t) := \mathcal{M}f\left(\sqrt{1-z^2}\,\boldsymbol{\sigma} + z\boldsymbol{\epsilon}^d, t\right), \qquad \boldsymbol{\sigma} \in \mathbb{S}^{d-2}\times\{0\},\ t \in [-1,1].$$

We denote by

$$L^2(\mathbb{S}^{d-2}\times(-1,1); w_d)$$

the Lebesgue space of square-integrable functions on $\mathbb{S}^{d-2}\times(-1,1)$ with the weight

$$w_d(\boldsymbol{\sigma},t) = (1-t^2)^{\frac{d-3}{2}}, \qquad \boldsymbol{\sigma} \in \mathbb{S}^{d-2},\ t \in (-1,1),$$

cf. (2.17).

Theorem 3.38. *For $m \in \mathbb{N}_0$, let $\{Y_{m,d-1}^k \; ; k = 1,\dots,N_{m,d-1}\}$ be an orthonormal basis of $\mathscr{Y}_{m,d-1}(\mathbb{S}^{d-2})$ and the spherical harmonic $Y_{n,d-1}^{m,k}$ be defined as in (2.35). The operator $\mathcal{V}_z$ admits the singular value decomposition*

$$\mathcal{V}_z Y_{n,d}^{m,k}(\boldsymbol{\sigma},t) = \lambda_{n,d}^m(z)\, B_{n,d}^{m,k}(\boldsymbol{\sigma},t) \tag{3.49}$$

for $n \in \mathbb{N}_0,\ m = 0,\dots,n$ and $k = 1,\dots,N_{m,d-1}$, with the singular values

$$\lambda_{n,d}^m(z) := \sqrt{\frac{|\mathbb{S}^{d-1}|}{N_{n,d}\,|\mathbb{S}^{d-2}|}}\,\widetilde{P}_{n,d}^m(z) \tag{3.50}$$

and the basis functions

$$B_{n,d}^{m,k}(\boldsymbol{\sigma},t) := Y_{m,d-1}^k(\boldsymbol{\sigma})\,\widetilde{P}_{n,d}(t), \qquad (\boldsymbol{\sigma},t) \in \mathbb{S}^{d-2}\times[-1,1]. \tag{3.51}$$

The functions $\{B_{n,d}^{m,k} \; ; n,m \in \mathbb{N}_0,\ k = 1,\dots,N_{m,d}\}$ form an orthonormal basis of the Hilbert space $L^2(\mathbb{S}^{d-2}\times(-1,1); w_d)$.

Proof. Let $n \in \mathbb{N}_0$ and $Y_{n,d} \in \mathscr{Y}_{n,d}(\mathbb{S}^{d-1})$. Then we have for $\boldsymbol{\sigma} \in \mathbb{S}^{d-2}$ and $t \in [-1,1]$ by Theorem 3.4

$$\mathcal{V}_z Y_{n,d}\,(\boldsymbol{\sigma}, t) = P_{n,d}(t)\, Y_{n,d}\left(\sqrt{1-z^2}\,\boldsymbol{\sigma} + z\boldsymbol{\epsilon}^d\right).$$

Plugging in the spherical harmonics of (2.35), we obtain

$$\mathcal{V}_z Y_{n,d}^{m,k}\,(\boldsymbol{\sigma}, t) = P_{n,d}(t)\, \widetilde{P}_{n,d}^m(z)\, Y_{m,d-1}^k(\boldsymbol{\sigma})$$

for $m = 0, \dots, n$ and $k = 1, \dots, N_{m,d-1}$, which shows (3.49).

The orthonormality of the functions $B_{n,d}^{m,k}$ follows from the orthonormality of the spherical harmonics $Y_{m,d-1}^k$ and the orthogonality (2.17) of the Legendre polynomials $P_{n,d}$ together with (2.18). ∎

Hence, we obtain the following "Freak Theorem" that resembles Proposition 3.19, which covered the spherical section transform $\mathcal{T}_z$, for $\mathcal{V}_z$.

Theorem 3.39. *The operator* $\mathcal{V}_z \colon L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-2} \times (-1,1); w_d)$ *is injective if and only if*

$$P_{n,d}^m(z) \neq 0 \qquad \forall n \in \mathbb{N}_0,\ m = 0, \dots, n.$$

In particular, the set of values $z \in (-1,1)$ *where* $\mathcal{V}_z$ *is not injective is countable and dense in* $(-1,1)$.

Theorem 3.39 looks almost like Proposition 3.19 for the spherical section transform $\mathcal{T}_z$, except that the Legendre polynomials $P_{n,d}$ are replaced by the associated Legendre functions $P_{n,d}^m$. So if $\mathcal{V}_z$ is injective for some $z \in (-1,1)$ then $P_{n,d}^m(z) \neq 0$ for all n, m and hence also the spherical section transform $\mathcal{T}_z$ is injective because $P_{n,d}(z) = P_{n,d}^0(z) \neq 0$ for all n. However, the converse does not hold. This makes an analysis whether $\mathcal{V}_z$ is injective for a given z more difficult than for $\mathcal{T}_z$.

Furthermore, we obtain the following continuity result of the operator $\mathcal{V}_z$. We recall the Sobolev spaces on $\mathbb{S}^{d-2} \times (-1,1)$ that were introduced in (3.8).

Theorem 3.40. *Let* $s \in \mathbb{R}$ *and* $z \in (-1,1)$. *The transform* $\mathcal{V}_z$ *is a continuous operator*

$$\mathcal{V}_z \colon H^s(\mathbb{S}^{d-1}) \to H_{\mathrm{mix}}^{\frac{d-3}{2},s}(\mathbb{S}^{d-2} \times (-1,1); w_d).$$

In particular, for $s = 0$, *we have that*

$$\mathcal{V}_z \colon L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-2} \times (-1,1); w_d)$$

is continuous.

Proof. Let $n \in \mathbb{N}_0$, $m \in \{0, \dots, n\}$, and $k \in \{1, \dots, N_{m,d-1}\}$. At first, we show a bound on the absolute value $\left|\lambda_{n,d}^m(z)\right|$ of the singular values (3.50). For $t \in [-1,1]$ and $\boldsymbol{\sigma} \in \mathbb{S}^{d-2}$, we apply the addition formula (2.25) to the spherical harmonic $Y_{n,d}^{m,j}(\sqrt{1-t^2}\,\boldsymbol{\sigma} + t\boldsymbol{\epsilon}^d) = \widetilde{P}_{n,d}^m(t)\, Y_{m,d-1}^k(\boldsymbol{\sigma})$ on $\mathbb{S}^{d-1}$ from (2.35). Then we have

$$\sum_{m=0}^{n} (\widetilde{P}_{n,d}^m(t))^2 \sum_{k=1}^{N_{m,d-1}} \left|Y_{m,d-1}^k(\boldsymbol{\sigma})\right|^2 = \frac{N_{n,d}}{|\mathbb{S}^{d-1}|}.$$

Using again the addition formula for the inner sum of $\left|Y^k_{m,d-1}(\boldsymbol{\sigma})\right|^2$, we have

$$\sum_{m=0}^{n} \widetilde{P}^m_{n,d}(t)^2 \, \frac{N_{m,d-1}}{|\mathbb{S}^{d-2}|} = \frac{N_{n,d}}{|\mathbb{S}^{d-1}|}.$$

Since all the summands are non-negative, we obtain

$$\widetilde{P}^m_{n,d}(t)^2 \, \frac{N_{m,d-1}}{|\mathbb{S}^{d-2}|} \le \frac{N_{n,d}}{|\mathbb{S}^{d-1}|}.$$

Hence, we have for the singular values (3.50)

$$\left|\lambda^m_{n,d}(z)\right| = \sqrt{\frac{|\mathbb{S}^{d-1}|}{N_{n,d}\,|\mathbb{S}^{d-2}|}}\,\left|\widetilde{P}^m_{n,d}(z)\right| \le \frac{1}{\sqrt{N_{m,d-1}}}. \tag{3.52}$$

We proceed in a similar manner to the proof of Theorem 3.5. By the definition of the Sobolev norm (3.8) and the singular value decomposition (3.49) of $\mathcal{V}_z$, we have

$$\begin{aligned}&\left\|\mathcal{V}_z f\right\|^2_{H^{\frac{d-3}{2},s}_{\mathrm{mix}}(\mathbb{S}^{d-2}\times(-1,1);w_d)} \\ &\qquad = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{k=0}^{N_{m,d-1}} \left|\lambda^m_{n,d}(z)\,\hat{f}^{m,k}_{n,d}\right|^2 \left(m+\frac{d-2}{2}\right)^{d-3}\left(n+\frac{d-2}{2}\right)^{2s}.\end{aligned}$$

By (3.52), we obtain the bound

$$\begin{aligned}&\left\|\mathcal{V}_z f\right\|^2_{H^{\frac{d-3}{2},s}_{\mathrm{mix}}(\mathbb{S}^{d-2}\times(-1,1);w_d)} \\ &\qquad \le \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{k=0}^{N_{m,d-1}} \frac{1}{N_{m,d-1}}\left|\hat{f}^{m,k}_{n,d}\right|^2 \left(m+\frac{d-2}{2}\right)^{d-3}\left(n+\frac{d-2}{2}\right)^{2s}.\end{aligned}$$

The last sum converges by the assumption $f \in H^s(\mathbb{S}^{d-1})$ and the asymptotic expansion (3.11), which states that $N_{m,d-1} \simeq \frac{2}{(d-3)!}\, m^{d-3}$ for $m \to \infty$. ■

Remark 3.41. In 2014, Volchkov and Volchkov [VV14] considered a so-called local two-radii problem, which yields another variation of the "Freak Theorem" 3.39. Let

$$B_R = \{\boldsymbol{\xi} \in \mathbb{S}^{d-1} \,;\, \xi_d > \cos R\}$$

denote the open spherical cap of radius $R \in (0,\pi)$ around the north pole $\boldsymbol{\epsilon}^d$. Furthermore, let $r_1, r_2 < R \le \pi$. We consider all subspheres whose spherical centers have latitude $\xi_d \in \{\cos r_1, \cos r_2\}$ and that are contained in B_R. Then the mean operator $\mathcal{M}\big|_{\mathscr{T}(r_1,r_2)}$ restricted to the set

$$\mathscr{T}(r_1,r_2) = \bigcup_{j\in\{1,2\}} \left\{(\boldsymbol{\xi},t) \in \mathbb{S}^{d-1}\times(-1,1) \,;\, \xi_d = \cos r_j,\ \cos(R-r_j) < t < 1\right\}$$

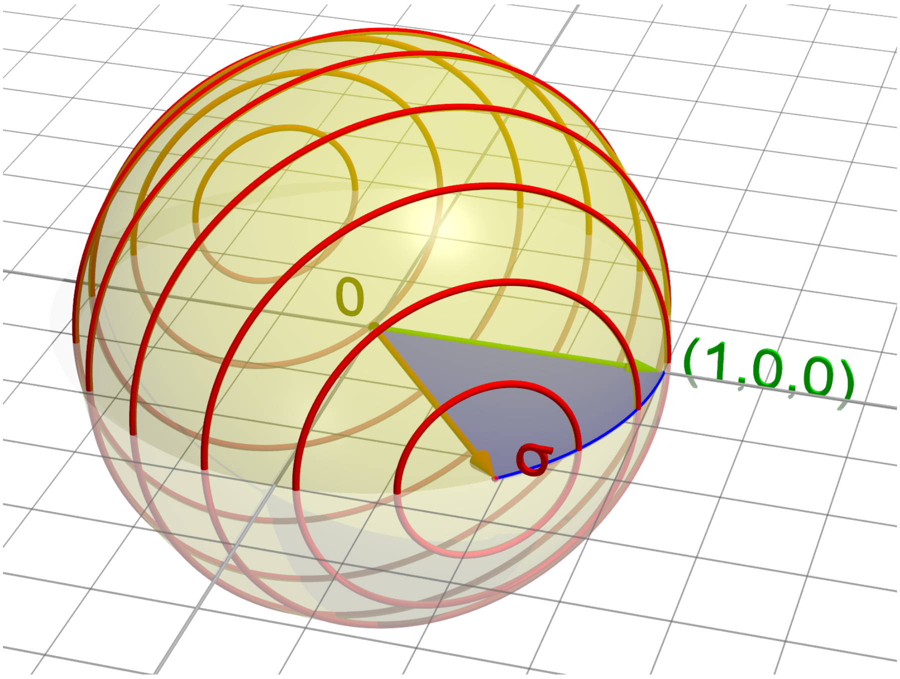

Figure 3.5: A vector $\boldsymbol{\sigma} \in \mathbb{S}^1 \times \{0\}$ on the equator of the two-sphere $\mathbb{S}^2$ and corresponding circles of the vertical slice transform $\mathcal{V}f(\boldsymbol{\sigma}, t)$ for several values of t

is injective for the class $L^1_{\mathrm{loc}}(B_R)$ of locally integrable functions on B_R if and only if $R \geq r_1 + r_2$ and

$$\bigcup_{k=0}^{\infty} (\mathscr{V}_k(r_1) \cap \mathscr{V}_k(r_2)) = \emptyset,$$

where

$$\mathscr{V}_k(r_j) = \left\{ \nu \in (k, \infty) \; ; \; P_{\nu + \frac{d-3}{2}}^{-k-\frac{d-3}{2}}(\cos r_j) = 0 \right\}, \qquad j = 1, 2,$$

and P_ν^μ denotes the associated Legendre function of orders $\nu, \mu \in \mathbb{R}$, which is an extension of the associated Legendre function $P_{\nu,3}^\mu$ from (2.32) to non-integers $\nu, \mu \in \mathbb{R}$, see [GR07, Section 8.7]. □

3.5.3 Centers on the equator: the vertical slice transform

In this section, we consider the special case that S is the equator of the sphere, i. e., we consider the operator $\mathcal{V}_z$ for $z = 0$. We call $\mathcal{V} := \mathcal{V}_0$ the vertical slice transform

$$\mathcal{V}f(\boldsymbol{\sigma}, t) = \begin{cases} \dfrac{1}{|\mathbb{S}^{d-2}| \, (1-t^2)^{\frac{d-2}{2}}} \displaystyle\int_{\boldsymbol{\eta}^\top \boldsymbol{\sigma} = t} f(\boldsymbol{\eta}) \, \mathrm{d}\boldsymbol{\eta}, & \boldsymbol{\sigma} \in \mathbb{S}^{d-2} \times \{0\}, \; t \in (-1, 1) \\ f(\pm \boldsymbol{\sigma}), & \boldsymbol{\sigma} \in \mathbb{S}^{d-2} \times \{0\}, \; t = \pm 1. \end{cases} \tag{3.53}$$

The vertical slice transform $\mathcal{V}$ computes the mean values of a function f along every section of the sphere $\mathbb{S}^{d-1}$ with a hyperplane that is parallel to the ξ_d–axis. An illustration can be found in Figure 3.5. Since all these subspheres are symmetric with respect to the reflection through the equatorial hyperplane $\{\boldsymbol{x} \in \mathbb{R}^d \ ; \ x_d = 0\}$, the vertical slice transform $\mathcal{V}f$ vanishes for functions f that are odd in the d-th coordinate, i. e., $\mathcal{V}f = 0$ if

$$f(\xi_1, \dots, \xi_{d-1}, \xi_d) = -f(\xi_1, \dots, \xi_{d-1}, -\xi_d), \qquad \boldsymbol{\xi} = (\xi_1, \dots, \xi_d)^\top \in \mathbb{S}^{d-1}. \tag{3.54}$$

Thus, by knowing $\mathcal{V}f$, only the even part of f could possibly be reconstructed. We define the space of symmetric functions

$$L^2_{\mathrm{sym}}(\mathbb{S}^{d-1}) := \left\{ f \in L^2(\mathbb{S}^{d-1}) \ ; \ f(\xi_1, \dots, \xi_{d-1}, \xi_d) = f(\xi_1, \dots, \xi_{d-1}, -\xi_d), \ \boldsymbol{\xi} \in \mathbb{S}^{d-1} \right\}.$$

Another symmetry property of the vertical slice transform is that $\mathcal{V}f$ is even in the sense that

$$\mathcal{V}f(\boldsymbol{\sigma}, t) = \mathcal{V}f(-\boldsymbol{\sigma}, -t), \quad \boldsymbol{\sigma} \in \mathbb{S}^{d-2}, \ t \in [-1, 1],$$

which is a special case of (3.3) and follows from the fact that the domain of integration, $\{\boldsymbol{\eta} \in \mathbb{S}^{d-1} \ ; \ \boldsymbol{\sigma}^\top \boldsymbol{\eta} = t\}$, is the same for both $\mathcal{V}f(\boldsymbol{\sigma}, t)$ and $\mathcal{V}f(-\boldsymbol{\sigma}, -t)$. Hence, we can define an equivalence relation $\sim$ on $\mathbb{S}^{d-2} \times [-1, 1]$ by saying $(\boldsymbol{\sigma}, t) \sim (-\boldsymbol{\sigma}, -t)$. For $d = 3$, the quotient space with respect to this equivalence relation is isomorphic to the Möbius strip, see [HQ16].

Remark 3.42. The vertical slice transform on the two-dimensional sphere $\mathbb{S}^2$ was discussed in an article by Gindikin et al. [GRS94] from 1994. That article includes an inversion formula of $\mathcal{V}$, which was achieved by projecting the vertical circles to line segments in the equatorial plane as we will see in Theorem 3.46.

In 2010, Zangerl and Scherzer [ZS10] considered the reconstruction of an image by photoacoustic tomography with detectors that resemble the vertical slices of the sphere. One step of this reconstruction approach consists of the inversion of the vertical slice transform $\mathcal{V}$, which they called the circular mean transform. Their inversion scheme of $\mathcal{V}$ is based on the stereographic projection from the point $\boldsymbol{\epsilon}^1 = (1, 0, 0)^\top$, which lies on the equator of the sphere, onto the ξ_2–ξ_3 plane. Then the vertical slices of the sphere are mapped to circles in the plane with centers on a line. The inversion of this circular Radon transform (3.48), which takes the integrals of a function in the plane along all circles with centers on a line, was previously considered by Andersson [And88]. □

In the following theorem, we show the singular value decomposition of the vertical slice transform $\mathcal{V}$ on the sphere $\mathbb{S}^{d-1}$ as a special case of Theorem 3.38, which considered $\mathcal{V}_z$. However, compared to the transform $\mathcal{V}_z$, we obtain an explicit expression of the singular values of the vertical slice transform $\mathcal{V} = \mathcal{V}_0$.

Theorem 3.43. *For $m \in \mathbb{N}_0$, let $\{Y^k_{m,d-1} \ ; \ k = 1, \dots, N_{m,d-1}\}$ be an orthonormal basis of $\mathscr{Y}_{m,d-1}(\mathbb{S}^{d-2})$ and let the spherical harmonic $Y^{m,k}_{n,d-1}$ be as defined in* (2.35)*. The vertical slice transform*

$$\mathcal{V} \colon L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-2} \times (-1, 1); w_d) \tag{3.55}$$

admits the singular value decomposition

$$\mathcal{V}Y_{n,d}^{m,k}(\boldsymbol{\xi},t) = \hat{\mathcal{V}}_{n,d}^{m}\, Y_{m,d-1}^{k}(\boldsymbol{\xi})\, \widetilde{P}_{n,d}(t), \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-2}\times\{0\},\ t\in[-1,1], \tag{3.56}$$

for $n \in \mathbb{N}_0$, $m = 0,\dots,n$ with $n-m$ even and $k = 1,\dots,N_{m,d-1}$, where the singular values are given by

$$\hat{\mathcal{V}}_{n,d}^{m} := (-1)^{\frac{n-m}{2}} \sqrt{\frac{(n+m+d-3)!\, n!}{(n-m)!\,(n+d-3)!}}\, \frac{(n-m-1)!!\,(d-3)!!}{(n+m+d-3)!!}. \tag{3.57}$$

The vertical slice transform $\mathcal{V}$ is injective for functions in $L^2_{\mathrm{sym}}(\mathbb{S}^{d-1})$, and its nullspace consists of the functions that are odd with respect to the last coordinate ξ_d.

Proof. Let $n \in \mathbb{N}_0$, $m = 0,\dots,n$ and $k = 1,\dots,N_{m,d-1}$. By Theorem 3.38 for $z=0$, we have

$$\mathcal{V}Y_{n,d}^{m,k}(\boldsymbol{\sigma},t) = \widetilde{P}_{n,d}^{m}(0)\, Y_{m,d-1}^{k}(\boldsymbol{\sigma})\, P_{n,d}(t). \tag{3.58}$$

Hence, we obtain (3.56) with

$$\hat{\mathcal{V}}_{n,d}^{m} = \widetilde{P}_{n,d}^{m}(0)\, \frac{P_{n,d}}{\widetilde{P}_{n,d}} = \widetilde{P}_{n,d}^{m}(0)\, \frac{2^{(d-2)/2}\,\Gamma(\frac{d-1}{2})\sqrt{n!}}{\sqrt{(2n+d-2)\,(n+d-3)!}}, \tag{3.59}$$

where we have used the definition (2.18) of the normalized Legendre functions $\widetilde{P}_{n,d}$. Next, we are going to calculate an explicit expression of $\widetilde{P}_{n,d}^{m}(0)$ in order to show (3.57). By the definition (2.33) and (2.32) of the normalized associated Legendre polynomial, we have

$$\begin{aligned}
\widetilde{P}_{n,d}^{m}(t) &= \frac{\sqrt{(2n+d-2)\,(n-m)!\,(n+d+m-3)!}}{2^{\frac{d-2}{2}}\, n!\,\Gamma(\frac{d-1}{2})} \\
&\quad \cdot \frac{n!\,\Gamma(\frac{d-1}{2})}{2^m\,(n-m)!\,\Gamma(m+\frac{d-1}{2})}\,(1-t^2)^{m/2} P_{n-m,d+2m}(t) \\
&= \frac{\sqrt{(2n+d-2)\,(n+d+m-3)!}}{2^{m+\frac{d-2}{2}}\sqrt{(n-m)!}\,\Gamma(m+\frac{d-1}{2})}\,(1-t^2)^{m/2} P_{n-m,d+2m}(t).
\end{aligned}$$

By (3.20), we have for $d \geq 3$

$$P_{n,d}(0) = \begin{cases} (-1)^{n/2}\frac{(n-1)!!\,(d-3)!!}{(n+d-3)!!}, & n \text{ even} \\ 0, & n \text{ odd.}\end{cases}$$

Thus, if $n-m$ is odd, we have $\widetilde{P}_{n,d}^{m}(0)=0$. If $n-m$ is even, we have

$$\begin{aligned}
\widetilde{P}_{n,d}^{m}(0) &= \frac{\sqrt{(2n+d-2)\,(n+d+m-3)!}}{2^{m+\frac{d-2}{2}}\sqrt{(n-m)!}\,\Gamma(m+\frac{d-1}{2})} P_{n-m,d+2m}(0) \\
&= (-1)^{\frac{n-m}{2}} \frac{\sqrt{(2n+d-2)\,(n+d+m-3)!}}{2^{m+\frac{d-2}{2}}\sqrt{(n-m)!}\,\Gamma(m+\frac{d-1}{2})}\, \frac{(n-m-1)!!\,(d+2m-3)!!}{(n+m+d-3)!!}.
\end{aligned} \tag{3.60}$$

Combining (3.59) and (3.60), we obtain

$$\begin{aligned}\hat{\mathcal{V}}_{n,d}^{m} &= (-1)^{\frac{n-m}{2}} \frac{\sqrt{(2n+d-2)\,(n+d+m-3)!}}{2^{m+\frac{d-2}{2}}\sqrt{(n-m)!}\,\Gamma(m+\frac{d-1}{2})} \frac{(n-m-1)!!\,(d+2m-3)!!}{(n+m+d-3)!!} \\ &\quad\cdot \frac{2^{(d-2)/2}\Gamma(\frac{d-1}{2})\sqrt{n!}}{\sqrt{(2n+d-2)(n+d-3)!}} \\ &= (-1)^{\frac{n-m}{2}} \frac{\sqrt{(n+m+d-3)!}}{2^{m}\sqrt{(n-m)!}\,\Gamma(m+\frac{d-1}{2})} \frac{(n-m-1)!!\,(d+2m-3)!!}{(n+d+m-3)!!} \frac{\sqrt{n!}\,\Gamma(\frac{d-1}{2})}{\sqrt{(n+d-3)!}}.\end{aligned}$$

By (2.68), we have

$$\hat{\mathcal{V}}_{n,d}^{m} = \frac{(-1)^{\frac{n-m}{2}}\,(d-3)!!}{(2m+d-3)!!}\sqrt{\frac{(n+m+d-3)!\,n!}{(n-m)!\,(n+d-3)!}}\frac{(n-m-1)!!\,(2m+d-3)!!}{(n+m+d-3)!!}. \tag{3.61}$$

By the recurrence relation (2.14), we see that the Legendre polynomial $P_{n,d}$ is even if and only if n is even. Hence, the spherical harmonic $Y_{n,d}^{m,k}(\boldsymbol{\xi})$ is even with respect to ξ_d if and only if $n-m$ is even by (2.35). So we see that

$$L^2_{\mathrm{sym}}(\mathbb{S}^{d-1}) = \overline{\operatorname{span}}\left\{Y_{n,d}^{m,k}\ ;\ n,m\in\mathbb{N}_0,\ n-m \text{ even},\ k=1,\dots,N_{m,d-1}\right\}.$$

Since the values $\hat{\mathcal{V}}_{n,d}^{m}$ are nonzero if and only if $n-m$ is even, we see that the vertical slice transform $\mathcal{V}$ is injective for all functions in $L^2_{\mathrm{sym}}(\mathbb{S}^{d-1})$. ∎

The singular value decomposition of the vertical slice transform $\mathcal{V}$ was shown for the case $d=3$ on $\mathbb{S}^2$ by the author in [HQ16]. In this case, we obtain the singular values

$$\begin{aligned}\hat{\mathcal{V}}_{n,3}^{m} &= \frac{(-1)^{\frac{n-m}{2}}\Gamma(1)}{2^{m}\Gamma(m+1)}\sqrt{\frac{(n+m)!\,n!}{(n-m)!\,n!}}\,\frac{(n-m-1)!!\,(2m)!!}{(n+m)!!} \\ &= (-1)^{\frac{n-m}{2}}\sqrt{\frac{(n+m)!}{(n-m)!}}\,\frac{(n-m-1)!!}{(n+m)!!}\end{aligned}$$

if $n\in\mathbb{N}_0$, $m=0,\dots,n$ with $n-m$ even and $\hat{\mathcal{V}}_{n,3}^{m}=0$ otherwise.

Asymptotics of the singular values

As we did for the Funk–Radon transform in Section 3.2.1, we take a closer look at the asymptotic behavior of the singular values $\hat{\mathcal{V}}_{n,d}^{m}$ of the vertical slice transform $\mathcal{V}$.

Theorem 3.44. *There exist constants $c_1, c_2 > 0$ such that for all $n\in\mathbb{N}_0$ and $m = 0,\dots,n$ with $n-m$ even, we have*

$$c_1\, n^{-\frac{d-2}{2}} \le \left|\hat{\mathcal{V}}_{n,d}^{m}\right| \le c_2\, n^{-\frac{d-2}{2}+\frac{1}{4}}. \tag{3.62}$$

These bounds are tight in the sense that for $n \to \infty$, we have

$$\left|\hat{\mathcal{V}}^0_{2n,d}\right| \simeq \Gamma\left(\frac{d-1}{2}\right) \pi^{-\frac{1}{2}} n^{-\frac{d-2}{2}} \tag{3.63}$$

and

$$\left|\hat{\mathcal{V}}^n_{n,d}\right| \simeq \Gamma\left(\frac{d-1}{2}\right) 2^{\frac{d-3}{2}} \pi^{-\frac{1}{4}} n^{-\frac{d-2}{2}+\frac{1}{4}}. \tag{3.64}$$

Proof. This proof is divided into three parts. We first show the asymptotic approximations (3.63) and (3.64) of $\left|\hat{\mathcal{V}}^0_{n,d}\right|$ and $\left|\hat{\mathcal{V}}^n_{n,d}\right|$, respectively. Then we show the inequalities

$$\left|\hat{\mathcal{V}}^m_{n,d}\right| \geq \begin{cases} \left|\hat{\mathcal{V}}^0_{n,d}\right|, & n \text{ even} \\ \left|\hat{\mathcal{V}}^0_{n+1,d}\right|, & n \text{ odd} \end{cases} \tag{3.65}$$

as well as

$$\left|\hat{\mathcal{V}}^n_{n,d}\right| \geq \left|\hat{\mathcal{V}}^m_{n,d}\right| \tag{3.66}$$

for $n \in \mathbb{N}_0$ and $m \in \{0, \dots, n\}$ with $n-m$ even, which imply the assertion (3.62).

We compute for even $n \in \mathbb{N}_0$

$$\left|\hat{\mathcal{V}}^0_{n,d}\right| = \frac{(n-1)!!\,(d-3)!!}{(n+d-3)!!}.$$

These are exactly the eigenvalues (3.20) of the Funk–Radon transform, which decay by (3.21) with the rate

$$\left|\hat{\mathcal{V}}^0_{2n,d}\right| \simeq \Gamma(\tfrac{d-1}{2})\, \pi^{-\frac{1}{2}} n^{\frac{2-d}{2}}$$

for $n \to \infty$. This shows (3.63).

We have by (2.68)

$$\begin{aligned} \left|\hat{\mathcal{V}}^n_{n,d}\right| &= \sqrt{\frac{(2n+d-3)!\,n!}{(n+d-3)!}}\ \frac{(d-3)!!}{(2n+d-3)!!} \\ &= \sqrt{\frac{(2n+d-3)!\,n!}{(n+d-3)!}}\ \frac{\Gamma(\frac{d-1}{2})}{2^n\,\Gamma(n+\frac{d-1}{2})}. \end{aligned}$$

Applying the asymptotic approximations (2.71) and (2.72) of the factorial and the Gamma function, respectively, we obtain for $n \to \infty$

$$\begin{aligned}
\left|\hat{\mathcal{V}}_{n,d}^{n}\right|^2 &\simeq \frac{\Gamma(\frac{d-1}{2})^2}{2^{2n}\sqrt{2\pi}} \frac{(2n+d-3)^{2n+d-5/2}\, n^{n+1/2}}{(n+d-3)^{n+d-5/2}\,(n+\frac{d-1}{2})^{2n+d-2}} \mathrm{e}^{-2n-d+3-n+n+d-3+2n+d-1} \\
&= \frac{\Gamma(\frac{d-1}{2})^2\, 2^{2n+d-2}}{2^{2n}\sqrt{2\pi}} \frac{(2n+d-3)^{2n+d-5/2}\, n^{n+1/2}}{(n+d-3)^{n+d-5/2}\,(2n+d-1)^{2n+d-2}} \mathrm{e}^{d-1} \\
&= \frac{\Gamma\left(\frac{d-1}{2}\right)^2 2^{d-2}}{\sqrt{2\pi}} \left(\frac{2n+d-3}{2n+d-1}\right)^{2n+d-2} (2n+d-3)^{-1/2} \\
&\quad \left(\frac{n}{n+d-3}\right)^{n+1/2} (n+d-3)^{-d+3}\,\mathrm{e}^{d-1} \\
&= \frac{\Gamma\left(\frac{d-1}{2}\right)^2 2^{d-2}}{\sqrt{2\pi}} \left(1+\frac{-2}{2n+d-1}\right)^{2n+d-2} (2n+d-3)^{-1/2} \\
&\quad \left(1+\frac{-d+3}{n+d-3}\right)^{n+1/2} (n+d-3)^{-d+3}\,\mathrm{e}^{d-1}.
\end{aligned}$$

With the series of the exponential function, we obtain

$$\begin{aligned}
\left|\hat{\mathcal{V}}_{n,d}^{n}\right|^2 &\simeq \Gamma\left(\frac{d-1}{2}\right)^2 2^{d-2}\,(2\pi)^{-1/2}\,\mathrm{e}^{-2}\,(2n)^{-1/2}\,\mathrm{e}^{-d+3}\,n^{-d+3}\,\mathrm{e}^{d-1} \\
&= \Gamma\left(\frac{d-1}{2}\right)^2 \frac{2^{d-3}}{\sqrt{\pi}}\, n^{-d+5/2},
\end{aligned}$$

which shows (3.64).

In the last part of this proof, we show the monotonicity of $\left|\hat{\mathcal{V}}_{n,d}^{m}\right|$ with respect to m. We have for $n \geq 1$ and $m = 0, \ldots, n-2$ with $n-m$ even

$$\begin{aligned}
\left|\frac{\hat{\mathcal{V}}_{n,d}^{m+2}}{\hat{\mathcal{V}}_{n,d}^{m}}\right|^2 &= \frac{(n+m+d-1)!\,(n-m)!\,(n-m-3)!!^2\,(n+m+d-3)!!^2}{(n+m+d-3)!\,(n-m-2)!\,(n-m-1)!!^2\,(n+m+d-1)!!^2} \\
&= \frac{(n+m+d-1)\,(n+m+d-2)\,(n-m)\,(n-m-1)}{(n-m-1)^2\,(n+m+d-1)^2} \\
&= \frac{(n+m+d-2)\,(n-m)}{(n-m-1)\,(n+m+d-1)}.
\end{aligned}$$

Expanding and collecting terms, we obtain

$$\begin{aligned}
\left|\frac{\hat{\mathcal{V}}_{n,d}^{m+2}}{\hat{\mathcal{V}}_{n,d}^{m}}\right|^2 &= \frac{(n+m+d-1)\,(n-m-1)+2m+d-1}{(n-m-1)\,(n+m+d-1)} \\
&= 1+\frac{2m+d-1}{(n-m-1)\,(n+m+d-1)}.
\end{aligned}$$

Because numerator and denominator of the last fraction are non-negative and positive, respectively, by the assumptions on n and m, we see that $\left|\hat{\mathcal{V}}_{n,d}^{m+2}\right| \geq \left|\hat{\mathcal{V}}_{n,d}^{m}\right|$. Hence, we obtain that

$$\left|\hat{\mathcal{V}}_{n,d}^{m}\right| \leq \left|\hat{\mathcal{V}}_{n,d}^{n}\right|,$$

which shows the inequality (3.66). Furthermore, we see that if n is even

$$\left|\hat{\mathcal{V}}_{n,d}^{m}\right| \geq \left|\hat{\mathcal{V}}_{n,d}^{0}\right|,$$

which shows the first line of the inequality (3.65). We have for $n \geq 1$

$$\begin{aligned}
\left|\frac{\hat{\mathcal{V}}_{n-1,d}^{1}}{\hat{\mathcal{V}}_{n,d}^{0}}\right|^2 &= \frac{(n-1)!\, n!\, (n+d-3)!\, (n-3)!!^2}{n!\, (n-2)!\, (n+d-4)!\, (n-1)!!^2} \\
&= \frac{(n-1)\,(n+d-3)}{(n-1)^2} \\
&= \frac{n+d-3}{n-1} \geq 1.
\end{aligned}$$

Hence, we obtain for odd n

$$\left|\hat{\mathcal{V}}_{n,d}^{m}\right| \geq \left|\hat{\mathcal{V}}_{n,d}^{1}\right| \geq \left|\hat{\mathcal{V}}_{n+1,d}^{0}\right|,$$

which finally shows the second line of the inequality (3.65). ■

Unlike for the Funk–Radon transform $\mathcal{F}$, the asymptotic behavior of the singular values $\hat{\mathcal{V}}_{n,d}^{m}$ of the vertical slice transform $\mathcal{V}$ depends not only on the degree n but also on the order m of the spherical harmonics $Y_{n,d}^{m,k}$. The spherical harmonics $Y_{n,d}^{0,k}$, for which the corresponding singular values $\hat{\mathcal{V}}_{n,d}^{0}$ decay faster than $\hat{\mathcal{V}}_{n,d}^{n}$ for $n \to \infty$, depend only on the d-th coordinate, i. e., $Y_{n,d}^{0,k}(\boldsymbol{\xi}) = Y_{n,d}^{0,k}(\xi_d)$, see (2.35). Conversely, the spherical harmonics $Y_{n,d}^{n,k}$, for which the corresponding singular values $\hat{\mathcal{V}}_{n,d}^{n}$ decay slower than $\hat{\mathcal{V}}_{n,d}^{0}$ for $n \to \infty$, have the highest oscillation in the equatorial hyperplane $\{\boldsymbol{\xi} \in \mathbb{S}^{d-1} ; \xi_d = 0\}$. This might be explained by the fact that the subspheres along which $\mathcal{V}$ integrates are smaller near the equator and thus the "resolution" seems to be higher near the equator. So, roughly speaking, the inversion of the vertical slice transform becomes a little less ill-posed if the function oscillates more near the equator.

Nevertheless, we still obtain the following continuity result of the vertical slice transform $\mathcal{V}$, which follows from the upper bound of the singular values $\hat{\mathcal{V}}_{n,d}^{m}$ we calculated in Theorem 3.44.

Theorem 3.45. *Let $s \in \mathbb{R}$. The vertical slice transform $\mathcal{V}$ is a continuous operator*

$$\mathcal{V} \colon H^s(\mathbb{S}^{d-1}) \to H_{\mathrm{mix}}^{0,s+\frac{d-2}{2}-\frac{1}{4}}(\mathbb{S}^{d-2} \times (-1,1); w_d).$$

Proof. We proceed in a similar manner to the proof of Theorem 3.40. By the definition of the Sobolev norm (3.8) and the singular value decomposition (3.56) of $\mathcal{V}$, we have

$$\|\mathcal{V}f\|^2_{H^{0,s+\frac{d-2}{2}-\frac{1}{4}}_{\mathrm{mix}}(\mathbb{S}^{d-2}\times(-1,1);w_d)} = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{k=0}^{N_{m,d-1}} \left|\hat{\mathcal{V}}^m_{n,d}\,\hat{f}^{m,k}_{n,d}\right|^2 \left(n+\frac{d-2}{2}\right)^{2s+d-2-\frac{1}{2}}.$$

With the upper bound (3.62) of $\left|\hat{\mathcal{V}}^m_{n,d}\right|$, we obtain

$$\|\mathcal{V}f\|^2_{H^{0,s+\frac{d-2}{2}-\frac{1}{4}}_{\mathrm{mix}}(\mathbb{S}^{d-2}\times(-1,1);w_d)} \le \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{k=0}^{N_{m,d-1}} \left|c_2\,n^{-\frac{d-2}{2}+\frac{1}{4}}\,\hat{f}^{m,k}_{n,d}\left(n+\frac{d-2}{2}\right)^{s+\frac{d-2}{2}-\frac{1}{4}}\right|^2.$$

The last sum converges by the assumption that $f \in H^s(\mathbb{S}^{d-1})$. ■

Connection with the Radon transform

The following theorem gives a connection between the vertical slice transform $\mathcal{V}$ on the unit sphere $\mathbb{S}^{d-1}$ and the Radon transform $\mathcal{R}$ on the unit ball

$$\mathbb{B}^{d-1} = \{\boldsymbol{x} \in \mathbb{R}^{d-1}\ ;\ \|\boldsymbol{x}\| < 1\},$$

which is defined for a continuous function $f\colon \mathbb{B}^{d-1} \to \mathbb{C}$ by

$$\mathcal{R}f(\boldsymbol{\omega}, s) = \int_{\boldsymbol{x}^\top\boldsymbol{\omega}=s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \qquad \boldsymbol{\omega} \in \mathbb{S}^{d-2},\ s \in \mathbb{R}, \tag{3.67}$$

see also (1.2). The idea is to project the sphere $\mathbb{S}^{d-1}$ orthogonally to the equatorial hyperplane $\{\boldsymbol{x} \in \mathbb{R}^d\ ;\ x_d = 0\}$. Then the vertical sections of the sphere $\mathbb{S}^{d-1}$ are transformed to hyperplanes in $\mathbb{R}^{d-1}$, which is illustrated for $d = 3$ in Figure 3.6.

For a function $f\colon \mathbb{S}^{d-1} \to \mathbb{C}$, we define its weighted projection $\widetilde{f}_{(d-1)}\colon \mathbb{B}^{d-1} \to \mathbb{C}$ to the unit ball $\mathbb{B}^{d-1}$ by

$$\widetilde{f}_{(d-1)}(\boldsymbol{x}) := \frac{1}{\sqrt{1-\|\boldsymbol{x}\|^2}}\left(f\left(\begin{matrix}\boldsymbol{x}\\ \sqrt{1-\|\boldsymbol{x}\|^2}\end{matrix}\right) + f\left(\begin{matrix}\boldsymbol{x}\\ -\sqrt{1-\|\boldsymbol{x}\|^2}\end{matrix}\right)\right), \qquad \boldsymbol{x} \in \mathbb{B}^{d-1}. \tag{3.68}$$

Theorem 3.46. *Let $f\colon \mathbb{S}^{d-1} \to \mathbb{C}$ for $d \ge 3$. Then, we have*

$$\mathcal{V}f(\boldsymbol{\sigma}, t) = \frac{(1-t^2)^{\frac{3-d}{2}}}{|\mathbb{S}^{d-2}|}\,\mathcal{R}\widetilde{f}_{(d-1)}(\boldsymbol{\sigma}, t), \qquad \boldsymbol{\sigma} \in \mathbb{S}^{d-2},\ t \in (-1,1). \tag{3.69}$$

Proof. We are going to denote the sphere with radius $r > 0$ centered in the origin by

$$\mathbb{S}^{d-1}(r) = r\,\mathbb{S}^{d-1} = \{\boldsymbol{\xi} \in \mathbb{R}^d\ ;\ \|\boldsymbol{\xi}\| = r\}.$$

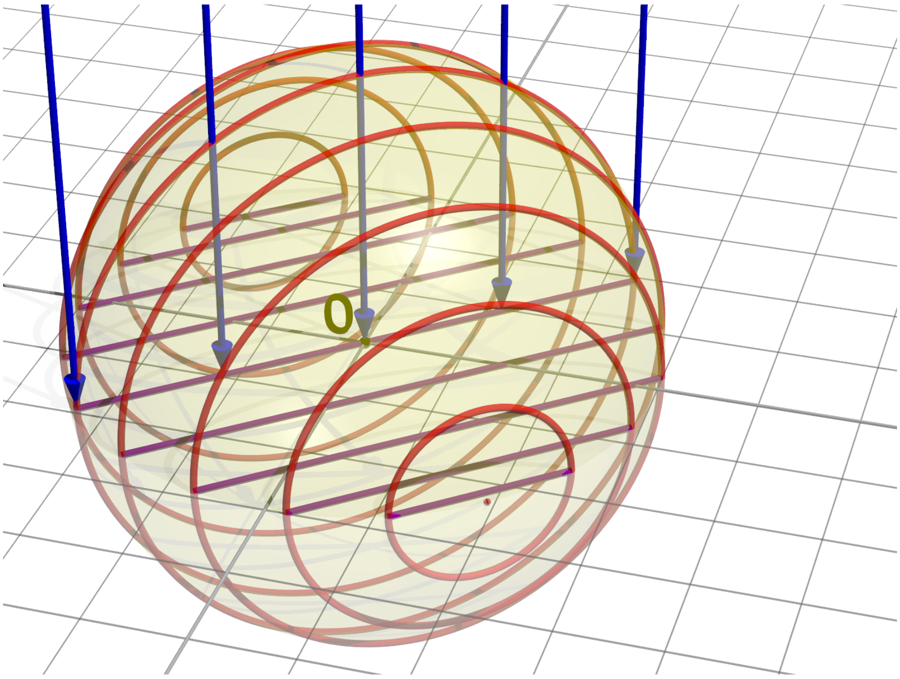

Figure 3.6: The circles (red) from the vertical slice transform on the two-sphere $\mathbb{S}^2$ are projected to line segments (purple) in the unit disk in the equatorial plane.

Since the claimed formula (3.69) is invariant with respect to any rotation in $\boldsymbol{\sigma}$, we state the proof without loss of generality for $\boldsymbol{\sigma} = \boldsymbol{\epsilon}^1$. Let $t \in (-1, 1)$. By the definition (3.55) of the vertical slice transform, we have

$$\mathcal{V}f(\boldsymbol{\epsilon}^1, t) = \frac{1}{|\mathbb{S}^{d-2}|\,(1-t^2)^{\frac{d-2}{2}}} \int_{\boldsymbol{\eta}\in\mathbb{S}^{d-1};\eta_1=t} f(\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta}.$$

We write the vector $\boldsymbol{\eta}$ in the form $\boldsymbol{\eta} = \begin{pmatrix} t \\ \boldsymbol{\eta}' \end{pmatrix} \in \mathbb{S}^{d-1}$. Since $1 = \|\boldsymbol{\eta}\|^2 = t^2 + \|\boldsymbol{\eta}'\|^2$, we see that $\boldsymbol{\eta}' \in \mathbb{S}^{d-2}(\sqrt{1-t^2})$. We obtain

$$\mathcal{V}f(\boldsymbol{\epsilon}^1, t) = \frac{1}{|\mathbb{S}^{d-2}|\,(1-t^2)^{\frac{d-2}{2}}} \int_{\boldsymbol{\eta}'\in\mathbb{S}^{d-2}(\sqrt{1-t^2})} f\begin{pmatrix} t \\ \boldsymbol{\eta}' \end{pmatrix} \mathrm{d}\boldsymbol{\eta}',$$

where $\mathrm{d}\boldsymbol{\eta}'$ denotes the standard surface measure on $\mathbb{S}^{d-2}(\sqrt{1-t^2})$. We perform the substitution $\boldsymbol{\eta}' \mapsto \boldsymbol{\xi} \in \mathbb{S}^{d-2}$ with $\boldsymbol{\eta}' = \sqrt{1-t^2}\,\boldsymbol{\xi}$. Then $\mathrm{d}\boldsymbol{\eta}' = (1-t^2)^{\frac{d-2}{2}}\,\mathrm{d}\boldsymbol{\xi}$ and we have

$$\mathcal{V}f(\boldsymbol{\epsilon}^1, t) = \frac{1}{|\mathbb{S}^{d-2}|} \int_{\boldsymbol{\xi}\in\mathbb{S}^{d-2}} f\begin{pmatrix} t \\ \sqrt{1-t^2}\,\boldsymbol{\xi} \end{pmatrix} \mathrm{d}\boldsymbol{\xi}.$$

We split the domain of integration into the upper $\mathbb{S}^{d-2}_{+} = \{\boldsymbol{\xi} \in \mathbb{S}^{d-2} ; \boldsymbol{\xi}_{d-1} > 0\}$ and lower half $\mathbb{S}^{d-2}_{-} = \{\boldsymbol{\xi} \in \mathbb{S}^{d-2} ; \boldsymbol{\xi}_{d-1} < 0\}$ of the sphere and obtain

$$\mathcal{V}f(\boldsymbol{\epsilon}^1, t) = \frac{1}{|\mathbb{S}^{d-2}|} \left(\int_{\boldsymbol{\xi}^+\in\mathbb{S}^{d-2}_{+}} f\begin{pmatrix} t \\ \sqrt{1-t^2}\,\boldsymbol{\xi}^+ \end{pmatrix} \mathrm{d}\boldsymbol{\xi}^+ + \int_{\boldsymbol{\xi}^-\in\mathbb{S}^{d-2}_{-}} f\begin{pmatrix} t \\ \sqrt{1-t^2}\,\boldsymbol{\xi}^- \end{pmatrix} \mathrm{d}\boldsymbol{\xi}^- \right).$$

Now we transform the integral to the ball $\mathbb{B}^{d-2}$. To this end, we consider the bijections $\varphi^{\pm}\colon \mathbb{B}^{d-2} \to \mathbb{S}^{d-2}_{\pm}$ from (2.5) given by

$$\boldsymbol{\xi}^{\pm} = \varphi^{\pm}(\boldsymbol{y}) = \begin{pmatrix} \boldsymbol{y} \\ \pm\sqrt{1-\|\boldsymbol{y}\|^2} \end{pmatrix}, \qquad \boldsymbol{y} \in \mathbb{B}^{d-2}.$$

Hence, we obtain by (2.6)

$$\begin{aligned}\mathcal{V}f(\boldsymbol{\epsilon}^1,t) = {} & \frac{1}{|\mathbb{S}^{d-2}|}\int_{\mathbb{B}^{d-2}} f\begin{pmatrix} t \\ \sqrt{1-t^2}\,\boldsymbol{y} \\ \sqrt{1-t^2}\,\sqrt{1-\|\boldsymbol{y}\|^2} \end{pmatrix} \frac{1}{\sqrt{1-\|\boldsymbol{y}\|^2}}\,\mathrm{d}\boldsymbol{y} \\ & + \frac{1}{|\mathbb{S}^{d-2}|}\int_{\mathbb{B}^{d-2}} f\begin{pmatrix} t \\ \sqrt{1-t^2}\,\boldsymbol{y} \\ -\sqrt{1-t^2}\,\sqrt{1-\|\boldsymbol{y}\|^2} \end{pmatrix} \frac{1}{\sqrt{1-\|\boldsymbol{y}\|^2}}\,\mathrm{d}\boldsymbol{y}.\end{aligned}$$

We write this in one integral

$$\begin{aligned}\mathcal{V}f(\boldsymbol{\epsilon}^1,t) = \frac{1}{|\mathbb{S}^{d-2}|}\int_{\mathbb{B}^{d-2}} & \left(f\begin{pmatrix} t \\ \sqrt{1-t^2}\,\boldsymbol{y} \\ \sqrt{1-t^2}\,\sqrt{1-\|\boldsymbol{y}\|^2} \end{pmatrix} + f\begin{pmatrix} t \\ \sqrt{1-t^2}\,\boldsymbol{y} \\ -\sqrt{1-t^2}\,\sqrt{1-\|\boldsymbol{y}\|^2} \end{pmatrix}\right) \\ & \cdot \frac{1}{\sqrt{1-\|\boldsymbol{y}\|^2}}\,\mathrm{d}\boldsymbol{y}.\end{aligned}$$

Now we perform the substitution $\boldsymbol{y} \mapsto \boldsymbol{x} \in \mathbb{B}^{d-2}(\sqrt{1-t^2})$ with $\boldsymbol{x} = \sqrt{1-t^2}\,\boldsymbol{y}$ and $\mathrm{d}\boldsymbol{x} = (1-t^2)^{\frac{d-2}{2}}\,\mathrm{d}\boldsymbol{y}$ and obtain

$$\begin{aligned}\mathcal{V}f(\boldsymbol{\epsilon}^1,t) = \frac{(1-t^2)^{\frac{3-d}{2}}}{|\mathbb{S}^{d-2}|}\int_{\mathbb{B}^{d-2}(\sqrt{1-t^2})} & \frac{1}{\sqrt{1-t^2-\|\boldsymbol{x}\|^2}} \\ & \cdot \left(f\begin{pmatrix} t \\ \boldsymbol{x} \\ \sqrt{1-t^2-\|\boldsymbol{x}\|^2} \end{pmatrix} + f\begin{pmatrix} t \\ \boldsymbol{x} \\ -\sqrt{1-t^2-\|\boldsymbol{x}\|^2} \end{pmatrix}\right)\mathrm{d}\boldsymbol{x}.\end{aligned}$$

By the definition of $\widetilde{f}_{(d-1)}$, we have

$$\begin{aligned}\mathcal{V}f(\boldsymbol{\epsilon}^1,t) &= \frac{(1-t^2)^{\frac{3-d}{2}}}{|\mathbb{S}^{d-2}|}\int_{\mathbb{B}^{d-2}(\sqrt{1-t^2})} \widetilde{f}_{(d-1)}\begin{pmatrix} t \\ \boldsymbol{x} \end{pmatrix}\mathrm{d}\boldsymbol{x} \\ &= \frac{(1-t^2)^{\frac{3-d}{2}}}{|\mathbb{S}^{d-2}|}\mathcal{R}\widetilde{f}(\boldsymbol{\epsilon}^1,t).\end{aligned}$$ ■

Theorem 3.46 in the case $d = 3$ was shown by the author in [HQ16].

We obtain the following result about the inversion of the vertical slice transform via the inverse Radon transform $\mathcal{R}^{-1}$.

Theorem 3.47. *Let $d \geq 3$, and let the function $f\colon \mathbb{S}^{d-1} \to \mathbb{C}$ be even in the last coordinate ξ_d as in (3.54). Then f can be reconstructed given its vertical slice transform $\mathcal{V}f$ with the help of the inverse Radon transform $\mathcal{R}^{-1}$ for $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$ via*

$$f(\boldsymbol{\xi}) = \begin{cases} \xi_d\, \mathcal{R}^{-1} g(\xi_1, \ldots, \xi_d), & \xi_d \neq 0, \\ \mathcal{V}f((\xi_1, \ldots, \xi_{d-1}), 1), & \xi_d = 0, \end{cases} \tag{3.70}$$

where

$$g(\boldsymbol{\sigma}, t) = (1-t^2)^{\frac{d-3}{2}} \left|\mathbb{S}^{d-2}\right| \mathcal{V}f(\boldsymbol{\sigma}, t), \qquad \boldsymbol{\sigma} \in \mathbb{S}^{d-2},\ t \in [-1,1].$$

Proof. If the function f is even in the last coordinate ξ_d as in (3.54), then there is a one-to-one relation between the function f and its weighted projection $\widetilde{f}_{(d-1)}$ defined in (3.68). More precisely, f can be uniquely determined from $\widetilde{f}_{(d-1)}$ via

$$f(\boldsymbol{\xi}) = \xi_d\, \widetilde{f}_{(d-1)}(\xi_1, \ldots, \xi_{d-1}), \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1},\ \xi_d \neq 0. \tag{3.71}$$

By (3.69), we have

$$g(\boldsymbol{\sigma}, t) = \mathcal{R}\widetilde{f}_{(d-1)}(\boldsymbol{\sigma}, t).$$

Applying the inverse Radon transform $\mathcal{R}^{-1}$ on the unit ball $\mathbb{B}^{d-1}$, we obtain

$$\widetilde{f}_{(d-1)}(\boldsymbol{x}) = \mathcal{R}^{-1} g(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{B}^{d-1},$$

from which we can calculate f by (3.71). This shows (3.70). ∎

There are many known inversion methods of the Radon transform $\mathcal{R}$, not necessarily restricted to the unit ball $\mathbb{B}^{d-1}$. An overview about such inversion methods is provided in the book of Natterer and Wübbeling [NW00].

The following singular value decomposition of the Radon transform $\mathcal{R}$ was shown by Davison [Dav81] and Louis [Lou84].

Proposition 3.48. *Let $\nu > \frac{d-2}{2}$, $m \in \mathbb{N}_0$, and $l = 0, \ldots, m$ with $m + l$ even, and let $k = 1, \ldots, N_{l,d-1}$. We set for $s \in [0,1]$ and $\boldsymbol{\omega} \in \mathbb{S}^{d-1}$*

$$V^{d,\nu}_{m,l,k}(s\boldsymbol{\omega}) := c(d,m,l,\nu)\,(1-s^2)^{\nu-\frac{d}{2}}\, s^l P^{\left(\nu-\frac{d}{2}, l+\frac{d-2}{2}\right)}_{\frac{m-l}{2}}(2s^2-1)\, Y^k_{l,d}(\boldsymbol{\omega}), \tag{3.72}$$

where

$$c(d,m,l,\nu) := 2^{1-2\nu}\, \pi^{1-\frac{d}{2}}\, \frac{\Gamma(m+2\nu)\,\Gamma(\frac{m-l+2}{2})}{\Gamma(m+1)\,\Gamma(\nu)\,\Gamma(\frac{m-l+2+2\nu-n}{2})},$$

and $P^{(\alpha,\beta)}_n$ denotes the Jacobi polynomial of degree n and orders $\alpha, \beta > -1$, see (2.22). Then, the Radon transform

$$\mathcal{R}\colon L^2\left(\mathbb{B}^d; (1-\|\boldsymbol{x}\|^2)^{\frac{d}{2}-\nu}\right) \to L^2\left(\mathbb{S}^{d-1} \times [-1,1]; (1-s^2)^{\frac{1}{2}-\nu}\right)$$

has the singular value decomposition

$$\mathcal{R}V^{d,\nu}_{m,l,k}(\boldsymbol{\omega}, s) = (1-s^2)^{\nu-\frac{1}{2}}\, C^{(\nu)}_m(s)\, Y^k_{l,d}(\boldsymbol{\omega}). \tag{3.73}$$

Remark 3.49. The orthogonal polynomials $V_{m,l,k}^{d,\nu}$ on the unit ball $\mathbb{B}^d$ with respect to the weight function $(1-\|\boldsymbol{x}\|^2)^{\frac{d}{2}-\nu}$ from (3.72) are well-known, cf. [DX14, Section 5.2]. In the case $d=2$ and $\nu=1$, they are also called disk polynomials or Zernike polynomials, see [Wün05]. $\square$

The singular value decomposition has proven to be a powerful tool for the inversion of the Radon transform on the unit ball $\mathbb{B}^{d-1}$, see e. g. [Xu07]. Using Theorem 3.46, we are going to apply this singular value decomposition for the vertical slice transform $\mathcal{V}$ in the following. We will see that this yields the singular value decomposition of the vertical slice transform we have already obtained in Theorem 3.43.

Remark 3.50. We apply Theorem 3.46 by plugging in the singular functions (3.72) of the Radon transform for $\nu=\frac{d-2}{2}$ as follows. Let $m\in\mathbb{N}_0$, $l=0,\ldots,m$ with $m+l$ even, and $k=1,\ldots,N_{l,d-1}$. We set

$$\widetilde{f}_{(d-1)}(\boldsymbol{x})=V_{m,l,k}^{d-1,\frac{d-2}{2}}(\boldsymbol{x}),\qquad \boldsymbol{x}\in\mathbb{B}^{d-1}.$$

Then we have by (3.71)

$$f(\boldsymbol{\xi})=\xi_d\,V_{m,l,k}^{d-1,\frac{d-2}{2}}(\xi_1,\ldots,\xi_{d-1}),\qquad \boldsymbol{\xi}\in\mathbb{S}^{d-1},\ \xi_d\neq 0.$$

Let $\boldsymbol{\sigma}\in\mathbb{S}^{d-1}$ and $t\in(-1,1)$. By (3.69) and the singular value decomposition (3.73) of the Radon transform for dimension $d-1$ and with $\nu=\frac{d-2}{2}$, we obtain

$$\begin{aligned}\mathcal{V}f(\boldsymbol{\sigma},t)&=\frac{(1-t^2)^{\frac{3-d}{2}}}{|\mathbb{S}^{d-2}|}\,\mathcal{R}\widetilde{f}_{(d-1)}(\boldsymbol{\sigma},t)\\&=\frac{1}{|\mathbb{S}^{d-2}|}\,C_m^{(\frac{d-2}{2})}(\boldsymbol{\sigma})\,Y_{l,d-1}^k(t).\end{aligned}$$

By the relation (2.19) between the Legendre and Gegenbauer polynomials, we see that

$$\mathcal{V}f(\boldsymbol{\sigma},t)=\frac{1}{|\mathbb{S}^{d-2}|}\binom{n+d-3}{n}P_{m,d}(t)\,Y_{l,d-1}^k(\boldsymbol{\sigma}).\tag{3.74}$$

On the other hand, the vertical slice transform applied to the spherical harmonic $Y_{m,d}^{l,k}$ is by (3.58)

$$\mathcal{V}Y_{m,d}^{l,k}=\widetilde{P}_{m,d}^l(0)\,P_{m,d}(t)\,Y_{l,d-1}^k(\boldsymbol{\sigma}).\tag{3.75}$$

We see that the right hand side of (3.74) is a nonzero multiple of the right hand side of (3.75). Considering the injectivity of the vertical slice transform $\mathcal{V}$, we obtain that f must be a nonzero multiple of the spherical harmonic $Y_{m,d}^{l,k}$. Hence, we have shown that inserting the singular value decomposition to Theorem 3.46 yields the singular value decomposition of the vertical slice transform that was already shown in a different way in Theorem 3.43.

In 2018, Rubin [Rub18] used the just-mentioned approach to calculate a singular value decomposition of the vertical slice transform $\mathcal{V}$. This result coincides with Theorem 3.43

for $L^2(\mathbb{S}^{d-1})$. Furthermore, Rubin's paper also contains the singular value decomposition of $\mathcal{V}$ for the weighted space $L^2(\mathbb{S}^{d-1}; W)$ with the weight function $W(\boldsymbol{\xi}) = \xi_d^{d-2\nu-2}$, $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$, for $\nu > \frac{d-3}{2}$. □

Remark 3.51. In the article [HQ16], we have compared three approaches for the numerical reconstruction of a function f on the two-sphere $\mathbb{S}^2$ given its vertical slice transform $\mathcal{V}f$. Firstly, the singular value decomposition (3.56) was implemented with the help of the fast spherical Fourier transform, see also Remark 3.15. Secondly, we used Theorem 3.47, where we inserted the algorithm [XT07] for the inverse Radon transform on the unit disk $\mathbb{B}^2$ via orthogonal polynomials. Thirdly, we used Theorem 3.47 again, but this time with the standard filtered back-projection for the inverse Radon transform.

The numerical tests showed that the filtered back-projection performed worst among these algorithms. It shows especially high errors if the test function f does not vanish near the equator, which corresponds to the fact that the projection $\widetilde{f}_{(2)}$, which was defined in (3.68), does not vanish near the boundary of the unit disk $\mathbb{B}^2$. The reconstruction errors of the other two algorithms are closer together, with the singular value decomposition (3.56) still performing a little ahead. □

3.6 Hyperplane sections through a common point

While the previous section considered the restriction of the mean operator $\mathcal{M}$ to the family of subspheres with centers on a great circle, in this section, we are going to take a look at subspheres obtained by the intersection of the sphere $\mathbb{S}^{d-1}$ with hyperplanes that meet in a fixed point $\boldsymbol{\zeta} \in \mathbb{R}^d$. By rotational symmetry, we can assume that this common point $\boldsymbol{\zeta}$ lies on the ξ_d axis, i. e.

$$\boldsymbol{\zeta} = z\boldsymbol{\epsilon}^d = (0, \dots, 0, z)^\top$$

with some $z \in \mathbb{R}$. We are going to call the mean operator restricted to these subspheres the *spherical transform*

$$\mathcal{U}_z f(\boldsymbol{\xi}) := \mathcal{M} f(\boldsymbol{\xi}, z\xi_d), \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1}. \tag{3.76}$$

The domains of integration of the spherical transform $\mathcal{U}_z$ are the subspheres

$$\mathscr{C}_z^{\boldsymbol{\xi}} := \{\boldsymbol{\eta} \in \mathbb{S}^{d-1} \,;\, \boldsymbol{\eta}^\top \boldsymbol{\xi} = z\xi_d\}, \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1},\ z \in \mathbb{R}.$$

A visualization of circles $\mathscr{C}_z^{\boldsymbol{\xi}}$ on the two-sphere $\mathbb{S}^2$ is found in Figure 3.7. We define an orientation on $\mathscr{C}_z^{\boldsymbol{\xi}}$ by saying that a basis $\boldsymbol{e}^1, \dots, \boldsymbol{e}^{d-2}$ of the tangent space $T_{\boldsymbol{\eta}} \mathscr{C}_z^{\boldsymbol{\xi}}$ at a point $\boldsymbol{\eta} \in C_z^{\boldsymbol{\xi}}$ is oriented positively if

$$\det\left[\boldsymbol{\eta}, \boldsymbol{\xi}, \boldsymbol{e}^1, \dots, \boldsymbol{e}^{d-2}\right] > 0.$$

We have

$$\mathcal{U}_z f(\boldsymbol{\xi}) = \frac{1}{\left|\mathscr{C}_z^{\boldsymbol{\xi}}\right|} \int_{\mathscr{C}_z^{\boldsymbol{\xi}}} f(\boldsymbol{\eta}) \,\mathrm{d}\mathscr{C}_z^{\boldsymbol{\xi}}(\boldsymbol{\eta}), \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1},$$

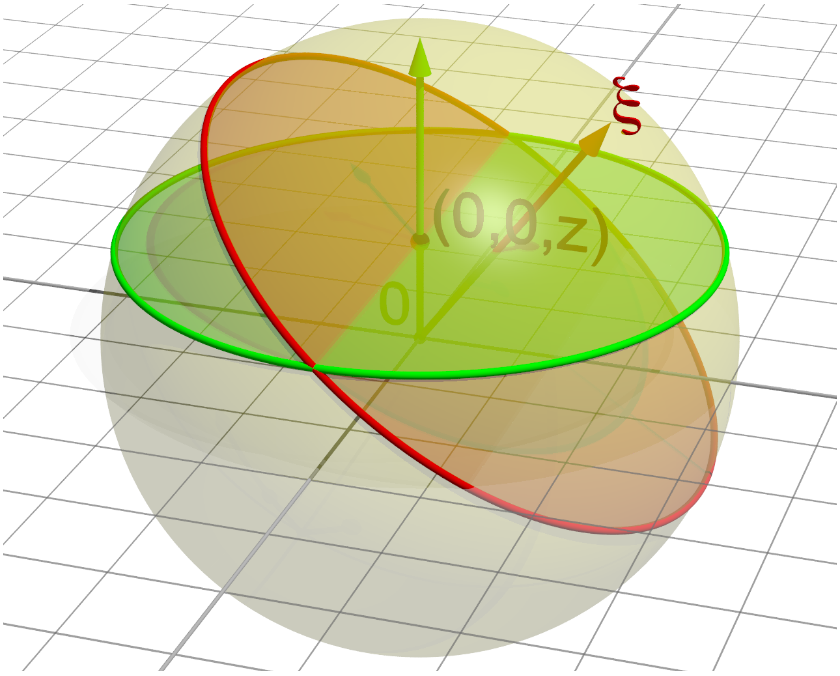

Figure 3.7: Visualization of the plane sections $\mathscr{C}_z^{\boldsymbol{\epsilon}^3}$ (green) and $\mathscr{C}_z^{\boldsymbol{\xi}}$ for some vector $\boldsymbol{\xi}$ (red) on the two-sphere $\mathbb{S}^2$.

where

$$\left|\mathscr{C}_z^{\boldsymbol{\xi}}\right| = (1 - z^2\xi_d^2)^{\frac{d-2}{2}} \left|\mathbb{S}^{d-2}\right|$$

denotes the $(d-2)$-dimensional volume of the subsphere $\mathscr{C}_z^{\boldsymbol{\xi}}$.

We notice the following special cases of the spherical transform $\mathcal{U}_z$. For $z = 0$, we see that $\boldsymbol{\zeta}$ is the center of the sphere and thus we have that $\mathcal{U}_0 = \mathcal{F}$ is the Funk–Radon transform $\mathcal{F}$ from Section 3.2. In the case $z = 1$, we see that $\boldsymbol{\zeta} = \boldsymbol{\epsilon}^d$ is the north pole of the sphere and, hence, the spherical transform $\mathcal{U}_1$ takes the integrals along all subspheres of $\mathbb{S}^{d-1}$ that contain the north pole $\boldsymbol{\epsilon}^d$. This case of $\mathcal{U}_1$ is known as the spherical slice transform, which will be the subject of Section 3.6.5. Furthermore, for $z \to \infty$, we remark that $\mathcal{U}_z$ approaches the vertical slice transform $\mathcal{V}$, see Section 3.5.3.

In this section, we consider the case that $z \in (-1, 1)$, i. e., that the point $\boldsymbol{\zeta} = z\boldsymbol{\epsilon}^d$ is in the unit ball $\mathbb{B}^d$. Most of the material in this section is submitted for publication, see the preprint [Que18].

The subspheres $\mathscr{C}_z^{\boldsymbol{\xi}}$, along which we integrate for the spherical transform $\mathcal{U}_z$, can also be imagined in the following way.

Lemma 3.52. *Let $z \in (-1, 1)$ and $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$. Then the center of the subsphere $\mathscr{C}_z^{\boldsymbol{\xi}}$ is located on the sphere with center $\frac{z}{2}\boldsymbol{\epsilon}^d$ and radius $\frac{z}{2}$, see Figure 3.8. Equivalently, this is exactly the sphere that contains the origin and $z\boldsymbol{\epsilon}^d$ and that is rotationally symmetric about the North–South axis.*

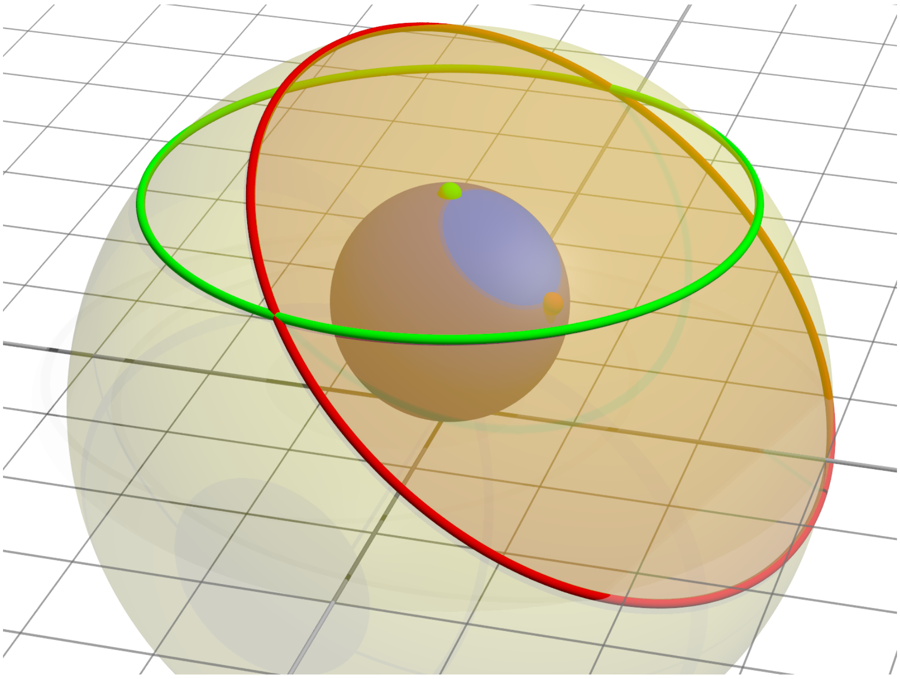

Figure 3.8: Two plane sections $\mathscr{C}_z^{\boldsymbol{\xi}}$ (red and green) of the unit sphere $\mathbb{S}^2$ and their respective centers, which are located on the blue sphere with center $\frac{z}{2}\boldsymbol{\epsilon}^{d+1}$ and radius $\frac{z}{2}$.

Proof. The center of the subsphere $\mathscr{C}_z^{\boldsymbol{\xi}}$ is given by $z\xi_d\boldsymbol{\xi}$. Then the squared distance between the point $z\xi_d\boldsymbol{\xi}$ and the center $\frac{z}{2}\boldsymbol{\epsilon}^d$ is

$$\begin{aligned}\left\|z\xi_d\boldsymbol{\xi}-\frac{z}{2}\boldsymbol{\epsilon}^d\right\|^2 &= \sum_{i=1}^{d}\xi_i^2z^2\xi_d^2+\left(z\xi_d^2-\frac{z}{2}\right)^2\\ &=(1-\xi_d^2)z^2\xi_d^2+\left(z\xi_d^2-\frac{z}{2}\right)^2\\ &=\frac{z^2}{4}.\end{aligned}$$

This implies that $z\xi_d\boldsymbol{\xi}$ is located on the sphere with center $\frac{z}{2}\boldsymbol{\epsilon}^d$ and radius $\frac{z}{2}$. ∎

The spherical transform $\mathcal{U}_z$ was first investigated in 2016 by Salman [Sal16], who showed an inversion formula. In 2017, this result was extended to general dimension $d\geq 3$ and also the smoothness requirement was lowered to functions in $C^1(\mathbb{S}^{d-1})$ instead of $C^\infty(\mathbb{S}^2)$, see [Sal17]. Following Salman's notation, we call $\mathcal{U}_z$ the spherical transform.

We present here the inversion formula of $\mathcal{U}_z$ obtained in [Sal17], which relies on the stereographic projection

$$\boldsymbol{\pi}\colon \mathbb{S}^{d-1}\setminus\{\boldsymbol{\epsilon}^d\}\to\mathbb{R}^{d-1},\quad \boldsymbol{\xi}\mapsto\sum_{i=1}^{d-1}\frac{\xi_i}{1-\xi_d}\boldsymbol{\epsilon}^i. \tag{3.77}$$

from the sphere to the equatorial hyperplane. The inverse stereographic projection is given by

$$\boldsymbol{\pi}^{-1} \colon \mathbb{R}^{d-1} \to \mathbb{S}^{d-1} \setminus \{\boldsymbol{\epsilon}^d\}, \quad \boldsymbol{x} \mapsto \frac{2\boldsymbol{x} + (\|\boldsymbol{x}\|^2 - 1)\,\boldsymbol{\epsilon}^d}{1 + \|\boldsymbol{x}\|^2}. \tag{3.78}$$

Proposition 3.53 ([Sal17]). *Let $0 \le z < 1$, $\sigma := \sqrt{\frac{1+z}{1-z}}$, and let $f \in C^1(\mathbb{S}^{d-1})$ have a compact support strictly inside the spherical cap $\{\boldsymbol{\xi} \in \mathbb{S}^{d-1}\,;\, -1 \le \xi_d \le z\}$. We write a vector $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$ as*

$$\boldsymbol{\xi}(\boldsymbol{\phi}, \theta) = (\cos\theta)\,\boldsymbol{\phi} + (\sin\theta)\,\boldsymbol{\epsilon}^d, \qquad \boldsymbol{\phi} \in \mathbb{S}^{d-2},\ \theta \in [0, \tfrac{\pi}{2}).$$

Then, for $d \ge 3$ odd and $\boldsymbol{x} \in \mathbb{R}^{d-1}$, we have

$$\begin{aligned}
&\left(f \circ \boldsymbol{\pi}^{-1}\right)\left(\frac{2\sigma\boldsymbol{x}}{1 + \sqrt{1 + 4|\boldsymbol{x}|^2}}\right) \\
&= \frac{(-1)^{\frac{d-3}{2}}(1-z)\sqrt{1+4|\boldsymbol{x}|^2}}{2^{3d-5}\,\pi^{d-1}\,\sigma^{d-3}} \left(\frac{\left(1+\sqrt{1+4|\boldsymbol{x}|^2}\right)^2 + 4\sigma^2|\boldsymbol{x}|^2}{1+\sqrt{1+4|\boldsymbol{x}|^2}}\right)^{d-2} \left|\mathbb{S}^{d-2}\right| \\
&\quad \Delta_{\boldsymbol{x}}^{\frac{d-1}{2}} \int_{\mathbb{S}^{d-2}} \int_0^{\frac{\pi}{2}} \mathcal{U}_z f(\boldsymbol{\xi}(\boldsymbol{\phi},\theta)) \log\left|\boldsymbol{x}^\top\boldsymbol{\phi} - \frac{\sqrt{1-z^2}}{2}\tan\theta\right| \frac{(1 - z^2\sin^2\theta)^{\frac{d-3}{2}}}{\cos\theta}\,\mathrm{d}\theta\,\mathrm{d}\boldsymbol{\phi},
\end{aligned}$$

and, for $d \ge 4$ even and $\boldsymbol{x} \in \mathbb{R}^{d-1}$, we have

$$\begin{aligned}
&\left(f \circ \boldsymbol{\pi}^{-1}\right)\left(\frac{2\sigma\boldsymbol{x}}{1 + \sqrt{1 + 4|\boldsymbol{x}|^2}}\right) \\
&= \frac{(-1)^{\frac{d-2}{2}}(1-z)\sqrt{1+4|\boldsymbol{x}|^2}}{2^{3d-4}\,\pi^{d-2}\,\sigma^{d-3}} \left(\frac{\left(1+\sqrt{1+4|\boldsymbol{x}|^2}\right)^2 + 4\sigma^2|\boldsymbol{x}|^2}{1+\sqrt{1+4|\boldsymbol{x}|^2}}\right)^{d-2} \left|\mathbb{S}^{d-2}\right| \\
&\quad \Delta_{\boldsymbol{x}}^{\frac{d}{2}} \int_{\mathbb{S}^{d-2}} \int_0^{\frac{\pi}{2}} \mathcal{U}_z f(\boldsymbol{\xi}(\boldsymbol{\phi},\theta)) \left|\boldsymbol{x}^\top\boldsymbol{\phi} - \frac{\sqrt{1-z^2}}{2}\tan\theta\right| (1 - z^2\sin^2\theta)^{\frac{d-3}{2}} \frac{\mathrm{d}\theta\,\mathrm{d}\boldsymbol{\phi}}{\cos\theta},
\end{aligned}$$

where $\Delta_{\boldsymbol{x}}$ denotes the Laplacian with respect to $\boldsymbol{x} \in \mathbb{R}^{d-1}$.

The main tool to obtain these inversion formulas in Proposition 3.53 is the stereographic projection, which turns the subspheres of the sphere $\mathbb{S}^{d-1}$ into spheres in the equatorial hyperplane $\mathbb{R}^{d-1}$. Then the problem of inverting $\mathcal{U}_z$ is converted to the inversion of the spherical-mean Radon transform in $\mathbb{R}^{d-1}$ on a certain family of spheres.

3.6.1 Connection with the Funk–Radon transform

In this section, we will derive a connection of the spherical transform $\mathcal{U}_z$ with the Funk–Radon transform $\mathcal{F}$. In particular, we will find that the factorization

$$\mathcal{U}_z = \mathcal{N}_z \mathcal{F} \mathcal{M}_z$$

holds, where $\mathcal{M}_z$ and $\mathcal{N}_z$ are operators from $L^2(\mathbb{S}^{d-1})$ to itself that obey a simple structure. The main advantage of this approach is that we can transfer many of the well-known results about the Funk–Radon transform to the spherical transform $\mathcal{U}_z$. In order to obtain this factorization, we start with investigating two transformations that map the sphere $\mathbb{S}^{d-1}$ to itself.

Two mappings on the sphere

Let $z \in (-1, 1)$. We define the transformations $\boldsymbol{h}_z, \boldsymbol{g}_z \colon \mathbb{S}^{d-1} \to \mathbb{S}^{d-1}$ by

$$\boldsymbol{h}_z(\boldsymbol{\eta}) := \sum_{i=1}^{d-1} \frac{\sqrt{1-z^2}}{1+z\eta_d} \eta_i \boldsymbol{\epsilon}^i + \frac{z+\eta_d}{1+z\eta_d} \boldsymbol{\epsilon}^d \tag{3.79}$$

and

$$\boldsymbol{g}_z(\boldsymbol{\xi}) := \frac{1}{\sqrt{1-z^2\xi_d^2}} \left(\sum_{i=1}^{d-1} \xi_i \boldsymbol{\epsilon}^i + \sqrt{1-z^2} \xi_d \boldsymbol{\epsilon}^d \right). \tag{3.80}$$

We note that $\boldsymbol{g}_z$ can also be defined for the complex numbers $z = \mathrm{i}x$, where $x \in \mathbb{R}$.

Corollary 3.54. The definitions of both $\boldsymbol{h}_z$ and $\boldsymbol{g}_z$ rely only on the d-th coordinate. The values in the other coordinates are just multiplied with the same factor in order to make the vectors stay on the sphere. Let $z, w \in (-1, 1)$. We have

$$\boldsymbol{h}_z(\boldsymbol{h}_w(\boldsymbol{\xi})) = \boldsymbol{h}_{\frac{z+w}{1+zw}}(\boldsymbol{\xi}), \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1} \tag{3.81}$$

and

$$\boldsymbol{g}_z(\boldsymbol{g}_w(\boldsymbol{\xi})) = \boldsymbol{g}_{\sqrt{z^2+w^2-z^2w^2}}(\boldsymbol{\xi}), \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1}. \tag{3.82}$$

Furthermore, the transformations $\boldsymbol{h}_z$ and $\boldsymbol{g}_z$ are bijective with their respective inverses given by

$$\boldsymbol{h}_z^{-1}(\boldsymbol{\omega}) = \boldsymbol{h}_{-z}(\boldsymbol{\omega}) = \sum_{i=1}^{d-1} \frac{\sqrt{1-z^2}}{1-z\omega_d} \omega_i \boldsymbol{\epsilon}^i + \frac{\omega_d - z}{1-z\omega_d} \boldsymbol{\epsilon}^d \tag{3.83}$$

and

$$\boldsymbol{g}_z^{-1}(\boldsymbol{\omega}) = \boldsymbol{g}_{\frac{\mathrm{i}z}{\sqrt{1-z^2}}}(\boldsymbol{\omega}) = \frac{1}{\sqrt{1-z^2+z^2\omega_d^2}} \left(\sum_{i=1}^{d-1} \sqrt{1-z^2}\, \omega_i \boldsymbol{\epsilon}^i + \omega_d \boldsymbol{\epsilon}^d \right). \tag{3.84}$$

In particular, both the set $\{\boldsymbol{h}_z \ ; \ z \in (-1, 1)\}$ and $\{\boldsymbol{g}_z \ ; \ z \in (-1, 1) \cup \mathrm{i}\mathbb{R}\}$ are groups together with the composition as group operation.

Proof. Let $z, w \in (-1, 1)$, and let $\boldsymbol{\eta} \in \mathbb{S}^{d-1}$. We set

$$h_z(t) := \frac{z+t}{1+zt}, \qquad t \in (-1, 1).$$

as the d-th component of the function $\boldsymbol{h}_z$. As in (2.7), we write

$$\boldsymbol{\eta} = \sqrt{1-t^2}\,\boldsymbol{\eta}_{(d-1)} + t\,\boldsymbol{\epsilon}^d, \qquad \boldsymbol{\eta}_{(d-1)} \in \mathbb{S}^{d-2},\ t \in [-1,1].$$

Then we see that

$$\boldsymbol{h}_z(\boldsymbol{\eta}) = \sqrt{1-h_z(t)^2}\,\boldsymbol{\eta}_{(d-1)} + h_z(t)\,\boldsymbol{\epsilon}^d.$$

We have for $t \in (-1,1)$

$$\begin{aligned} h_w(h_z(t)) &= \frac{h_z(t)+w}{1+wh_z(t)} \\ &= \frac{\frac{z+t}{1+zt}+w}{1+w\frac{z+\eta_d}{1+zt}} \\ &= \frac{z+\eta_d+w(1+zt)}{1+zt+w(z+t)} \\ &= \frac{z+w+(1+wz)\eta_d}{1+zw+(z+w)t} \\ &= \frac{\frac{z+w}{1+zw}+t}{1+\frac{z+w}{1+zw}t} = h_{\frac{z+w}{1+zw}}(t). \end{aligned}$$

Since $\boldsymbol{h}_z$ is already determined by its d-th coordinate, this implies (3.81). We see that also $h_z(h_{-z}(t)) = h_{-z}(h_z(t)) = h_0(t) = t$, which implies (3.83). Hence, $\{\boldsymbol{h}_z\ ;\ z \in (-1,1)\}$ is a group since for every element there exists an inverse, where the neutral element $\boldsymbol{h}_0$ is the identical map and the composition $\circ$ is associative.

Analogously to the first part of the proof, we define

$$g_z(t) := \sqrt{\frac{1-z}{1-zt^2}}\,t, \qquad t \in (-1,1),$$

which is the d-th component of $\boldsymbol{g}_z$. We have for $t \in (-1,1)$

$$\begin{aligned} g_w(g_z(t)) &= \sqrt{\frac{1-w^2}{1-w^2 g_z(t)^2}}\,g_z(t) \\ &= \sqrt{\frac{1-w^2}{1-w^2\frac{1-z^2}{1-z^2t^2}\,t}}\sqrt{\frac{1-z^2}{1-z^2t^2}}\,t \\ &= \sqrt{\frac{(1-w^2)(1-z^2)}{1-z^2t^2-w^2(1-z^2)\,t^2}}\,t \\ &= \sqrt{\frac{1-(z^2+w^2-z^2w^2)}{1-z^2t^2-w^2(1-z^2)\,t^2}}\,t \\ &= g_{\sqrt{z^2+w^2-z^2w^2}}(t), \end{aligned}$$

which implies (3.82). Furthermore, we obtain that

$$g_{\frac{\mathrm{i}z}{\sqrt{1-z^2}}}(g_z(t)) = g_{\sqrt{z^2-\frac{z^2}{1-z^2}-\frac{z^4}{1-z^2}}}(t) = g_0(t) = 1, \qquad t \in (-1,1),$$

which implies (3.84).

We see that if $z \in (-1,1)$, then we have $\frac{\mathrm{i}z}{\sqrt{1-z^2}} \in \mathrm{i}\,\mathbb{R}$. Conversely, if $z \in \mathrm{i}\,\mathbb{R}$, i. e., $z = \mathrm{i}x$ for some $x \in \mathbb{R}$, then we have $\frac{\mathrm{i}z}{\sqrt{1-z^2}} = \frac{-x}{\sqrt{1+x^2}} \in (-1,1)$. Hence, every element of the set $\{\boldsymbol{g}_z \,;\, z \in (-1,1) \cup \mathrm{i}\mathbb{R}\}$ has an inverse with respect to composition, so this set is a group. ■

The following lemma is the key observation that connects the domains of integration of the spherical transform $\mathcal{U}_z$ and the Funk–Radon transform $\mathcal{F}$. It shows that the inverse of $\boldsymbol{h}_z$ applied to the subsphere $\mathscr{C}_z^{\boldsymbol{\xi}}$ yields a maximal subsphere of $\mathbb{S}^{d-1}$ with the normal vector $\boldsymbol{g}_z(\boldsymbol{\xi})$.

Lemma 3.55. *Let* $z \in (-1,1)$ *and* $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$. *Then*

$$\boldsymbol{h}_z^{-1}(\mathscr{C}_z^{\boldsymbol{\xi}}) = \mathscr{C}_0^{\boldsymbol{g}_z(\boldsymbol{\xi})}. \tag{3.85}$$

Proof. Let $\boldsymbol{\eta} \in \mathbb{S}^{d-1}$. Then $\boldsymbol{\eta}$ lies in $\boldsymbol{h}_z^{-1}(\mathscr{C}_z^{\boldsymbol{\xi}})$ if and only if $\boldsymbol{h}_z(\boldsymbol{\eta}) \in \mathscr{C}_z^{\boldsymbol{\xi}}$, i. e.,

$$\langle \boldsymbol{h}_z(\boldsymbol{\eta}), \boldsymbol{\xi} \rangle = z\xi_d.$$

By the definition of $\boldsymbol{h}_z$ in (3.79), this is equivalent to

$$\langle \boldsymbol{h}_z(\boldsymbol{\eta}), \boldsymbol{\xi} \rangle = \sum_{i=1}^{d-1} \frac{\sqrt{1-z^2}}{1+z\eta_d}\eta_i\xi_i + \frac{z+\eta_d}{1+z\eta_d}\xi_d = z\xi_d.$$

After subtracting the right-hand side from the last equation, we have

$$\sum_{i=1}^{d-1} \frac{\sqrt{1-z^2}}{1+z\eta_d}\eta_i\xi_i + \frac{1-z^2}{1+z\eta_d}\eta_d\xi_d = 0.$$

Multiplication with $(1+z\eta_d)(1-z^2)^{-1/2}(1-z^2\xi_d^2)^{-1/2}$ yields

$$\sum_{i=1}^{d-1} \frac{1}{\sqrt{1-z^2\xi_d^2}}\eta_i\xi_i + \frac{\sqrt{1-z^2}}{\sqrt{1-z^2\xi_d^2}}\eta_d\xi_d = 0,$$

which is equivalent to $\langle \boldsymbol{\eta}, \boldsymbol{g}_z(\boldsymbol{\xi}) \rangle = 0$, so we obtain that $\boldsymbol{\eta} \in \mathscr{C}_0^{\boldsymbol{g}_z(\boldsymbol{\xi})}$. ■

The second ingredient of the connection with the Funk–Radon transform is the following lemma. It says how the application of $\boldsymbol{h}_z$ transfers the volume measure.

Lemma 3.56. *Let $z \in (-1,1)$ and $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$. Denote by $\mathrm{d}\mathscr{C}_z^{\boldsymbol{\xi}}$ and $\mathrm{d}\mathscr{C}_0^{\boldsymbol{g}_z(\boldsymbol{\xi})}$ the volume forms on the manifolds $\mathscr{C}_z^{\boldsymbol{\xi}}$ and $\mathscr{C}_0^{\boldsymbol{g}_z(\boldsymbol{\xi})}$, respectively. Then the following relation between the pullback of the volume form $\mathrm{d}\mathscr{C}_z^{\boldsymbol{\xi}}$ over $\boldsymbol{h}_z$ and $\mathrm{d}\mathscr{C}_0^{\boldsymbol{g}_z(\boldsymbol{\xi})}$ holds. For $\boldsymbol{\eta} \in \mathscr{C}_0^{\boldsymbol{g}_z(\boldsymbol{\xi})}$, we have*

$$[\boldsymbol{h}_z^*(\mathrm{d}\mathscr{C}_z^{\boldsymbol{\xi}})](\boldsymbol{\eta}) = \left(\frac{\sqrt{1-z^2}}{1+z\eta_d}\right)^{d-2} \mathrm{d}\mathscr{C}_0^{g_z(\boldsymbol{\xi})}(\boldsymbol{\eta}). \tag{3.86}$$

For the volume form $\mathrm{d}\mathbb{S}^{d-1}$ on the sphere and $\boldsymbol{\eta} \in \mathbb{S}^{d-1}$, we have

$$[\boldsymbol{h}_z^*(\mathrm{d}\mathbb{S}^{d-1})](\boldsymbol{\eta}) = \left(\frac{\sqrt{1-z^2}}{1+z\eta_d}\right)^{d-1} \mathrm{d}\mathbb{S}^{d-1}(\boldsymbol{\eta}). \tag{3.87}$$

Furthermore, the map $\boldsymbol{h}_z \colon \mathbb{S}^{d-1} \to \mathbb{S}^{d-1}$ is conformal.

Proof. Let $\boldsymbol{\eta} \in \mathscr{C}_0^{\boldsymbol{g}_z(\boldsymbol{\xi})}$. We compute the Jacobian $J_{\boldsymbol{h}_z} = J_{\boldsymbol{h}_z}(\boldsymbol{\eta})$ of $\boldsymbol{h}_z$ at $\boldsymbol{\eta}$, which comprises the partial derivatives of $\boldsymbol{h}_z$. For all $l, m \in \{1, \ldots, d-1\}$, we have

$$\begin{aligned} \frac{\partial [\boldsymbol{h}_z]_l}{\partial \eta_m} &= \frac{\sqrt{1-z^2}}{1+z\eta_d}\delta_{l,m}, & \frac{\partial [\boldsymbol{h}_z]_l}{\partial \eta_d} &= -\eta_l \frac{z\sqrt{1-z^2}}{(1+z\eta_d)^2}, \\ \frac{\partial [\boldsymbol{h}_z]_d}{\partial \eta_m} &= 0, & \frac{\partial [\boldsymbol{h}_z]_d}{\partial \eta_d} &= \frac{1-z^2}{(1+z\eta_d)^2}. \end{aligned} \tag{3.88}$$

Let $[\boldsymbol{e}^i]_{i=1}^{d-2}$ be an orthonormal basis of the tangent space $T_{\boldsymbol{\eta}}\mathscr{C}_0^{\boldsymbol{g}_z(\boldsymbol{\xi})}$. Then $J_{\boldsymbol{h}_z}\boldsymbol{e}^i \in T_{\boldsymbol{h}_z(\boldsymbol{\eta})}\mathscr{C}_z^{\boldsymbol{\xi}}$ for $i = 1, \ldots, d-1$ is given by

$$J_{\boldsymbol{h}_z}\boldsymbol{e}^i = \sum_{l=1}^{d-1} \left(\frac{\sqrt{1-z^2}}{1+z\eta_d} e_l^i - \eta_l \frac{z\sqrt{1-z^2}}{(1+z\eta_d)^2} e_d^i \right) \boldsymbol{\epsilon}^l + \frac{1-z^2}{(1+z\eta_d)^2} e_d^i \boldsymbol{\epsilon}^d.$$

Hence, we have for all $i, j \in \{1, \ldots, d-2\}$

$$\begin{aligned} \left\langle J_{h_z}\boldsymbol{e}^i, J_{h_z}\boldsymbol{e}^j \right\rangle = \sum_{l=1}^{d-1} \Bigg(&\frac{1-z^2}{(1+z\eta_d)^2} e_l^i e_l^j - \frac{z(1-z^2)}{(1+z\eta_d)^3} \eta_l \left(e_l^i e_d^j + e_l^j e_d^i \right) \\ &+ \frac{z^2(1-z^2)}{(1+z\eta_d)^4} \eta_l^2 e_d^i e_d^j \Bigg) + \frac{(1-z^2)^2}{(1+z\eta_d)^4} e_d^i e_d^j. \end{aligned}$$

Expanding the sum, we obtain

$$\begin{aligned} \left\langle J_{\boldsymbol{h}_z}\boldsymbol{e}^i, J_{\boldsymbol{h}_z}\boldsymbol{e}^j \right\rangle = &\frac{1-z^2}{(1+z\eta_d)^2} \sum_{l=1}^{d-1} e_l^i e_l^j - \frac{z(1-z^2)}{(1+z\eta_d)^3} \left(e_d^j \sum_{l=1}^{d-1} \eta_l e_l^i + e_d^i \sum_{l=1}^{d-1} \eta_l e_l^j \right) \\ &+ \frac{z^2(1-z^2)}{(1+z\eta_d)^4} e_d^i e_d^j \sum_{l=1}^{d-1} \eta_l^2 + \frac{(1-z^2)^2}{(1+z\eta_d)^4} e_d^i e_d^j. \end{aligned}$$

Since the vectors $\boldsymbol{e}^i$ and $\boldsymbol{e}^j$ are elements of an orthonormal basis, we have $\langle \boldsymbol{e}^i, \boldsymbol{e}^j \rangle = \sum_{l=1}^d e_l^i e_l^j = \delta_{i,j}$. Furthermore, we know that $\langle \boldsymbol{e}^i, \boldsymbol{\eta} \rangle = \langle \boldsymbol{e}^j, \boldsymbol{\eta} \rangle = 0$ because $\boldsymbol{e}^i$ and $\boldsymbol{e}^j$ are in the tangent space $T_{\boldsymbol{\eta}} \mathscr{C}_0^{g(\boldsymbol{\xi})} \subset T_{\boldsymbol{\eta}} \mathbb{S}^{d-1}$, and also $\|\boldsymbol{\eta}\|^2 = 1$. Hence, we have

$$\begin{aligned}
&\left\langle J_{\boldsymbol{h}_z} \boldsymbol{e}^i, J_{\boldsymbol{h}_z} \boldsymbol{e}^j \right\rangle \\
&= \frac{1-z^2}{(1+z\eta_d)^2} (\delta_{i,j} - e_d^i e_d^j) + 2 \frac{z(1-z^2)}{(1+z\eta_d)^3} \eta_d e_d^i e_d^j \\
&\quad + \frac{z^2(1-z^2)}{(1+z\eta_d)^4} e_d^i e_d^j (1-\eta_d^2) + \frac{(1-z^2)^2}{(1+z\eta_d)^4} e_d^i e_d^j \\
&= \frac{1-z^2}{(1+z\eta_d)^4} e_d^i e_d^j \left(-(1+z\eta_d)^2 + 2z\eta_d(1+z\eta_d) + z^2(1-\eta_d^2) + 1 - z^2 \right) \\
&\quad + \frac{1-z^2}{(1+z\eta_d)^2} \delta_{i,j} \\
&= \frac{1-z^2}{(1+z\eta_d)^2} \delta_{i,j}.
\end{aligned}$$

The above computation shows that the vectors $\{J_{\boldsymbol{h}_z} \boldsymbol{e}^i\}_{i=1}^d$ are orthogonal with length

$$\left\| J_{\boldsymbol{h}_z} \boldsymbol{e}^i \right\| = \frac{\sqrt{1-z^2}}{1+z\eta_d}.$$

Because of the orthogonality of the vectors $J_{\boldsymbol{h}_z} \boldsymbol{e}^i$, the map $\boldsymbol{h}_z$ is conformal. By the definition of the pullback $\boldsymbol{h}_z^*$ in (2.1) and the fact that the volume form $\mathrm{d}\mathscr{C}_z^{\boldsymbol{\xi}}$ is a multilinear $(d-2)$-form, we obtain

$$[\boldsymbol{h}_z^*(\mathrm{d}\mathscr{C}_z^{\boldsymbol{\xi}})](\boldsymbol{\eta})([\boldsymbol{e}^i]_{i=1}^{d-2}) = [\mathrm{d}\mathscr{C}_z^{\boldsymbol{\xi}}](\boldsymbol{h}_z(\boldsymbol{\eta}))([J_{\boldsymbol{h}_z} \boldsymbol{e}^i]_{i=1}^{d-2}) = \left(\frac{\sqrt{1-z^2}}{1+z\eta_d} \right)^{d-2}. \tag{3.89}$$

If we set $[\boldsymbol{e}^i]_{i=1}^{d-1}$ as a basis of the tangent space $T_{\boldsymbol{\eta}}(\mathbb{S}^{d-1})$ in order to obtain (3.87), the previous calculations still hold except that the exponent $d-2$ is replaced by $d-1$ in equation (3.89).

Finally, we prove that the basis $\left[J_{\boldsymbol{h}_z} \boldsymbol{e}^1, \ldots, J_{\boldsymbol{h}_z} \boldsymbol{e}^d \right]$ of the tangent space $T_{\boldsymbol{h}_z(\boldsymbol{\eta})} \mathscr{C}_z^{\boldsymbol{\xi}}$ is oriented positively, i. e., that

$$d(z) := \det \left(\boldsymbol{h}_z(\boldsymbol{\eta}), \boldsymbol{\xi}, J_{\boldsymbol{h}_z} \boldsymbol{e}^1, \ldots, J_{\boldsymbol{h}_z} \boldsymbol{e}^d \right) > 0, \qquad z \in (-1,1).$$

By the formula (3.88) of $J_{\boldsymbol{h}_z}$, the function $d\colon [0,1) \to \mathbb{R}$ is continuous, and it satisfies

$$d(0) = \det \left(\boldsymbol{h}_0(\boldsymbol{\eta}), \boldsymbol{\xi}, J_{\boldsymbol{h}_0} \boldsymbol{e}^1, \ldots, J_{\boldsymbol{h}_0} \boldsymbol{e}^{d-2} \right) = \det \left(\boldsymbol{\eta}, \boldsymbol{g}_0(\boldsymbol{\xi}), \boldsymbol{e}^1, \ldots, \boldsymbol{e}^{d-2} \right) > 0$$

since both $\boldsymbol{h}_0$ and $\boldsymbol{g}_0$ are equal to the identity map and we assumed the orthonormal basis $\left[\boldsymbol{e}^1, \ldots, \boldsymbol{e}^{d-2} \right]$ be oriented positively. By the orthogonality of the vectors $\boldsymbol{\xi}$, $\boldsymbol{h}_z(\boldsymbol{\eta})$ and $J_{\boldsymbol{h}_z} \boldsymbol{e}^i$, we see that $d(z)$ vanishes nowhere and, hence, we obtain that $d(z) > 0$ for all $z \in [0,1)$. The assertion follows by the uniqueness of the volume form $\mathrm{d}\mathscr{C}_0^{\boldsymbol{g}_z(\boldsymbol{\xi})}(\boldsymbol{\eta})$. ■

Factorization

Let $z \in (-1,1)$ and $f \in C(\mathbb{S}^{d-1})$. We define the two transformations

$$\mathcal{M}_z, \mathcal{N}_z \colon C(\mathbb{S}^{d-1}) \to C(\mathbb{S}^{d-1})$$

by

$$\mathcal{M}_z f(\boldsymbol{\xi}) := \left(\frac{\sqrt{1-z^2}}{1+z\xi_d}\right)^{d-2} f \circ \boldsymbol{h}_z(\boldsymbol{\xi}), \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1}, \tag{3.90}$$

and

$$\mathcal{N}_z f(\boldsymbol{\xi}) := (1 - z^2\xi_d^2)^{-\frac{d-2}{2}} f \circ \boldsymbol{g}_z(\boldsymbol{\xi}), \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1}. \tag{3.91}$$

Corollary 3.57. Let $z \in (-1,1)$. The transformation $\mathcal{M}_z$ is inverted by

$$f(\boldsymbol{\eta}) = \mathcal{M}_{-z}\mathcal{M}_z f(\boldsymbol{\eta}), \qquad \boldsymbol{\eta} \in \mathbb{S}^{d-1}, \tag{3.92}$$

which expands to

$$f(\boldsymbol{\eta}) = \left(\frac{\sqrt{1-z^2}}{1-z\eta_d}\right)^{d-2} \mathcal{M}_z f(\boldsymbol{h}_z^{-1}(\boldsymbol{\eta})), \qquad \boldsymbol{\eta} \in \mathbb{S}^{d-1}. \tag{3.93}$$

Furthermore, the inverse of $\mathcal{N}_z$ is given by

$$f(\boldsymbol{\eta}) = \left(\frac{1-z^2}{1-z^2+z^2\eta_d^2}\right)^{\frac{d-2}{2}} \mathcal{N}_z f(\boldsymbol{g}_z^{-1}(\boldsymbol{\eta})), \qquad \boldsymbol{\eta} \in \mathbb{S}^{d-1}. \tag{3.94}$$

Proof. Let $f \in C(\mathbb{S}^{d-1})$ and $\boldsymbol{\eta} \in \mathbb{S}^{d-1}$. We have

$$\begin{aligned}\mathcal{M}_{-z}\mathcal{M}_z f(\boldsymbol{\eta}) &= \left(\frac{\sqrt{1-z^2}}{1-z\eta_d}\right)^{d-2} \mathcal{M}_z f(\boldsymbol{h}_{-z}(\boldsymbol{\eta})) \\ &= \left(\frac{\sqrt{1-z^2}}{1-z\eta_d}\right)^{d-2} \left(\frac{\sqrt{1-z^2}}{1+z[\boldsymbol{h}_{-z}(\boldsymbol{\eta})]_d}\right)^{d-2} f(\boldsymbol{h}_z(\boldsymbol{h}_{-z}(\boldsymbol{\eta}))).\end{aligned}$$

By the definition of $\boldsymbol{h}_z$ in (3.79) and the relation (3.83) for $\boldsymbol{h}_z^{-1}$, we obtain

$$\begin{aligned}\mathcal{M}_{-z}\mathcal{M}_z f(\boldsymbol{\eta}) &= \left(\frac{1-z^2}{1-z\eta_d}\right)^{d-2} \left(\frac{1}{1+z\frac{-z+\eta_d}{1-z\eta_d}}\right)^{d-2} f(\boldsymbol{\eta}) \\ &= \left(\frac{1-z^2}{1-z\eta_d}\right)^{d-2} \left(\frac{1-z\eta_d}{1-z\eta_d-z^2+z\eta_d}\right)^{d-2} f(\boldsymbol{\eta}) = f(\boldsymbol{\eta}).\end{aligned}$$

We come to the inverse of $\mathcal{N}_z$. Inserting the definition (3.91) of $\mathcal{N}_z$, we have

$$\begin{aligned}&\left(\frac{1-z^2}{1-z^2+z^2\eta_d^2}\right)^{\frac{d-2}{2}} \mathcal{N}_z f(\boldsymbol{g}_z^{-1}(\boldsymbol{\eta})) \\ &\qquad = \left(\frac{1-z^2}{1-z^2+z^2\eta_d^2}\right)^{\frac{d-2}{2}} (1-z^2[\boldsymbol{g}_z^{-1}(\boldsymbol{\eta})]_d^2)^{-\frac{d-2}{2}} f(\boldsymbol{g}_z(\boldsymbol{g}_z^{-1}(\boldsymbol{\eta}))).\end{aligned}$$

We insert equation (3.84) for $\boldsymbol{g}_z^{-1}$ and obtain

$$
\begin{aligned}
\left(\frac{1-z^2}{1-z^2+z^2\eta_d^2}\right)^{\frac{d-2}{2}} \mathcal{N}_z f(\boldsymbol{g}_z^{-1}(\boldsymbol{\eta})) &= \left(\frac{1-z^2}{1-z^2+z^2\eta_d^2}\right)^{\frac{d-2}{2}} \left(\frac{1}{1-\frac{z^2\eta_d^2}{1-z^2+z^2\eta_d^2}}\right)^{\frac{d-2}{2}} f(\boldsymbol{\eta}) \\
&= \left(\frac{1-z^2}{1-z^2+z^2\eta_d^2}\right)^{\frac{d-2}{2}} \left(\frac{1-z^2+z^2\eta_d^2}{1-z^2}\right)^{\frac{d-2}{2}} f(\boldsymbol{\eta}) \\
&= f(\boldsymbol{\eta}),
\end{aligned}
$$

which shows (3.94). ■

Now we are able to prove the Factorization Theorem about the spherical transform $\mathcal{U}_z$ as follows.

Theorem 3.58. *Let $z \in [0,1)$. Then the factorization of the spherical transform*

$$
\mathcal{U}_z = \mathcal{N}_z \mathcal{F} \mathcal{M}_z \tag{3.95}
$$

holds, where $\mathcal{F}$ is the Funk–Radon transform (3.18).

Proof. Let $f \in C(\mathbb{S}^2)$ and $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$. By the definition of $\mathcal{U}_z$ in (3.76), we have

$$
\left|\mathbb{S}^{d-2}\right| (1-z^2\xi_d^2)^{\frac{d-2}{2}}\, \mathcal{U}_z f(\boldsymbol{\xi}) = \int_{\mathscr{C}_z^{\boldsymbol{\xi}}} f \,\mathrm{d}\mathscr{C}_z^{\boldsymbol{\xi}}. \tag{3.96}
$$

Then we have by the substitution rule (2.2)

$$
\int_{\mathscr{C}_z^{\boldsymbol{\xi}}} f \,\mathrm{d}\mathscr{C}_z^{\boldsymbol{\xi}} = \int_{\boldsymbol{h}_z^{-1}(\mathscr{C}_z^{\boldsymbol{\xi}})} (f \circ \boldsymbol{h}_z)\, \boldsymbol{h}_z^*(\mathrm{d}\mathscr{C}_z^{\boldsymbol{\xi}}).
$$

By (3.86) and (3.85), we obtain

$$
\int_{\mathscr{C}_z^{\boldsymbol{\xi}}} f \,\mathrm{d}\mathscr{C}_z^{\boldsymbol{\xi}} = \int_{\mathscr{C}_0^{\boldsymbol{g}_z(\boldsymbol{\xi})}} f(\boldsymbol{h}_z(\boldsymbol{\eta})) \left(\frac{\sqrt{1-z^2}}{1+z\eta_d}\right)^{d-2} \mathrm{d}\mathscr{C}_0^{\boldsymbol{g}_z(\boldsymbol{\xi})}(\boldsymbol{\eta}).
$$

By the definition of $\mathcal{M}_z$ in (3.90), we see that

$$
\int_{\mathscr{C}_z^{\boldsymbol{\xi}}} f \,\mathrm{d}\mathscr{C}_z^{\boldsymbol{\xi}} = \int_{\mathscr{C}_0^{\boldsymbol{g}_z(\boldsymbol{\xi})}} \mathcal{M}_z f \,\mathrm{d}\mathscr{C}_0^{\boldsymbol{g}_z(\boldsymbol{\xi})}.
$$

The definition of the Funk–Radon transform (3.18) shows that

$$
\int_{\mathscr{C}_z^{\boldsymbol{\xi}}} f \,\mathrm{d}\mathscr{C}_z^{\boldsymbol{\xi}} = \mathcal{F}\mathcal{M}_z f(\boldsymbol{g}_z(\boldsymbol{\xi})),
$$

which implies (3.95). ■

Theorem 3.58 was proven by the author in [Que17] for dimension $d = 3$ and in [Que18] for general $d \geq 3$. Shortly after the preprint [Que18] was published in October 2018, Rubin released the preprint of [Rub19], in which Theorem 3.58 was also proven, but with a method different from ours.

The Factorization Theorem 3.58 enables us to investigate the properties of the spherical transform $\mathcal{U}_z$. Because the operators $\mathcal{M}_z$ and $\mathcal{N}_z$ are relatively simple, we can transfer many properties from the Funk–Radon transform, see Section 3.2, to the spherical transform $\mathcal{U}_z$.

A geometric interpretation of the factorization

We give geometric interpretations of the two mappings $\boldsymbol{g}_z$ and $\boldsymbol{h}_z \colon \mathbb{S}^{d-1} \to \mathbb{S}^{d-1}$, which were defined in (3.80) and (3.79), respectively. The mapping $\boldsymbol{g}_z$ consists of two parts: The first is a scaling with the factor $\sqrt{1-z^2}$ along the d-th coordinate, namely

$$\boldsymbol{s}_z(\boldsymbol{\xi}) := \sum_{i=1}^{d-1} \xi_i \boldsymbol{\epsilon}^i + \sqrt{1-z^2}\, \xi_d \boldsymbol{\epsilon}^d, \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1},$$

which maps the sphere to an ellipsoid that is symmetric with respect to rotations about $\boldsymbol{\epsilon}_d$. Then the central projection

$$\boldsymbol{p}(\boldsymbol{x}) := \frac{1}{\|\boldsymbol{x}\|}\, \boldsymbol{x}, \qquad \boldsymbol{x} \in \mathbb{R}^d \setminus \{\boldsymbol{0}\},$$

maps this ellipsoid onto the sphere $\mathbb{S}^{d-1}$ again. Recapitulating, we can write $\boldsymbol{g}_z$ as the composition

$$\boldsymbol{g}_z(\boldsymbol{\xi}) = \boldsymbol{p}(\boldsymbol{s}_z(\boldsymbol{\xi})), \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1}.$$

Moreover, we obtain a geometric description of the mapping $\boldsymbol{h}_z$ as follows. We recall the stereographic projection $\boldsymbol{\pi}$ and its inverse $\boldsymbol{\pi}^{-1}$ from (3.77) and (3.78), respectively.

Corollary 3.59. Let $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$ and $z \in (0,1)$. Then we have

$$\boldsymbol{h}_z(\boldsymbol{\xi}) = \boldsymbol{\pi}^{-1}\left(\sqrt{\frac{1+z}{1-z}}\, \boldsymbol{\pi}(\boldsymbol{\xi})\right).$$

Proof. Let $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$ and $z \in (0,1)$. We are going to show that

$$\boldsymbol{\pi}(\boldsymbol{h}_z(\boldsymbol{\xi})) = \sqrt{\frac{1+z}{1-z}}\, \boldsymbol{\pi}(\boldsymbol{\xi}).$$

We have on the one hand

$$\sqrt{\frac{1+z}{1-z}}\, \boldsymbol{\pi}(\boldsymbol{\xi}) = \sqrt{\frac{1+z}{1-z}} \sum_{i=1}^{d-1} \frac{\xi_i}{1-\xi_d} \boldsymbol{\epsilon}^i$$

and the other hand

$$\boldsymbol{\pi}(\boldsymbol{h}_z(\boldsymbol{\xi})) = \sum_{i=1}^{d-1} \frac{\frac{\sqrt{1-z^2}\,\xi_i}{1+z\xi_d}}{1-\frac{z+\xi_d}{1+z\xi_d}}\,\boldsymbol{\epsilon}^i = \sum_{i=1}^{d-1} \frac{\sqrt{1-z^2}\,\xi_i}{1+z\xi_d-(z+\xi_d)}\,\boldsymbol{\epsilon}^i$$
$$= \sum_{i=1}^{d-1} \frac{\sqrt{1-z^2}\,\xi_i}{(1-z)\,(1-\xi_d)}\,\boldsymbol{\epsilon}^i.$$

The assertion follows by canceling $\sqrt{1-z}$ in the last fraction. ∎

The last corollary states that, under the stereographic projection $\boldsymbol{\pi}$, the mapping $\boldsymbol{h}_z$ on the sphere $\mathbb{S}^{d-1}$ corresponds to a uniform scaling in the equatorial hyperplane $\mathbb{R}^{d-1}$ with the scaling factor $\sqrt{\frac{1+z}{1-z}}$. Since the stereographic projection and the uniform scaling are conformal, the corollary gives an alternative proof that $\boldsymbol{h}_z$ is conformal, which we already showed in Lemma 3.56.

3.6.2 Nullspace

With the help of the factorization (3.95) obtained in the previous section, we can show the following characterization of the nullspace of the spherical transform $\mathcal{U}_z$.

Theorem 3.60. *Let $z \in (-1,1)$ and $f \in C(\mathbb{S}^{d-1})$. Then $\mathcal{U}_z f = 0$ if and only if*

$$f(\boldsymbol{\omega}) = -\left(\frac{1-z^2}{1-2z\omega_d+z^2}\right)^{d-2} f \circ \boldsymbol{r}_z(\boldsymbol{\omega}), \qquad \boldsymbol{\omega} \in \mathbb{S}^{d-1}, \tag{3.97}$$

where $\boldsymbol{r}_z \colon \mathbb{S}^{d-1} \to \mathbb{S}^{d-1}$ is given by

$$\boldsymbol{r}_z(\boldsymbol{\omega}) := \sum_{i=1}^{d-1} \frac{z^2-1}{1+z^2-2z\omega_d}\omega_i\boldsymbol{\epsilon}^i + \frac{2z-z^2\omega_d-\omega_d}{1+z^2-2z\omega_d}\boldsymbol{\epsilon}^d, \qquad \boldsymbol{\omega} \in \mathbb{S}^{d-1}. \tag{3.98}$$

Proof. Let $f \in C(\mathbb{S}^{d-1})$. Since the operator $\mathcal{N}_z$ is bijecive by Corollary 3.54, we see that $\mathcal{U}_z f = \mathcal{N}_z\mathcal{F}\mathcal{M}_z f = 0$ if and only if $\mathcal{F}\mathcal{M}_z f = 0$. The nullspace of the Funk–Radon transform $\mathcal{F}$ consists of the odd functions, see Section 3.2, so we obtain that

$$\mathcal{U}_f = 0 \Leftrightarrow \mathcal{M}_z f(\boldsymbol{\eta}) = -\mathcal{M}_z f(-\boldsymbol{\eta}), \qquad \boldsymbol{\eta} \in \mathbb{S}^{d-1}.$$

By the definition of $\mathcal{M}_z$ in (3.90), this is equivalent to

$$\left(\frac{\sqrt{1-z^2}}{1+z\eta_d}\right)^{d-2} f \circ \boldsymbol{h}_z(\boldsymbol{\eta}) = -\left(\frac{\sqrt{1-z^2}}{1-z\eta_d}\right)^{d-2} f \circ \boldsymbol{h}_z(-\boldsymbol{\eta}).$$

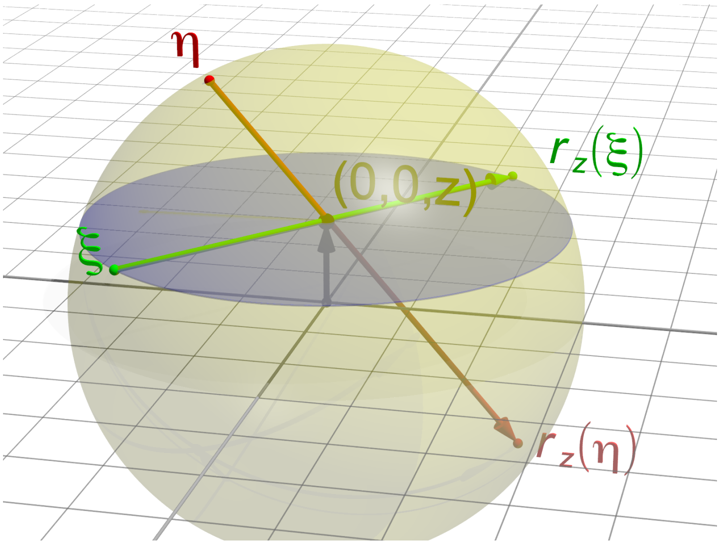

Figure 3.9: The point reflection $\boldsymbol{r}_z$ about the point $z\boldsymbol{\epsilon}^d$ acting on two vectors $\boldsymbol{\xi}$, $\boldsymbol{\eta} \in \mathbb{S}^2$.

We substitute $\boldsymbol{\omega} = \boldsymbol{h}_z(\boldsymbol{\eta})$ and obtain the equivalent formulation

$$\begin{aligned} f(\boldsymbol{\omega}) &= -\left(\frac{1 + z\frac{\omega_d - z}{1 - z\omega_d}}{1 - z\frac{\omega_d - z}{1 - z\omega_d}}\right)^{d-2} f \circ \boldsymbol{h}_z(-\boldsymbol{h}_z^{-1}(\boldsymbol{\omega})) \\ &= -\left(\frac{1 - z\omega_d + z(\omega_d - z)}{1 - z\omega_d - z(\omega_d - z)}\right)^{d-2} f \circ \boldsymbol{h}_z(-\boldsymbol{h}_z^{-1}(\boldsymbol{\omega})) \\ &= -\left(\frac{1 - z^2}{1 - 2z\omega_d + z^2}\right)^{d-2} f \circ \boldsymbol{h}_z(-\boldsymbol{h}_z^{-1}(\boldsymbol{\omega})). \end{aligned}$$

In order to show that $\boldsymbol{r}_z = \boldsymbol{h}_z(-\boldsymbol{h}_z^{-1})$, we compute the d-th component

$$[\boldsymbol{h}_z(-\boldsymbol{h}_z^{-1}(\boldsymbol{\omega}))]_d = \frac{z + \frac{z - \omega_d}{1 - z\omega_d}}{1 + z\frac{z - \omega_d}{1 - z\omega_d}} = \frac{z - z^2\omega_d + z - \omega_d}{1 - z\omega_d + z^2 - z\omega_d} = \frac{2z - z^2\omega_d - \omega_d}{1 - 2z\omega_d + z^2}.$$

For the i-th component, $i \in \{1, \ldots, d-1\}$, we have

$$[\boldsymbol{h}_z(-\boldsymbol{h}_z^{-1}(\boldsymbol{\omega}))]_i = \frac{\sqrt{1 - z^2}}{1 - z\frac{\omega_d - z}{1 - z\omega_d}} \frac{-\sqrt{1 - z^2}}{1 - z\omega_d}\omega_i = \frac{z^2 - 1}{1 - 2z\omega_d + z^2}\omega_i. \qquad \blacksquare$$

Remark 3.61. The map $\boldsymbol{r}_z$ from (3.98) is the point reflection of the sphere $\mathbb{S}^{d-1}$ about the point $z\boldsymbol{\epsilon}^d$, see Figure 3.9. This can be seen as follows. Let $\boldsymbol{\omega} \in \mathbb{S}^{d-1}$. The vectors $\boldsymbol{\omega} - z\boldsymbol{\epsilon}^d$ and $\boldsymbol{r}_z(\boldsymbol{\omega}) - z\boldsymbol{\epsilon}^d$ are parallel if we have for all $i \in \{1, \ldots, d-1\}$

$$\frac{[\boldsymbol{r}_z(\boldsymbol{\omega})]_i}{\omega_i} = \frac{[\boldsymbol{r}_z(\boldsymbol{\omega})]_d - z}{\omega_d - z}.$$

We have

$$\frac{\omega_i}{[\boldsymbol{r}_z(\boldsymbol{\omega})]_i}\frac{[\boldsymbol{r}_z(\boldsymbol{\omega})]_d - z}{\omega_d - z} = \frac{2z - z^2\omega_d - \omega_d - z(1+z^2-2z\omega_d)}{(z^2-1)(\omega_d - z)}$$
$$= \frac{z + z^2\omega_d - \omega_d - z^3}{(z^2-1)(\omega_d - z)} = 1,$$

provided all denominators are nonzero. □

Theorem 3.60 shows that the spherical transform $\mathcal{U}_z$ vanishes for all functions that are odd with respect to a "weighted" point reflection in $z\,\boldsymbol{\epsilon}^d$. This can be explained by the fact that if a point $\boldsymbol{\omega} \in \mathbb{S}^{d-1}$ is contained in a hyperplane passing through $z\,\boldsymbol{\epsilon}^d$, then so is its point reflection $\boldsymbol{r}_z(\boldsymbol{\omega})$.

Remark 3.62. Since the spherical transform $\mathcal{U}_z$ is not injective as we have seen in Theorem 3.60, Agranovsky and Rubin [AR19] recently suggested to use two instead of only one center $z\boldsymbol{\epsilon}^d$ and found that any continuous function f can be uniquely reconstructed given $\mathcal{U}_z f$ and $\mathcal{U}_w f$, where $z \neq w$. □

3.6.3 The range in terms of Sobolev spaces

In this section, we show that for $s \geq 0$, the spherical transform

$$\mathcal{U}_z\colon H^s(\mathbb{S}^{d-1}) \to H^{s+\frac{d-2}{2}}(\mathbb{S}^{d-1})$$

is continuous. To this end, we separately investigate the three parts of the decomposition obtained in Theorem 3.58,

$$\mathcal{U}_z = \mathcal{N}_z\mathcal{F}\mathcal{M}_z.$$

We have already shown in Theorem 3.13 that the Funk–Radon transform

$$\mathcal{F}\colon H^s_{\mathrm{even}}(\mathbb{S}^{d-1}) \to H^{s+\frac{d-2}{2}}_{\mathrm{even}}(\mathbb{S}^{d-1})$$

is continuous and bijective.

For the remaining operators $\mathcal{M}_z$ and $\mathcal{N}_z$, we have the following lemma, which relies on the theorems about bounded operators in Sobolev spaces from Section 2.1.5.

Lemma 3.63. *Let $z \in [0,1)$ and $s \in \mathbb{R}$ with $s \geq 0$. The operators*

$$\mathcal{M}_z\colon H^s(\mathbb{S}^{d-1}) \to H^s(\mathbb{S}^{d-1})$$

and

$$\mathcal{N}_z\colon H^s(\mathbb{S}^{d-1}) \to H^s(\mathbb{S}^{d-1}),$$

as defined in (3.90) *and* (3.91)*, are continuous and open.*

Proof. We first perform the proof for $\mathcal{M}_z$. Initially, we consider only the situation $s \in \mathbb{N}_0$. Let $f \in H^s(\mathbb{S}^{d-1})$ and $z \in (-1,1)$. We write

$$\mathcal{M}_z f(\boldsymbol{\xi}) = u_z(\boldsymbol{\xi}) \, [f \circ \boldsymbol{h}_z](\boldsymbol{\xi}), \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1},$$

where

$$u_z \colon \mathbb{S}^{d-1} \to \mathbb{R}, \quad u_z(\boldsymbol{\xi}) := \left(\frac{\sqrt{1-z^2}}{1+z\xi_d} \right)^{d-2}$$

and $\boldsymbol{h}_z$ is given in (3.79). As we did in (2.40), we extend the function u_z to the surrounding space by

$$u_z^\bullet(\boldsymbol{x}) = u_z\left(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\right) = \left(\sqrt{1-z^2}\, \frac{\|\boldsymbol{x}\|}{\|\boldsymbol{x}\| + zx_d} \right)^{d-2}, \qquad \boldsymbol{x} \in \mathbb{R}^d \setminus \{\boldsymbol{0}\},$$

We see that the extension $u_z^\bullet$ is smooth except in the origin, i. e., $u_z^\bullet \in C^\infty(\mathbb{R}^d \setminus \{\boldsymbol{0}\})$. Hence, $u_z \in C^\infty(\mathbb{S}^{d-1})$. Then Theorem 2.6 implies that

$$\|u_z\,(f \circ \boldsymbol{h}_z)\|_{H^s(\mathbb{S}^{d-1})} \le (2d+2)^{s/2}\, \|u_z\|_{C^s(\mathbb{S}^{d-1})}\, \|f \circ \boldsymbol{h}_z\|_{H^s(\mathbb{S}^{d-1})}\,. \tag{3.99}$$

Moreover, the extension of $\boldsymbol{h}_z$,

$$\boldsymbol{h}_z^\bullet \colon \mathbb{R}^d \setminus \{\boldsymbol{0}\} \to \mathbb{R}^d, \quad \boldsymbol{h}_z^\bullet(\boldsymbol{x}) = \sum_{i=1}^{d-1} \frac{\sqrt{1-z^2}}{\|\boldsymbol{x}\| + zx_d} x_i \boldsymbol{\epsilon}^i + \frac{z\,\|\boldsymbol{x}\| + x_d}{\|\boldsymbol{x}\| + zx_d} \boldsymbol{\epsilon}^d,$$

is also smooth, so $\boldsymbol{h}_z \in C^\infty(\mathbb{S}^{d-1} \to \mathbb{S}^{d-1})$. This implies that also the inverse $\boldsymbol{h}_z^{-1} = \boldsymbol{h}_{-z}$, see (3.83), is smooth. So $\boldsymbol{h}_z$ is a diffeomorphism and Theorem 2.7 together with (3.99) implies that

$$\begin{aligned}
\|\mathcal{M}_z f\|_{H^s(\mathbb{S}^{d-1})} &\le (2d+2)^{s/2}\, \|u_z\|_{C^s(\mathbb{S}^{d-1})}\, \|f \circ \boldsymbol{h}_z\|_{H^s(\mathbb{S}^{d-1})} \\
&\le (2d+2)^{s/2}\, \|u_z\|_{C^s(\mathbb{S}^{d-1})}\, b_{d,s}(\boldsymbol{h}_z)\, \|f\|_{H^s(\mathbb{S}^{d-1})}\,.
\end{aligned}$$

Thus, the operator $\mathcal{M}_z \colon H^s(\mathbb{S}^{d-1}) \to H^s(\mathbb{S}^{d-1})$ is continuous.

Now let $s \in \mathbb{R}$ with $s \ge 0$. The above proof shows that both the restrictions of $\mathcal{M}_z$ to $H^{\lfloor s \rfloor}$ and to $H^{\lfloor s \rfloor + 1}$ are continuous, where $\lfloor s \rfloor$ denotes the largest integer that is smaller than or equal to s. The continuity of $\mathcal{M}_z$ on $H^s(\mathbb{S}^{d-1})$ follows by the interpolation result Proposition 2.5.

In order to prove the openness of $\mathcal{M}_z$, we show that the inverse $\mathcal{M}_z^{-1}$ restricted to $H^s(\mathbb{S}^{d-1})$ is continuous. However, we have already done this because $\mathcal{M}_z^{-1} = \mathcal{M}_{-z}$ by (3.93).

The same argumentation as above also works for the operator $\mathcal{N}_z$ as follows. Let $z \in [0,1)$ and $s \in \mathbb{N}_0$. We write

$$\mathcal{N}_z f(\boldsymbol{\xi}) = v_z(\boldsymbol{\xi}) \, [f \circ \boldsymbol{g}_z](\boldsymbol{\xi}), \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1},$$

where
$$v_z(\boldsymbol{\xi}) := (1 - z^2\xi_d^2)^{-\frac{d-2}{2}}, \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1}.$$
As in (2.40), we extend the function v_z to the surrounding space $\mathbb{R}^d \setminus \{\mathbf{0}\}$ by
$$v_z^\bullet(\boldsymbol{x}) = \left(\frac{\|\boldsymbol{x}\|^2}{\|\boldsymbol{x}\|^2 - z^2x_d^2}\right)^{\frac{d-2}{2}}, \qquad \boldsymbol{x} \in \mathbb{R}^d \setminus \{\mathbf{0}\},$$
We see that the extension $v_z^\bullet$ is smooth and hence $v_z \in C^s(\mathbb{S}^d)$. Theorem 2.6 yields
$$\|v_z\,(f \circ \boldsymbol{g}_z)\|_{H^s(\mathbb{S}^{d-1})} \le (2d+2)^{s/2}\,\|v_z\|_{C^s(\mathbb{S}^{d-1})}\,\|f \circ \boldsymbol{g}_z\|_{H^s(\mathbb{S}^{d-1})}\,.$$
Since the extensions of both
$$\boldsymbol{g}_z^\bullet(\boldsymbol{x}) = \boldsymbol{g}_z\left(\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}\right) = \frac{1}{\sqrt{\|\boldsymbol{x}\|^2 - z^2x_d^2}}\left(\sum_{i=1}^{d-1} x_i\boldsymbol{\epsilon}^i + \sqrt{1-z^2}\,x_d\boldsymbol{\epsilon}^d\right), \qquad \boldsymbol{x} \in \mathbb{R}^d \setminus \{\mathbf{0}\},$$
and its inverse (3.84)
$$[(\boldsymbol{g}_z^{-1})^\bullet](\boldsymbol{x}) = \frac{1}{\sqrt{\|\boldsymbol{x}\|^2 - z^2 + z^2x_d^2}}\left(\sqrt{1-z^2}\sum_{i=1}^{d-1}\omega_i\boldsymbol{\epsilon}^i + x_d\boldsymbol{\epsilon}^d\right), \qquad \boldsymbol{x} \in \mathbb{R}^d \setminus \{\mathbf{0}\},$$
are smooth functions on $\mathbb{R}^d\setminus\{\mathbf{0}\}$, we see that $\boldsymbol{g}_z$ is a smooth diffeomorphism in $C^s(\mathbb{S}^{d-1})$. By Theorem 2.7, there exists a constant $b_{d,s}(\boldsymbol{g}_z)$ independent of f such that
$$\|f \circ \boldsymbol{g}_z\|_{H^s(\mathbb{S}^{d-1})} \le b_{d,s}(\boldsymbol{g}_z)\,\|f\|_{H^s(\mathbb{S}^{d-1})}\,.$$
Hence, $\mathcal{N}_z$ is a bounded operator on $H^s(\mathbb{S}^{d-1})$. An analogue computation shows that the inverse operator (3.94)
$$\mathcal{N}_z^{-1}f(\boldsymbol{\eta}) = \frac{f(\boldsymbol{g}_z^{-1}(\boldsymbol{\eta}))}{v_z(\boldsymbol{g}_z^{-1}(\boldsymbol{\eta}))} = \left(\frac{1-z^2}{1-z^2+z^2\eta_d^2}\right)^{\frac{d-2}{2}}(f \circ \boldsymbol{g}_z^{-1})(\boldsymbol{\eta}), \qquad \boldsymbol{\eta} \in \mathbb{S}^{d-1}$$
is also bounded on $H^s(\mathbb{S}^{d-1})$. The assertion for general s follows by the same interpolation argument as for $\mathcal{M}_z$. ■

Finally, we have collected all ingredients in order to prove the continuity of the spherical transform $\mathcal{U}_z$ in Sobolev spaces.

Theorem 3.64. *Let $z \in (0,1)$ and $s \in \mathbb{R}$ with $s \ge 0$. We set $H^s_{\mathrm{even},z}(\mathbb{S}^{d-1})$ as the subspace of all functions $f \in H^s(\mathbb{S}^{d-1})$ that satisfy*
$$f(\boldsymbol{\omega}) = \left(\frac{1-z^2}{1-2z\omega_d+z^2}\right)^{d-2} f \circ \boldsymbol{r}_z(\boldsymbol{\omega}), \qquad \boldsymbol{\omega} \in \mathbb{S}^{d-1}, \tag{3.100}$$
almost everywhere, where the point reflection $\boldsymbol{r}_z$ about the point $z\boldsymbol{\epsilon}^d$ is given in (3.98). Then the spherical transform
$$\mathcal{U}_z\colon H^s_{\mathrm{even},z}(\mathbb{S}^{d-1}) \to H^{s+\frac{d-2}{2}}_{\mathrm{even}}(\mathbb{S}^{d-1})$$
is continuous and bijective and its inverse operator is also continuous.

Proof. In Theorem 3.58, we obtained the decomposition

$$\mathcal{U}_z = \mathcal{N}_z \mathcal{F} \mathcal{M}_z.$$

We are going to look at the parts of this decomposition separately. By Lemma 3.63, we obtain that

$$\mathcal{M}_z \colon H^s(\mathbb{S}^{d-1}) \to H^s(\mathbb{S}^{d-1})$$

is continuous and bijective. The same holds for the restriction

$$\mathcal{M}_z \colon H^s_{\mathrm{even},z}(\mathbb{S}^{d-1}) \to H^s_{\mathrm{even}}(\mathbb{S}^{d-1}),$$

which follows from the characterization of the nullspace in Theorem 3.60. By Theorem 3.13, the Funk–Radon transform

$$\mathcal{F} \colon H^s_{\mathrm{even}}(\mathbb{S}^{d-1}) \to H^{s+\frac{d-2}{2}}_{\mathrm{even}}(\mathbb{S}^{d-1})$$

is continuous and bijective. Finally, Lemma 3.63 and the observation that any function $f \colon \mathbb{S}^{d-1} \to \mathbb{C}$ is even if and only if $\mathcal{N}_z f$ is even show that

$$\mathcal{N}_z \colon H^{s+\frac{d-2}{2}}_{\mathrm{even}}(\mathbb{S}^{d-1}) \to H^{s+\frac{d-2}{2}}_{\mathrm{even}}(\mathbb{S}^{d-1})$$

is continuous and bijective. The continuity of the inverse operator of $\mathcal{U}_z$ follows from the open mapping theorem. ∎

Theorem 3.64 is a generalization of Theorem 3.13 for the Funk–Radon transform $\mathcal{F}$; the main difference is that the space $H^s_{\mathrm{even}}(\mathbb{S}^{d-1})$ is replaced by $H^s_{\mathrm{even},z}(\mathbb{S}^{d-1})$, which contains functions that satisfy the symmetry condition (3.100) with respect to the point reflection in $z\,\boldsymbol{\epsilon}^d$. Furthermore, the spherical transform $\mathcal{U}_z$ is smoothing of degree $\frac{d-2}{2}$, which comes from the fact that $\mathcal{U}_z$ takes the integrals along $(d-2)$-dimensional submanifolds.

3.6.4 An inversion formula

In this section, we are going to show an explicit inversion formula of the spherical transform $\mathcal{U}_z$. The idea is to use the Factorization Theorem 3.58 of the spherical transform and apply it to Helgason's inversion formula (3.23) of the Funk–Radon transform $\mathcal{F}$.

Theorem 3.65. *Let $z \in (-1,1)$, and let the function $f \colon \mathbb{S}^{d-1} \to \mathbb{C}$ fulfill the symmetry condition* (3.100)*. Then f can be reconstructed from its spherical transform $\mathcal{U}_z f$ for any $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$ by*

$$f(\boldsymbol{\xi}) = \left(\frac{1-z^2}{1-z\xi_d}\right)^{d-2} \frac{2^{d-2}}{(d-3)!\,|\mathbb{S}^{d-2}|} \left[\left(\frac{\mathrm{d}}{\mathrm{d}(u^2)}\right)^{d-2} \int_0^u (1-v^2)^{\frac{2-d}{2}} \right.$$
$$\left. \cdot \int_{\langle \boldsymbol{h}_z^{-1}(\boldsymbol{\xi}), \boldsymbol{\eta}\rangle = v} \left(1-z^2+z^2\eta_d^2\right)^{\frac{2-d}{2}} \mathcal{U}_z f(\boldsymbol{g}_z^{-1}(\boldsymbol{\eta}))\,\mathrm{d}\boldsymbol{\eta}\, v^{d-2}\,(u^2-v^2)^{\frac{d-4}{2}}\,\mathrm{d}v \right]_{u=1}.$$

Proof. By the factorization (3.95) of the spherical transform $\mathcal{U}_z$, we have

$$f = \mathcal{U}_z^{-1}\mathcal{U}_z f = \mathcal{M}_z^{-1}\mathcal{F}^{-1}\mathcal{N}_z^{-1}\mathcal{U}_z f.$$

Let $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$. With (3.93) for $\mathcal{M}_z^{-1}$, we obtain

$$f(\boldsymbol{\xi}) = \left(\frac{\sqrt{1-z^2}}{1-z\xi_d}\right)^{d-2} \mathcal{F}^{-1}\mathcal{N}_z^{-1}\mathcal{U}_z f(\boldsymbol{h}_z^{-1}(\boldsymbol{\xi})).$$

Since $\mathcal{U}_z f$ is always even and hence also $\mathcal{N}_z\mathcal{U}_z f$ is even, we can apply Helgason's inversion formula (3.23). Then we have

$$f(\boldsymbol{\xi}) = \left(\frac{\sqrt{1-z^2}}{1-z\xi_d}\right)^{d-2} \frac{2^{d-2}}{(d-3)!} \cdot \left[\left(\frac{\mathrm{d}}{\mathrm{d}(u^2)}\right)^{d-2} \int_0^u \mathcal{M}[\mathcal{N}_z^{-1}\mathcal{U}_z f](\boldsymbol{h}_z^{-1}(\boldsymbol{\xi}), v)\, v^{d-2}\,(u^2-v^2)^{\frac{d-4}{2}}\,\mathrm{d}v\right]_{u=1}.$$

Inserting the definition (3.2) of the mean operator $\mathcal{M}$, we have

$$f(\boldsymbol{\xi}) = \left(\frac{\sqrt{1-z^2}}{1-z\xi_d}\right)^{d-2} \frac{2^{d-2}}{(d-3)!} \cdot \left[\left(\frac{\mathrm{d}}{\mathrm{d}(u^2)}\right)^{d-2} \int_0^u \frac{(1-v^2)^{\frac{2-d}{2}}}{|\mathbb{S}^{d-2}|} \int_{\langle \boldsymbol{h}_z^{-1}(\boldsymbol{\xi}),\boldsymbol{\eta}\rangle = v} \mathcal{N}_z^{-1}\mathcal{U}_z f(\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta}\, v^{d-2}\,(u^2-v^2)^{\frac{d-4}{2}}\,\mathrm{d}v\right]_{u=1}.$$

With (3.94), we obtain

$$f(\boldsymbol{\xi}) = \left(\frac{\sqrt{1-z^2}}{1-z\xi_d}\right)^{d-2} \frac{2^{d-2}}{(d-3)!} \left[\left(\frac{\mathrm{d}}{\mathrm{d}(u^2)}\right)^{d-2} \int_0^u \frac{(1-v^2)^{\frac{2-d}{2}}}{|\mathbb{S}^{d-2}|} \cdot \int_{\langle \boldsymbol{h}_z^{-1}(\boldsymbol{\xi}),\boldsymbol{\eta}\rangle = v} \left(\frac{1-z^2}{1-z^2+z^2\eta_d^2}\right)^{\frac{d-2}{2}} \mathcal{U}_z f(\boldsymbol{g}_z^{-1}(\boldsymbol{\eta}))\,\mathrm{d}\boldsymbol{\eta}\, v^{d-2}\,(u^2-v^2)^{\frac{d-4}{2}}\,\mathrm{d}v\right]_{u=1}.$$

■

Remark 3.66. In dimension $d = 3$, the inversion formula of the spherical transform $\mathcal{U}_z$ from Theorem 3.65 can be simplified for $\boldsymbol{\xi} \in \mathbb{S}^2$ to

$$f(\boldsymbol{\xi}) = \frac{1}{2\pi}\frac{1-z^2}{1-z\xi_3}\left[\frac{\mathrm{d}}{\mathrm{d}u}\int_0^u \frac{v}{\sqrt{1-v^2}} \cdot \int_{\boldsymbol{h}_z^{-1}(\boldsymbol{\xi})^\top\boldsymbol{\eta}=v} \frac{1}{\sqrt{1-(1-z^2)\eta_3^2}}\,\mathcal{U}_z f(\boldsymbol{g}_z^{-1}(\boldsymbol{\eta}))\,\mathrm{d}\boldsymbol{\eta}\,\frac{1}{\sqrt{u^2-v^2}}\,\mathrm{d}v\right]_{u=1},$$

where we have made use of the simplification $\frac{\mathrm{d}}{\mathrm{d}(u^2)} = \frac{1}{2u}\frac{\mathrm{d}}{\mathrm{d}u}$. For $z = 0$, the inversion formula from Theorem 3.65 becomes Helgason's inversion formula (3.23) of the Funk–Radon transform $\mathcal{F} = \mathcal{U}_0$. Furthermore, we note that the inversion formula of $\mathcal{U}_z$ in Theorem 3.65 has the same form for both even and odd dimensions d, unlike the one in Proposition 3.53. □

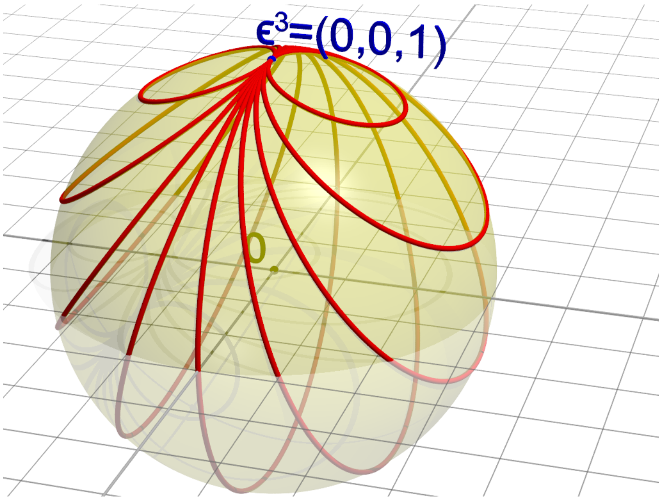

Figure 3.10: Circles through the north pole $\boldsymbol{\epsilon}^3$ of the two-sphere $\mathbb{S}^2$ from the spherical slice transform $\mathcal{U}_1$.

3.6.5 The Spherical slice transform

The spherical transform $\mathcal{U}_1$, in the case $z = 1$, integrates a function along all subspheres that pass through the north pole $\boldsymbol{\epsilon}^d = (0, \dots, 0, 1)^\top$ of $\mathbb{S}^{d-1}$, see Figure 3.10. However, the approach from Section 3.6.1 cannot be applied in this case, because the mapping $\boldsymbol{h}_z$ for $z \to 1$ then contracts the sphere to the north pole $\boldsymbol{\epsilon}^d$.

In 1993, Abouelaz and Daher [AD93] considered the transform $\mathcal{U}_1$ calling it the *Radon transformation on the sphere* (literally "*la transformation de Radon sur la sphère*"). They found an inversion formula for this transform on the two-dimensional sphere $\mathbb{S}^2$ for radial functions $f(\boldsymbol{\xi}) = \tilde{f}(\xi_3)$. Gindikin et al. [GRS94] constructed an inversion formula on $\mathbb{S}^2$ utilizing the kappa operator.

The name *spherical slice transform* for $\mathcal{U}_1$ is due to Helgason [Hel99, Section II.1.C]. In 1999, he proved that every function $f \in C^1(\mathbb{S}^2)$ that vanishes at the north pole $\boldsymbol{\epsilon}^d$ can be reconstructed from its spherical slice transform. This result was obtained by stereographic projection from the north pole onto the equatorial hyperplane $\mathbb{R}^{d-1} \times \{0\} \subset \mathbb{R}^d$. Then all $(d-2)$-dimensional subspheres passing through the north pole, which can be described as the intersection of the sphere with a hyperplane containing the north pole, become $(d-2)$-dimensional hyperplanes in the Euclidean space $\mathbb{R}^{d-1}$. In 2001, Daher [Dah01] showed the injectivity of the spherical slice transform for functions $f \in L^2(\mathbb{S}^{d-1})$ that vanish in a neighborhood of the north pole. A similar result that shows the injectivity for Lipschitz-continuous functions vanishing at the equator was obtained by the author [Que17, Section 7] via a connection with the spherical transform $\mathcal{U}_z$ for $z \to 1$ with $z < 1$. In 2017, Rubin [Rub17b] showed the injectivity of the operator $\mathcal{U}_1$ for functions $f \in L^\infty(\mathbb{S}^{d-1})$, which was obtained via stereographic projection.

3.6.6 Nongeodesic hyperplane sections

The following inversion formula for the spherical transform $\mathcal{U}_z$ is due to Palamodov [Pal16, Section 5.2] in 2016, see also [Pal17]. It covers some nongeodesic hyperplane sections of the sphere $\mathbb{S}^{d-1}$. We consider hyperplanes that have a fixed distance $r \geq 0$ to the fixed point $z\boldsymbol{\epsilon}^d$ inside the sphere, where we assume that $r + z < 1$. Such hyperplanes are the tangent planes to the sphere with center $z\boldsymbol{\epsilon}^d$ and radius r, which lies inside $\mathbb{S}^{d-1}$. The mean values of a function $f \in C(\mathbb{S}^{d-1})$ along these hyperplane sections with the sphere $\mathbb{S}^{d-1}$ are parameterized as a restriction of the mean operator $\mathcal{M}$ by

$$\mathcal{M}f(\boldsymbol{\xi}, z\,\xi_d + r), \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1}.$$

As a particular case for $r = 0$ and $z \in [0, 1)$, we have the spherical transform $\mathcal{U}_z$ from Section 3.6. In the case $z = 0$ and $r \in [0, 1)$, we have the spherical section transform $\mathcal{T}_r$ from Section 3.3.

Proposition 3.67 ([Pal16, Theorem 5.4])**.** *Let $\boldsymbol{\tau} \in \mathbb{S}^{d-1}$, and let $z, r \in [0, 1)$ such that $r + z < 1$. We define the spherical cap*

$$X := \{\boldsymbol{\eta} \in \mathbb{S}^{d-1} \,;\, \langle \boldsymbol{\eta} - z\boldsymbol{\epsilon}^d, \boldsymbol{\tau} \rangle > r\}$$

and the function

$$D_d(\boldsymbol{\eta}) := \frac{\left(\left\| \boldsymbol{\eta} - z\boldsymbol{\epsilon}^d \right\|^2 - r^2 \right)^{\frac{d-3}{2}}}{\left\| \boldsymbol{\eta} - z\boldsymbol{\epsilon}^d \right\|^{d-2}}, \qquad \boldsymbol{\eta} \in \mathbb{S}^{d-1}.$$

If d is even, any function $f \in C_0^{d-2}(X)$ can be reconstructed via

$$f(\boldsymbol{\eta}) = \frac{1}{2(2\pi \mathrm{i})^{d-2}\, D_d(\boldsymbol{\eta})} \int_{\mathbb{S}^{d-1}} \delta^{(d-2)}(\langle \boldsymbol{\eta}, \boldsymbol{\xi} \rangle - z\xi_d - r)\, \mathcal{M}f(\boldsymbol{\xi}, z\xi_d + r)\, \mathrm{d}\boldsymbol{\xi}, \qquad \boldsymbol{\eta} \in X.$$

If d is odd, any function $f \in C_0^{d-1}(X)$ can be reconstructed via

$$f(\boldsymbol{\eta}) = \frac{(d-2)!}{(2\pi \mathrm{i})^{d-1}\, D_d(\boldsymbol{\eta})} \int_{\mathbb{S}^{d-1}} \frac{\mathcal{M}f(\boldsymbol{\xi}, z\xi_d + r)}{(\langle \boldsymbol{\eta}, \boldsymbol{\xi} \rangle - z\xi_d - r)^{d-1}}\, \mathrm{d}\boldsymbol{\xi}, \qquad \boldsymbol{\eta} \in X.$$

4 Applications

In this chapter, we consider two particular applications of the spherical transforms that we have investigated in Chapter 3.

At first, in Section 4.1, we are going to take a look at the cone-beam transform, which integrates a function that is defined on the Euclidean space $\mathbb{R}^d$ along all rays that start in a certain set. The cone-beam transform on $\mathbb{R}^3$ provides the mathematical background of the three-dimensional X-ray computed tomography. Grangeat's formula gives a connection of the cone-beam transform with the generalized Funk–Radon transform $\mathcal{S}^{(j)}$ and the Radon transform $\mathcal{R}$. Hence, the inversion of the cone-beam transform is splitted up into the inverse Radon transform $\mathcal{R}^{-1}$ and the inverse generalized Funk–Radon transform $(\mathcal{S}^{(j)})^{-1}$. We utilize the results obtained in Section 3.4 about the transform $\mathcal{S}^{(j)}$ in order to show the singular value decomposition of the cone-beam transform.

In the second section, 4.2, we consider the integrals along incomplete great circles of the two-sphere $\mathbb{S}^2$. Interestingly, this problem mainly appeared in the geophysical publications with only a limited coverage in the mathematical literature, which is due to the fact that these integrals play an important role in the modeling of the spherical surface wave tomography. We are going to call the transform that maps to a function on $\mathbb{S}^2$ these integrals along incomplete great circles the arc transform on the sphere. After finding a suitable parameterization of the circle arcs, we show the singular value decomposition of the arc transform in Theorem 4.19.

The task of recovering a function f from its arc transform is overdetermined. In Section 4.2.4, we focus on the special case that only the integrals along certain great circle arcs are known, namely those arcs having a fixed length. Even from this limited data, it is still possible to recover the original function on the sphere. Furthermore, we obtain a singular value decomposition for the arc transform with fixed length in Theorem 4.22, where we also describe the asymptotic behavior of the singular values depending on the length of the circle arcs.

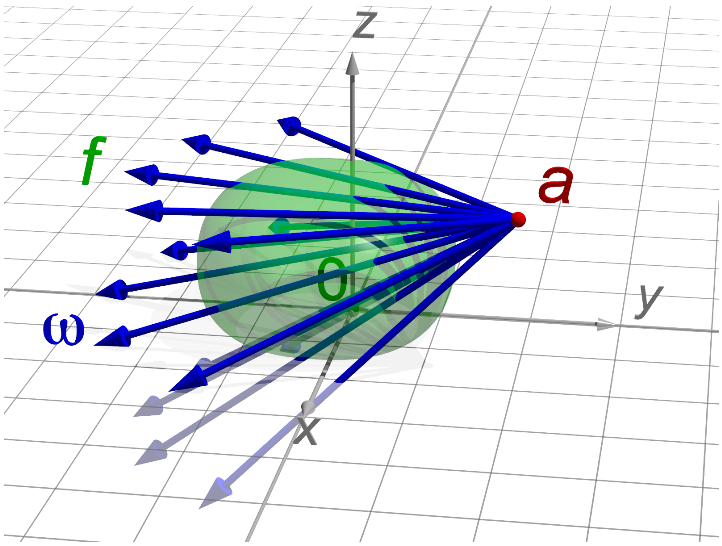

Figure 4.1: Illustration of the cone-beam transform in 3D of a function f, whose support is marked in green. The rays (blue) start in the point $\boldsymbol{a} \in \Gamma$ (red).

4.1 Cone-beam transform

In this section, we are going to derive a singular value decomposition of the cone-beam transform. Most of the material that is presented in this section is contained in the second part of our article [QHL18]. We start with the definition of the relevant integral transforms.

The cone-beam transform integrates a function $f\colon \mathbb{R}^d \to \mathbb{R}$ along every ray that starts in some scanning set $\Gamma \subset \mathbb{R}^d$. We define the *cone-beam transform* $\mathcal{D}$, which is also known as the *divergent beam X-ray transform*, by

$$\mathcal{D}f(\boldsymbol{a}, \boldsymbol{\omega}) := \int_0^\infty f(\boldsymbol{a} + t\boldsymbol{\omega})\,\mathrm{d}t, \qquad \boldsymbol{\omega} \in \mathbb{S}^{d-1},\ \boldsymbol{a} \in \Gamma.$$

An illustration of the rays of integration is provided in Figure 4.1.

The cone-beam transform in 3D is widely used in medical imaging and nondestructive testing of three-dimensional objects, cf. [Smi90]. The injectivity of the cone-beam transform was shown under rather weak assumptions, namely that Γ is an infinite set with positive distance to the convex hull of the support of f [HSSW80]. An explicit inversion formula [Tuy83] is known for the case that the Tuy–Kirillov completeness condition is satisfied, which states that the scanning set Γ intersects every hyperplane hitting $\operatorname{supp} f$ transversally, see [NW00, Chapter 2]. In 3D, if the Tuy–Kirillov condition is not satisfied, one can stably detect singularities of f only along planes that meet the scanning curve Γ, see [Qui93].

In the following, we consider the setting that the function f is supported on the unit ball $\mathbb{B}^d$ and the scanning set Γ is the whole unit sphere $\mathbb{S}^{d-1}$. However, in many practical applications in 3D, the scanning set Γ is a circle [Fin85].

4.1.1 Connection of Radon and cone-beam transform

Radon transform. The Radon transform $\mathcal{R}$ on the d-dimensional unit ball $\mathbb{B}^d = \{\boldsymbol{x} \in \mathbb{R}^d \,;\, \|x\| \le 1\}$ is defined by [Nat86, Section II.1]

$$\begin{aligned} &\mathcal{R}\colon L^2(\mathbb{B}^d) \to L^2\big(\mathbb{S}^{d-1} \times [-1,1]; (1-s^2)^{\frac{1-d}{2}}\big), \\ &\mathcal{R}f(\boldsymbol{\omega}, s) = \int_{\boldsymbol{x}^\top \boldsymbol{\omega} = s} f(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}, \end{aligned} \tag{4.1}$$

which is a restriction of the the Radon transform (3.67) on $\mathbb{R}^d$. The Radon transform $\mathcal{R}$ on the unit ball $\mathbb{B}^d$ has the following singular value decomposition, which we obtain from Proposition 3.48 by inserting $\nu = \frac{d}{2}$. For $m \in \mathbb{N}_0$, $l = 0, \dots, m$ with $m + l$ even and $k = 1, \dots, N_{l,d}$, we have

$$\mathcal{R}\widetilde{V}_{m,l,k}(\boldsymbol{\omega}, s) = \frac{\sqrt{2m+d}\,\Gamma(\frac{d}{2})\, m!}{2^{1-d}\, \pi^{1-\frac{d}{2}}\, (m+d-1)!} \,(1-s^2)^{\frac{d-1}{2}}\, C_m^{(\frac{d}{2})}(s)\, Y_{l,d}^k(\boldsymbol{\omega}), \tag{4.2}$$

where

$$\widetilde{V}_{m,l,k}(s\boldsymbol{\omega}) := \sqrt{2m+d}\, s^l P_{\frac{m-l}{2}}^{\left(0, l+\frac{d-2}{2}\right)}(2s^2-1)\, Y_{l,d}^k(\boldsymbol{\omega}), \qquad s \in [0,1],\ \boldsymbol{\omega} \in \mathbb{S}^{d-1}, \tag{4.3}$$

and $P_n^{(\alpha,\beta)}$ denotes the Jacobi polynomial of degree n and orders $\alpha, \beta > -1$, see (2.22). The set

$$\Big\{ \widetilde{V}_{m,l,k} \,;\, l \in \mathbb{N}_0,\ m \in \{l, l+2, l+4, \dots\},\ k \in \{1, \dots, N_{l,d}\} \Big\}$$

is an orthonormal basis of $L^2(\mathbb{B}^d)$ consisting of polynomials of degree $m \in \mathbb{N}_0$.

Remark 4.1. We quickly show the orthonormality of the functions $\widetilde{V}_{m,l,k}$ in $L^2(\mathbb{B}^d)$. Let $m, m' \in \mathbb{N}_0$, $l = 0, \dots, m$, $l' = 0, \dots, m'$, $k = 1, \dots, N_{l,d}$, $k' = 1, \dots, N_{l',d}$ such that $m + l$ and $m' + l'$ are even. We have

$$\begin{aligned} &\left\langle \widetilde{V}_{m,l,k}, \widetilde{V}_{m',l',k'} \right\rangle^2_{L^2(\mathbb{B}^d)} \\ &\quad = \sqrt{2m+d}\sqrt{2m+d} \int_0^1 s^{l+l'} P_{\frac{m-l}{2}}^{\left(0,l+\frac{d-2}{2}\right)}(2s^2-1)\, P_{\frac{m'-l'}{2}}^{\left(0,l'+\frac{d-2}{2}\right)}(2s^2-1) s^{d-1} \,\mathrm{d}s \\ &\qquad\qquad \cdot \int_{\mathbb{S}^{d-1}} Y_{l,d}^k(\boldsymbol{\omega})\, \overline{Y_{l',d}^{k'}(\boldsymbol{\omega})} \,\mathrm{d}\boldsymbol{\omega}. \end{aligned}$$

By the orthonormality of the spherical harmonics $Y_{l,d}^k$ and the substitution $t = 2s^2 - 1$ with $\mathrm{d}t = 4s \,\mathrm{d}s$, we obtain

$$\begin{aligned} &\left\langle \widetilde{V}_{m,l,k}, \widetilde{V}_{m',l',k'} \right\rangle^2_{L^2(\mathbb{B}^d)} \\ &\quad = \frac{\sqrt{2m+d}\sqrt{2m+d}}{4} \delta_{l,l'} \delta_{k,k'} \int_{-1}^1 \left(\frac{t+1}{2}\right)^{l+\frac{d-2}{2}} P_{\frac{m-l}{2}}^{\left(0,l+\frac{d-2}{2}\right)}(t)\, P_{\frac{m'-l}{2}}^{\left(0,l+\frac{d-2}{2}\right)}(t) \,\mathrm{d}t. \end{aligned}$$

By the orthogonality (2.23) of the Jacobi polynomials, we have

$$\left\langle \widetilde{V}_{m,l,k}, \widetilde{V}_{m',l',k'} \right\rangle^2_{L^2(\mathbb{B}^d)} = \delta_{m,m'}\delta_{l,l'}\delta_{k,k'} \frac{2m+d}{4} 2^{-l-\frac{d-2}{2}} \frac{2^{l+\frac{d}{2}}}{m+\frac{d}{2}} \frac{\Gamma(\frac{m-l}{2}+1)\,\Gamma(\frac{m+l+d}{2})}{(\frac{m-l}{2})!\,\Gamma(\frac{m+l+d}{2})} = 1.$$

□

Grangeat's formula. There is a relation between the cone-beam transform $\mathcal{D}$, the Radon transform $\mathcal{R}$ and the generalized Funk–Radon transform $\mathcal{S}^{(j)}$ from Section 3.4. Let $h\colon \mathbb{R} \to \mathbb{R}$ be a function that is homogeneous of degree $1-d$. It was essentially shown by Hamaker et al. [HSSW80] in 1980 (see also [NW00, Section 2.3] and [Pal16, Section 2.2.1]) that for $f\colon \mathbb{R}^d \to \mathbb{C}$

$$\int_{-\infty}^{\infty} \mathcal{R}f(\boldsymbol{\omega}, s)\, h(s - \boldsymbol{a}^\top \boldsymbol{\omega})\, \mathrm{d}s = \int_{\mathbb{S}^{d-1}} \mathcal{D}f(\boldsymbol{a}, \boldsymbol{\xi})\, h(\boldsymbol{\omega}^\top \boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi}. \tag{4.4}$$

Inserting $h = \delta^{(d-2)}$, we obtain Grangeat's formula, which was originally proved for $d = 3$ by Grangeat [Gra91] in 1991. It states that

$$(-1)^d \left(\frac{\partial}{\partial s}\right)^{d-2} \mathcal{R}f(\boldsymbol{\omega}, \boldsymbol{a}^\top \boldsymbol{\omega}) = \mathcal{S}^{(d-2)} \mathcal{D}f(\boldsymbol{a}, \boldsymbol{\omega}), \tag{4.5}$$

where the differentiation is performed with respect to the second argument $\mathcal{R}f$ and the generalized Funk–Radon transform $\mathcal{S}^{(d-2)}$, which was defined in Section 3.4, is applied with respect to $\boldsymbol{\omega}$.

In the planar case $d = 2$, Grangeat's formula (4.5) does not contain any derivatives. The Radon transform $\mathcal{R}$ in 2D takes the integral along all lines, which can be expressed as the sum of two ray integrals of the cone-beam transform as follows. For a vector $\boldsymbol{\omega} = (\omega_1, \omega_2) \in \mathbb{S}^1$, we denote by $\boldsymbol{\omega}^\perp = (\omega_2, -\omega_1)$ its orthogonal complement. Then we have

$$\mathcal{R}f(\boldsymbol{\omega}, \boldsymbol{a}^\top \boldsymbol{\omega}) = \mathcal{D}f(\boldsymbol{a}, \boldsymbol{\omega}^\perp) + \mathcal{D}f(\boldsymbol{a}, -\boldsymbol{\omega}^\perp), \qquad \boldsymbol{a}, \boldsymbol{\omega} \in \mathbb{S}^1. \tag{4.6}$$

The following theorem gives an alternative version of Grangeat's formula for $d = 3$. However, it is not a special case of (4.4), because the function h is not homogeneous of degree -2.

Theorem 4.2. *Let $d = 3$, $\boldsymbol{\omega} \in \mathbb{S}^2$ and $\boldsymbol{a} \in \mathbb{R}^3$. We have*

$$-\mathcal{S}^{(-1)} \frac{\partial}{\partial s} \mathcal{R}f(\boldsymbol{\omega}, \boldsymbol{a}^\top \boldsymbol{\omega}) = \int_{\mathbb{S}^2} h(\boldsymbol{\xi}^\top \boldsymbol{\omega})\, \mathcal{D}f(\boldsymbol{a}, \boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi}, \tag{4.7}$$

where

$$h(t) := \frac{2t}{\sqrt{1-t^2}}, \qquad t \in (-1, 1).$$

and $\mathcal{S}^{(-1)}$ is the modified hemispherical transform (3.45) *applied with respect to $\boldsymbol{\omega}$.*

Proof. Grangeat's formula (4.5) with $d = 3$ reads

$$-\frac{\partial}{\partial s}\mathcal{R}f(\boldsymbol{\omega}, \boldsymbol{a}^\top\boldsymbol{\omega}) = \mathcal{S}^{(1)}\mathcal{D}f(\boldsymbol{a}, \boldsymbol{\omega}).$$

We apply the generalized Funk–Radon transform $\mathcal{S}^{(-1)}$ with respect to $\boldsymbol{\omega}$ on both sides and obtain

$$-\mathcal{S}^{(-1)}\frac{\partial}{\partial s}\mathcal{R}f(\boldsymbol{\omega}, \boldsymbol{a}^\top\boldsymbol{\omega}) = \mathcal{S}^{(-1)}\mathcal{S}^{(1)}\mathcal{D}f(\boldsymbol{a}, \boldsymbol{\omega}).$$

Hence, the assertion (4.7) is equivalent to

$$\mathcal{S}^{(-1)}\mathcal{S}^{(1)}\mathcal{D}f(\boldsymbol{a}, \boldsymbol{\omega}) = \int_{\mathbb{S}^2} h(\boldsymbol{\xi}^\top\boldsymbol{\omega})\,\mathcal{D}f(\boldsymbol{a}, \boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}. \tag{4.8}$$

Let $n \in \mathbb{N}_0$ and $k \in \{-n, \ldots, n\}$. We also fix $\boldsymbol{a} \in \mathbb{S}^2$. We are going to plug in the spherical harmonic $\mathcal{D}f(\boldsymbol{a}, \boldsymbol{\omega}) = Y_n^k(\boldsymbol{\omega})$ into (4.8). On the left-hand side of (4.8), we obtain by the singular value decomposition of $\mathcal{S}^{(j)}$ from Theorem 3.24

$$\mathcal{S}^{(-1)}\,\mathcal{S}^{(1)}\,Y_n^k = \hat{\mathcal{S}}^{(-1)}(n)\,\hat{\mathcal{S}}^{(1)}(n)\,Y_n^k = \begin{cases} 4\pi^2\frac{(n-2)!!\,n!!}{(n-1)!!\,(n+1)!!}\,Y_n^k, & \text{n odd,} \\ 0, & \text{n even.} \end{cases} \tag{4.9}$$

Now, we come to the right-hand side of (4.8). We make use of the identity [GR07, 8.922.4]

$$h(t) = \frac{2t}{\sqrt{1-t^2}} = \pi \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} (2n+1)\frac{(n-2)!!\,n!!}{(n-1)!!\,(n+1)!!}\,P_n(t), \qquad t \in (-1, 1).$$

We have by the Funk–Hecke formula (2.30)

$$\int_{\mathbb{S}^2} h(\boldsymbol{\xi}^\top\boldsymbol{\omega})\,Y_n^k(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} = 2\pi\int_{-1}^{1} h(t)\,P_n(t)\,\mathrm{d}t\,Y_n^k(\boldsymbol{\omega}).$$

With (4.1.1) and the help of the orthogonality (2.17) of the Legendre polynomials P_n, we see that

$$\int_{\mathbb{S}^2} h(\boldsymbol{\xi}^\top\boldsymbol{\omega})\,Y_n^k(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} = \begin{cases} 4\pi^2\frac{(n-2)!!\,n!!}{(n-1)!!\,(n+1)!!}\,Y_n^k(\boldsymbol{\omega}), & n \text{ odd} \\ 0, & n \text{ even.} \end{cases} \tag{4.10}$$

Combining (4.9) and (4.10), we obtain

$$\mathcal{S}^{(-1)}\,\mathcal{S}^{(1)}\,Y_n^k = \int_{\mathbb{S}^2} h(\boldsymbol{\xi}^\top\boldsymbol{\omega})\,Y_n^k(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi},$$

which holds for all $n \in \mathbb{N}_0$ and $k \in \{-n, \ldots, n\}$. Then (4.8) follows by the density of the spherical harmonics Y_n^k in the space $L^2(\mathbb{S}^{d-1})$. ■

4.1.2 Singular value decomposition

In the following, we consider the cone-beam transform $\mathcal{D}$ with the scanning set $\Gamma = \mathbb{S}^{d-1}$ and we assume that the function f is supported in the unit ball $\mathbb{B}^d$. We see that $\mathcal{D}f(\boldsymbol{a},\boldsymbol{\omega}) = 0$ for all $\boldsymbol{\omega} \in \mathbb{S}^{d-1}$ with $\boldsymbol{a}^\top\boldsymbol{\omega} \geq 0$ since the ray of integration is outside $\mathbb{B}^d$. We denote the odd part of the cone-beam transform $\mathcal{D}(\boldsymbol{a},\cdot)$ by

$$\mathcal{D}^{(\mathrm{odd})}f(\boldsymbol{a},\boldsymbol{\omega}) := \frac{Df(\boldsymbol{a},\boldsymbol{\omega}) - Df(\boldsymbol{a},-\boldsymbol{\omega})}{2}.$$

Then

$$\mathcal{D}f(\boldsymbol{a},\boldsymbol{\omega}) = 2\mathcal{D}^{(\mathrm{odd})}f(\boldsymbol{a},\boldsymbol{\omega})$$

for all $\boldsymbol{\omega} \in \mathbb{S}^{d-1}$ with $\boldsymbol{a}^\top\boldsymbol{\omega} < 0$ and $\mathcal{D}f(\boldsymbol{a},\boldsymbol{\omega}) = 0$ otherwise.

Lemma 4.3. *Let $m \in \mathbb{N}_0$ and $d \geq 3$ be odd. Then the following differentiation identity for the Gegenbauer polynomial $C_m^{(\frac{d}{2})}$ from (2.21) holds. We have for $s \in \mathbb{R}$*

$$(-1)^{\frac{d-1}{2}} \frac{(d-2)\,m!}{(m+d-1)!} \left(\frac{\partial}{\partial s}\right)^{d-2} (1-s^2)^{\frac{d-1}{2}}\, C_m^{(\frac{d}{2})}(s) = C_{m+1}^{(\frac{d-2}{2})}(s). \tag{4.11}$$

Proof. We denote the right-hand side of (4.11) by $B_{m+1}^{(\frac{d-2}{2})}(s)$, i. e., we set

$$B_{m+1}^{(\frac{d-2}{2})}(s) = (-1)^{\frac{d-1}{2}} \frac{(d-2)\,m!}{(m+d-1)!} \left(\frac{\partial}{\partial s}\right)^{d-2} (1-s^2)^{\frac{d-1}{2}}\, C_m^{(\frac{d}{2})}(s).$$

We obtain with Rodrigues' formula (2.21) for the Gegenbauer polynomials

$$\begin{aligned} B_{m+1}^{(\frac{d-2}{2})}(s) &= (-1)^{\frac{d-1}{2}} \frac{(d-2)\,m!}{(m+d-1)!} \frac{(-1)^m\,(\frac{d-1}{2})!\,(m+d-1)!}{2^m\,m!\,(d-1)!\,(m+\frac{d-1}{2})!} \left(\frac{\partial}{\partial s}\right)^{m+d-2} (1-s^2)^{m+\frac{d-1}{2}} \\ &= \frac{(-1)^{m+\frac{d-1}{2}}\,(d-2)\,(\frac{d-1}{2})!}{2^m\,(d-1)!\,(m+\frac{d-1}{2})!} \left(\frac{\partial}{\partial s}\right)^{m+d-2} (1-s^2)^{m+\frac{d-1}{2}}. \end{aligned}$$

We compute with the binomial theorem (3.38)

$$B_{m+1}^{(\frac{d-2}{2})}(s) = \frac{(-1)^{m+\frac{d-1}{2}}\,(d-2)\,(\frac{d-1}{2})!}{2^m\,(d-1)!\,(m+\frac{d-1}{2})!} \sum_{i=0}^{m+\frac{d-1}{2}} (-1)^i \binom{m+\frac{d-1}{2}}{i} \left(\frac{\partial}{\partial s}\right)^{m+d-2} s^{2i}.$$

Considering

$$\left(\frac{\partial}{\partial s}\right)^{k} s^{2i} = \frac{(2i)!}{(2i-k)!} s^{2i-k},$$

we have

$$\begin{aligned} &B_{m+1}^{(\frac{d-2}{2})}(s) \\ &\quad= \frac{(-1)^{m+\frac{d-1}{2}}\,(d-2)\,(\frac{d-1}{2})!}{2^m\,(d-1)!\,(m+\frac{d-1}{2})!} \sum_{i=\lceil\frac{m+d-2}{2}\rceil}^{m+\frac{d-1}{2}} \binom{m+\frac{d-1}{2}}{i} \frac{(-1)^i\,(2i)!}{(2i-m-d+2)!} s^{2i-m-d+2}. \end{aligned}$$

Shifting the index $i \mapsto l$ with $i = m - l + \frac{d-1}{2}$, we obtain

$$
\begin{aligned}
&B_{m+1}^{(\frac{d-2}{2})}(s) \\
&= \frac{(-1)^{m+\frac{d-1}{2}}\,(d-2)\,(\frac{d-1}{2})!}{2^m\,(d-1)!\,(m+\frac{d-1}{2})!} \sum_{l=0}^{\lfloor\frac{m+1}{2}\rfloor} \frac{(-1)^{m-l+\frac{d-1}{2}}\,(m+\frac{d-1}{2})!}{(m-l+\frac{d-1}{2})!\,l!} \frac{(2m-2l+d-1)!}{(m+1-2l)!}\, s^{m+1-2l} \\
&= \frac{(d-2)\,(\frac{d-1}{2})!}{2^m\,(d-1)!} \sum_{l=0}^{\lfloor\frac{m+1}{2}\rfloor} \frac{(-1)^l}{(m-l+\frac{d-1}{2})!\,l!} \frac{(2m-2l+d-1)!}{(m+1-2l)!}\, s^{m+1-2l}.
\end{aligned}
$$

Because

$$
\frac{(2m)!}{m!} = \frac{2^m\,(2m)!}{(2m)!!} = 2^m\,(2n-1)!!,
$$

we have

$$
\begin{aligned}
B_{m+1}^{(\frac{d-2}{2})}(s) &= \frac{(d-2)}{2^m\,(d-2)!!\,2^{\frac{d-1}{2}}} \sum_{l=0}^{\lfloor\frac{m+1}{2}\rfloor} \frac{(-1)^l\,2^{m-l+\frac{d-1}{2}}\,(2m-2l+d-2)!!}{l!\,(m+1-2l)!}\, s^{m+1-2l} \\
&= \frac{1}{(d-4)!!} \sum_{l=0}^{\lfloor\frac{m+1}{2}\rfloor} \frac{(-1)^l\,2^{-l}\,(2m-2l+d-2)!!}{l!\,(m+1-2l)!}\, s^{m+1-2l}.
\end{aligned}
$$

We rewrite the quotient of double factorials with the Gamma function

$$
\frac{(m+2k)!!}{m!!} = 2^k \left(\frac{m+2k}{2}\right)\left(\frac{m+2k-2}{2}\right)\cdots\left(\frac{m+2}{2}\right) = 2^k \frac{\Gamma\left(\frac{m+2k+2}{2}\right)}{\Gamma\left(\frac{2k+2}{2}\right)}
$$

and obtain

$$
B_{m+1}^{(\frac{d-2}{2})}(s) = \sum_{l=0}^{\lfloor\frac{m+1}{2}\rfloor} \frac{(-1)^l\,\Gamma(m-l+\frac{d}{2})}{\Gamma(\frac{d-2}{2})\,l!\,(m+1-2l)!}\,(2s)^{m+1-2l},
$$

which is exactly the formula (2.20) for the Gegenbauer polynomial $C_{m+1}^{(\frac{d-2}{2})}(s)$. ■

Theorem 4.4. *Let $m \in \mathbb{N}_0$, $l = 0,\ldots,m$ with $l+m$ even, $k \in \{1,\ldots,N_{l,d}\}$ and $d \geq 3$ odd. The odd cone-beam transform $\mathcal{D}^{(odd)}\colon C(\mathbb{B}^d) \to C(\mathbb{S}^{d-1} \times \mathbb{S}^{d-1})$ satisfies for $\boldsymbol{a},\boldsymbol{\omega} \in \mathbb{S}^{d-1}$*

$$
\mathcal{D}^{(\mathrm{odd})}\widetilde{V}_{m,l,k}(\boldsymbol{a},\boldsymbol{\omega}) = \mu_{m,d} \sum_{j=1}^{N_{m+1,d}} \overline{Y_{m+1,d}^j(\boldsymbol{a})} \sideset{}{'}\sum_{n=m+1-l}^{l+m+1} \nu_{n,d} \sum_{i=1}^{N_{n,d}} G_{m+1,j,l,k}^{n,i,d}\, Y_{n,d}^i(\boldsymbol{\omega}),
$$

where $\sum'$ denotes the summation over odd indices, $\widetilde{V}_{m,l,k}$ is given in (4.3) *and*

$$
\mu_{m,d} := \sqrt{\frac{2^{d+1}\,\pi^{d-1}}{2m+d}}, \tag{4.12}
$$

$$
\nu_{n,d} := \frac{(-1)^{\frac{n+1}{2}}\,(n-1)!!}{(n+d-3)!!}. \tag{4.13}
$$

Proof. Let $m \in \mathbb{N}_0$, $l \in \{0, \dots, m\}$ with $m + l$ even, $k \in \{1, \dots, N_{l,d}\}$ and d be odd. We have by the singular value decomposition (4.2) of the Radon transform and Lemma 4.3

$$\begin{aligned}
&\left(\frac{\partial}{\partial s}\right)^{d-2} \mathcal{R}\widetilde{V}_{m,l,k}(\boldsymbol{\omega}, s) \\
&= \frac{2^{d-1}\,\pi^{\frac{d}{2}-1}\sqrt{2m+d}\,\Gamma(\frac{d}{2})\,m!}{(m+d-1)!} \left(\frac{\partial}{\partial s}\right)^{d-2} (1-s^2)^{\frac{d-1}{2}}\, C_m^{(\frac{d}{2})}(s)\, Y_{l,d}^k(\boldsymbol{\omega}) \\
&= \frac{2^{d-1}\,\pi^{\frac{d}{2}-1}\sqrt{2m+d}\,\Gamma(\frac{d}{2})\,m!}{(m+d-1)!} (-1)^{\frac{d-1}{2}} \frac{(m+d-1)!}{(d-2)\,m!} C_{m+1}^{(\frac{d-2}{2})}(s)\, Y_{l,d}^k(\boldsymbol{\omega}) \\
&= \frac{2^{d-1}\,\pi^{\frac{d}{2}-1}\sqrt{2m+d}\,\Gamma(\frac{d}{2})}{d-2} (-1)^{\frac{d-1}{2}}\, C_{m+1}^{(\frac{d-2}{2})}(s)\, Y_{l,d}^k(\boldsymbol{\omega}).
\end{aligned}$$

By Grangeat's formula (4.5) and the relation (2.19) between the Gegenbauer and the Legendre polynomials, we obtain

$$\begin{aligned}
&\mathcal{S}^{(d-2)}\mathcal{D}\widetilde{V}_{m,l,k}(\boldsymbol{a}, \boldsymbol{\omega}) \\
&\qquad = (-1)^{\frac{d+1}{2}} \frac{2^{d-1}\,\pi^{\frac{d}{2}-1}\sqrt{2m+d}\,\Gamma(\frac{d}{2})\,(m+d-2)!}{(m+1)!\,(d-2)!} P_{m+1,d}(\boldsymbol{a}^\top\boldsymbol{\omega})\, Y_{l,d}^k(\boldsymbol{\omega}).
\end{aligned}$$

By the addition formula (2.25) for spherical harmonics, we have

$$\mathcal{S}^{(d-2)}\mathcal{D}\widetilde{V}_{m,l,k}(\boldsymbol{a}, \boldsymbol{\omega}) = (-1)^{\frac{d+1}{2}} \left|\mathbb{S}^{d-1}\right| \frac{2^{d-1}\,\pi^{\frac{d}{2}-1}\,\Gamma(\frac{d}{2})}{\sqrt{2m+d}} \sum_{j=1}^{N_{m+1,d}} \overline{Y_{m+1,d}^j(\boldsymbol{a})}\, Y_{m+1,d}^j(\boldsymbol{\omega})\, Y_{l,d}^k(\boldsymbol{\omega}).$$

By the linearization formula (2.58) for spherical harmonics, we see that

$$\begin{aligned}
&\mathcal{S}^{(d-2)}\mathcal{D}\widetilde{V}_{m,l,k}(\boldsymbol{a}, \boldsymbol{\omega}) \\
&\quad = (-1)^{\frac{d-1}{2}} \left|\mathbb{S}^{d-1}\right| \frac{2^{d+1}\,\pi^{\frac{d}{2}-1}\,\Gamma(\frac{d}{2})}{\sqrt{2m+d}} \sum_{j=1}^{N_{m+1,d}} \overline{Y_{m+1,d}^j(\boldsymbol{a})} \sideset{}{'}\sum_{n=m+1-l}^{m+1+l} \sum_{i=1}^{N_{n,d}} G_{m+1,j,l,k}^{n,i,d} Y_{n,d}^i(\boldsymbol{\omega}).
\end{aligned}$$

Since d is assumed to be odd and the generalized Funk–Radon transform $\mathcal{S}^{(d-2)}$ acts only on odd functions, we have $\mathcal{S}^{(d-2)}\mathcal{D} = \mathcal{S}^{(d-2)}\mathcal{D}^{(\mathrm{odd})}$. Then the eigenvalue decomposition (3.34) of $\mathcal{S}^{(d-2)}$ yields

$$\begin{aligned}
\mathcal{D}^{(\mathrm{odd})}\widetilde{V}_{m,l,k}(\boldsymbol{a}, \boldsymbol{\omega}) = (-1)^{\frac{d+1}{2}} \frac{\left|\mathbb{S}^{d-1}\right|}{\left|\mathbb{S}^{d-2}\right|} \frac{2^{d-1}\,\pi^{\frac{d}{2}-1}\,\Gamma(\frac{d}{2})}{\sqrt{2m+d}} \sum_{j=1}^{N_{m+1,d}} \overline{Y_{m+1,d}^j(\boldsymbol{a})} \\
\sideset{}{'}\sum_{n=m+1-l}^{l+m+1} (-1)^{\frac{n+d-2}{2}} \frac{(n-1)!!}{(n+d-3)!!\,(d-3)!!} \sum_{i=1}^{N_{n,d}} G_{m+1,j,l,k}^{n,i,d} Y_{n,d}^i(\boldsymbol{\omega}).
\end{aligned}$$

Since, by (2.10),

$$\frac{\left|\mathbb{S}^{d-1}\right|}{\left|\mathbb{S}^{d-2}\right|} = \frac{\sqrt{\pi}\,\Gamma(\frac{d-1}{2})}{\Gamma(\frac{d}{2})} = \frac{\sqrt{\pi}\,(\frac{d-3}{2})!}{\Gamma(\frac{d}{2})} = \frac{\sqrt{\pi}\,2^{-\frac{d-3}{2}}\,(d-3)!!}{\Gamma(\frac{d}{2})},$$

we obtain

$$\mathcal{D}^{(\mathrm{odd})}\widetilde{V}_{m,l,k}(\boldsymbol{a},\boldsymbol{\omega})$$
$$= \frac{2^{\frac{d+1}{2}}\pi^{\frac{d-1}{2}}}{\sqrt{2m+d}} \sum_{j=1}^{N_{m+1,d}} \overline{Y^j_{m+1,d}(\boldsymbol{a})} \sum_{n=m+1-l}^{l+m+1}{}' (-1)^{\frac{n+1}{2}} \frac{(n-1)!!}{(n+d-3)!!} \sum_{i=1}^{N_{n,d}} G^{n,i,d}_{m+1,j,l,k} Y^i_{n,d}(\boldsymbol{\omega}). \quad \blacksquare$$

Theorem 4.4 shows how the cone-beam transform $\mathcal{D}^{(\mathrm{odd})}$ acts on the orthogonal polynomials $\widetilde{V}_{m,l,d}$ on the ball B^d. In order to show that this is indeed a singular value decomposition, we still have to show that the functions $\mathcal{D}^{(\mathrm{odd})}\widetilde{V}_{m,l,d}$ are orthogonal.

Theorem 4.5. *The functions*

$$\mathcal{D}^{(\mathrm{odd})}\widetilde{V}_{m,l,k}, \qquad m \in \mathbb{N}_0,\ l = 0,\dots,m,\ k = 1,\dots,N_{l,d},\ l+m \text{ even}$$

are orthogonal in the space $L^2(\mathbb{S}^{d-1}\times\mathbb{S}^{d-1})$. They have the norm

$$\hat{\mathcal{D}}_{m,l,d} := \left\|\mathcal{D}^{(\mathrm{odd})}\widetilde{V}_{m,l,k}\right\|_{L^2(\mathbb{S}^{d-1}\times\mathbb{S}^{d-1})}$$
$$= \sqrt{\frac{N_{m+1,d}\,\mu^2_{m,d}\,|\mathbb{S}^{d-2}|}{|\mathbb{S}^{d-1}|^2} \sum_{n=m+1-l}^{m+1+l}{}' \nu^2_{n,d}\, N_{n,d}\, \langle P_{m+1,d}\,P_{l,d}, P_{n,d}\rangle_{w_d}}, \tag{4.14}$$

where

$$\langle P_{m+1,d}\,P_{l,d}, P_{n,d}\rangle_{w_d} = \int_{-1}^{1} P_{m+1,d}(t)\,P_{l,d}(t)\,P_{n,d}(t)\,(1-t^2)^{\frac{d-3}{2}}\,\mathrm{d}t.$$

Proof. Let $m, m' \in \mathbb{N}_0,\ l = 0,\dots,m,\ l' = 0,\dots,m',\ k = 1,\dots,N_{l,d},\ k' = 1,\dots,N_{l',d}$ such that $m+l$ and $m'+l'$ are even. We have

$$\left\langle \mathcal{D}^{(\mathrm{odd})}\widetilde{V}_{m,l,k}, \mathcal{D}^{(\mathrm{odd})}\widetilde{V}_{m',l',k'}\right\rangle_{L^2(\mathbb{S}^{d-1}\times\mathbb{S}^{d-1})}$$
$$= \mu_{m,d}\,\mu_{m',d} \sum_{j=1}^{N_{m+1,d}} \sum_{j'=1}^{N_{m'+1,d}} \int_{\mathbb{S}^{d-1}} \overline{Y^j_{m+1,d}(\boldsymbol{a})}\, Y^{j'}_{m'+1,d}(\boldsymbol{a})\,\mathrm{d}\boldsymbol{a}$$
$$\sum_{n=m+1-l}^{m+1+l}{}' \sum_{n'=m'+1-l'}^{m'+1+l'}{}' \nu_{n,d}\,\nu_{n',d} \sum_{i=1}^{N_{n,d}} \sum_{i'=1}^{N_{n',d}} G^{n,i,d}_{m+1,j,l,k}\, \overline{G^{n',i',d}_{m'+1,j',l',k'}} \int_{\mathbb{S}^{d-1}} Y^i_{n,d}(\boldsymbol{\omega})\, \overline{Y^{i'}_{n',d}(\boldsymbol{\omega})}\,\mathrm{d}\boldsymbol{\omega}.$$

By the orthonormality of the spherical harmonics, we obtain

$$\left\langle \mathcal{D}^{(\mathrm{odd})}\widetilde{V}_{m,l,k}, \mathcal{D}^{(\mathrm{odd})}\widetilde{V}_{m',l',k'}\right\rangle_{L^2(\mathbb{S}^{d-1}\times\mathbb{S}^{d-1})}$$
$$= \delta_{m,m'}\,\mu^2_{m,d} \sum_{j=1}^{N_{m+1,d}} \sum_{n=m+1-l}^{m+1+l}{}' \nu^2_{n,d} \sum_{i=1}^{N_{n,d}} G^{n,i,d}_{m+1,j,l,k}\, \overline{G^{n,i,d}_{m+1,j,l',k'}}.$$

We have by the definition of the Gaunt coefficients in (2.57)

$$\begin{aligned}
&\sum_{j=1}^{N_{m+1,d}} \sum_{i=1}^{N_{n,d}} G^{n,i,d}_{m+1,j,l,k} \overline{G^{n,i,d}_{m+1,j,l',k'}} \\
&= \sum_{j=1}^{N_{m+1,d}} \sum_{i=1}^{N_{n,d}} \int_{\mathbb{S}^{d-1}} Y^j_{m+1,d}(\boldsymbol{\xi})\, Y^k_{l,d}(\boldsymbol{\xi})\, \overline{Y^i_{n,d}(\boldsymbol{\xi})}\, \mathrm{d}\boldsymbol{\xi} \int_{\mathbb{S}^{d-1}} \overline{Y^j_{m+1,d}(\boldsymbol{\eta})\, Y^{k'}_{l',d}(\boldsymbol{\eta})}\, Y^i_{n,d}(\boldsymbol{\eta})\, \mathrm{d}\boldsymbol{\eta} \\
&= \frac{N_{m+1,d}\, N_{n,d}}{\left|\mathbb{S}^{d-1}\right|^2} \int_{\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} P_{m+1,d}(\boldsymbol{\xi}^\top \boldsymbol{\eta})\, P_{n,d}(\boldsymbol{\xi}^\top \boldsymbol{\eta})\, Y^k_{l,d}(\boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi}\, \overline{Y^{k'}_{l',d}(\boldsymbol{\eta})}\, \mathrm{d}\boldsymbol{\eta},
\end{aligned}$$

where the last equality follows from the addition formula (2.25) for spherical harmonics. Applying the Funk–Hecke formula (2.30) to the inner integral, we obtain

$$\begin{aligned}
&\sum_{j=1}^{N_{m+1,d}} \sum_{i=1}^{N_{n,d}} G^{n,i,d}_{m+1,j,l,k} \overline{G^{n,i,d}_{m+1,j,l',k'}} \\
&= \frac{N_{m+1,d}\, N_{n,d} \left|\mathbb{S}^{d-2}\right|}{\left|\mathbb{S}^{d-1}\right|^2} \int_{-1}^{1} P_{m+1,d}(t)\, P_{n,d}(t)\;\, P_{l,d}(t)\, (1-t^2)^{\frac{d-3}{2}}\, \mathrm{d}t \int_{\mathbb{S}^{d-1}} Y^k_{l,d}(\boldsymbol{\eta})\, Y^k_{l',d}(\boldsymbol{\eta})\, \mathrm{d}\boldsymbol{\eta} \\
&= \delta_{l,l'}\, \delta_{k,k'}\, \frac{N_{m+1,d}\, N_{n,d}}{\left|\mathbb{S}^{d-1}\right|^2} \left|\mathbb{S}^{d-2}\right| \int_{-1}^{1} P_{m+1,d}(t)\, P_{n,d}(t)\;\, P_{l,d}(t)\, (1-t^2)^{\frac{d-3}{2}}\, \mathrm{d}t,
\end{aligned}$$

where we used again the orthonormality of the spherical harmonics. By [CK10], the value of the integral $\langle P_{m+1,d}\, P_{n,d}, P_{l,d}\rangle_{w_d}$ is nonzero if and only if

$$n \in \{|m+1-l|\,,\, |m+1-l|+2,\, \ldots,\, m+1+l\}.$$

Hence, we have

$$\begin{aligned}
&\left\langle \mathcal{D}^{(\mathrm{odd})} \widetilde{V}_{m,l,k}, \mathcal{D}^{(\mathrm{odd})} \widetilde{V}_{m',l',k'} \right\rangle_{L^2(\mathbb{S}^{d-1}\times\mathbb{S}^{d-1})} \\
&\qquad = \delta_{m,m'}\, \delta_{l,l'}\, \delta_{k,k'}\, \frac{N_{m+1,d}\, \mu^2_{m,d}\, \left|\mathbb{S}^{d-2}\right|}{\left|\mathbb{S}^{d-1}\right|^2} \sideset{}{'}\sum_{n=m+1-l}^{m+1+l} \nu^2_{n,d}\, N_{n,d}\, \langle P_{m+1,d}\, P_{n,d}, P_{l,d}\rangle_{w_d}.
\end{aligned}$$

■

In the notation of (3.12), Theorem 4.5 implies that the cone-beam transform $\mathcal{D}^{(\mathrm{odd})}$ has the singular value decomposition

$$\left\{ \left(\widetilde{V}_{m,l,k},\, \hat{\mathcal{D}}_{m,l,d},\, \frac{1}{\hat{\mathcal{D}}_{m,l,d}} \mathcal{D}^{(\mathrm{odd})} \widetilde{V}_{m,l,k} \right) ;\, m \in \mathbb{N}_0,\ l = 0, \ldots, m,\ k = 1, \ldots, N_{l,d},\ l+m \text{ even} \right\}$$

with the singular values $\hat{\mathcal{D}}_{m,l,d}$ given in (4.14).

Remark 4.6. In Theorem 4.4, we have shown the singular value decomposition of the cone-beam transform $\mathcal{D}^{(\mathrm{odd})}$ for the case that the dimension $d \geq 3$ is odd. If d is even, Lemma 4.3 does not hold any more, which follows from the fact that the left-hand side of (4.11) is not a polynomial for d even, but the right-hand side is a polynomial. □

4.1.3 Bounds on the singular values

Upper bound

In this section, we show that the singular values $\hat{\mathcal{D}}_{m,l,d}$ of the cone-beam transform $\mathcal{D}^{(\mathrm{odd})}$, which are given in (4.14), are bounded independently of m and l, which implies that the cone-beam transform as operator $\mathcal{D}^{(\mathrm{odd})} \colon L^2(\mathbb{B}^d) \to L^2(\mathbb{S}^{d-1} \times \mathbb{S}^{d-1})$ is bounded.

Lemma 4.7. *Let $n \geq 1$ and $d \geq 3$ be odd integers. Then the coefficients $\nu_{n,d}$ given in (4.13) are bounded by*

$$\nu_{n,d}^2 N_{n,d} \leq \begin{cases} \pi, & d = 3 \\ \frac{d}{((d-2)!!)^2}, & d \geq 5. \end{cases}$$

Furthermore, we have

$$\lim_{\substack{n\to\infty \\ n \text{ odd}}} \nu_{n,d}^2 N_{n,d} = \frac{\pi}{(d-2)!}. \tag{4.15}$$

Proof. We have by (4.13) and (2.12)

$$\begin{aligned} \nu_{n,d}^2 N_{n,d} &= \frac{(n-1)!!^2}{(n+d-3)!!^2} \frac{(2n+d-2)\,(n+d-3)!}{n!\,(d-2)!} \\ &= \frac{(n-1)!!}{n!!} \frac{(n+d-4)!!}{(n+d-3)!!} \frac{2n+d-2}{(d-2)!}. \end{aligned}$$

We have

$$\begin{aligned} \frac{\nu_{n+2,d}^2 N_{n+2,d}}{\nu_{n,d}^2 N_{n,d}} &= \frac{n+1}{n+2} \frac{n+d-2}{n+d-1} \frac{2n+d+2}{2n+d-2} \\ &= \frac{2n^3 + 3dn^2 + (d^2+3d-6)n + d^2 - 4}{2n^3 + 3dn^2 + (d^2+3d-6)n + 2d^2 - 6d + 4}. \end{aligned} \tag{4.16}$$

Comparing the numerator with the denominator of (4.16), we see that the sequence $n \mapsto \nu_{n,d}^2 N_{n,d}$ is increasing for $d = 3$ and decreasing for $d \geq 5$. The fact that

$$\nu_{1,d}^2 N_{1,d} = \frac{d}{((d-2)!!)^2}$$

shows the upper bound for $d \geq 5$.

We note that, by (2.73), we have for $n \to \infty$ with n odd

$$(2n-1)!! \simeq \frac{(2n)!!}{\sqrt{\pi(n+\frac{1}{2})}}.$$

Hence, we obtain

$$\nu_{n,d}^2 N_{n,d} \simeq \frac{\sqrt{\pi(\frac{n}{2}+1)}}{n+1} \frac{\sqrt{\pi(\frac{n+d-1}{2})}}{n+d-4} \frac{2n+d-2}{(d-2)!} \simeq \frac{\pi}{(d-2)!},$$

which implies (4.15). Since, for $d = 3$, the sequence $n \mapsto \nu_{n,3}^2 N_{n,3}$ for $n \to \infty$ is increasing, its limit

$$\lim_{\substack{n\to\infty \\ n \text{ odd}}} \nu_{n,3}^2 N_{n,3} = \pi$$

is an upper bound. ■

Theorem 4.8. *Let $d \geq 3$ be an odd integer and $m, l \in \mathbb{N}_0$ such that $l \leq m$ and $m + l$ is even. Then the singular values $\hat{\mathcal{D}}_{m,l,d}$ of the cone-beam transform $\mathcal{D}$ satisfy*

$$\begin{aligned} \left|\hat{\mathcal{D}}_{m,l,d}\right| &\leq 2^{\frac{d+1}{4}} \pi^{\frac{d-1}{4}} \sqrt{C_d\,(d-2)!!} \sqrt{\frac{l+1}{2m+d}} \\ &\leq (2\pi)^{\frac{d-1}{4}} \sqrt{C_d\,(d-2)!!}, \end{aligned}$$

where

$$C_d := \begin{cases} \pi, & d = 3 \\ \frac{d}{((d-2)!!)^2}, & d \geq 5. \end{cases}$$

In particular, we have $\lim_{m\to\infty} \hat{\mathcal{D}}_{m,l,d} = 0$ for all $l \in \mathbb{N}_0$.

Proof. Because of the orthogonality (2.17) of the Legendre polynomials, we have for $n, l \in \mathbb{N}_0$

$$P_{l,d}\, P_{n,d} = \sum_{m=|l-n|-1}^{l+n-1} \frac{N_{m+1,d} \left|\mathbb{S}^{d-2}\right|}{\left|\mathbb{S}^{d-1}\right|} \left\langle P_{m+1,d} P_{l,d}, P_{n,d}\right\rangle_{w_d} P_{m+1,d}.$$

Utilizing the fact that $P_{i,d}(1) = 1$ for all $i \in \mathbb{N}_0$, we obtain

$$1 = \sum_{m=|l-n|-1}^{l+n-1} \frac{N_{m+1,d} \left|\mathbb{S}^{d-2}\right|}{\left|\mathbb{S}^{d-1}\right|} \left\langle P_{m+1,d} P_{l,d}, P_{n,d}\right\rangle_{w_d}. \tag{4.17}$$

Since all summands in the above sum (4.17) are non-negative, they are bounded by

$$\frac{N_{m+1,d} \left|\mathbb{S}^{d-2}\right|}{\left|\mathbb{S}^{d-1}\right|} \left\langle P_{m+1} P_l, P_n\right\rangle_{w_d} \leq 1. \tag{4.18}$$

Inserting the bound from Lemma 4.7 into the definition of the singular values (4.14), we have

$$\begin{aligned} \left|\hat{\mathcal{D}}_{m,l,d}\right|^2 &= \frac{N_{m+1,d}\, \mu_{m,d}^2 \left|\mathbb{S}^{d-2}\right|}{\left|\mathbb{S}^{d-1}\right|^2} \sum_{n=m+1-l}^{m+1+l}{}' \nu_{n,d}^2\, N_{n,d} \left\langle P_{m+1,d}\, P_{l,d}, P_{n,d}\right\rangle_{w_d} \\ &\leq \frac{N_{m+1,d}\, \mu_{m,d}^2 \left|\mathbb{S}^{d-2}\right|}{\left|\mathbb{S}^{d-1}\right|^2} C_d \sum_{n=m+1-l}^{m+1+l}{}' \left\langle P_{m+1,d}\, P_{l,d}, P_{n,d}\right\rangle_{w_d}. \end{aligned}$$

With (4.18), we obtain

$$\begin{aligned}\left|\hat{\mathcal{D}}_{m,l,d}\right|^2 &\le C_d \, \frac{\mu_{m,d}^2}{\left|\mathbb{S}^{d-1}\right|} \sum_{n=m+1-l}^{m+1+l}{}' \; 1 \\ &= C_d \, \frac{\mu_{m,d}^2}{\left|\mathbb{S}^{d-1}\right|} \,(l+1) \\ &= C_d \, 2^{\frac{d+1}{2}} \, \pi^{\frac{d-1}{2}} \,(d-2)!! \, \frac{l+1}{2m+d},\end{aligned}$$

where we inserted the formulas of $\mu_{m,d}$ from (4.12) and $\left|\mathbb{S}^{d-1}\right|$ from (2.10). ■

With the help of the singular value decomposition in Theorem 4.5 and the bound of the singular values in Theorem 4.8, we obtain the continuity of the cone-beam transform $\mathcal{D}^{(\mathrm{odd})}$ in the space $l^2(\mathbb{B}^d)$ as follows.

Corollary 4.9. The cone-beam transform $\mathcal{D}^{(\mathrm{odd})}$ is a continuous operator

$$\mathcal{D}^{(\mathrm{odd})} \colon L^2(\mathbb{B}^d) \to L^2(\mathbb{S}^{d-1} \times \mathbb{S}^{d-1}).$$

Lower bound

As we just did for the upper bound in Theorem 4.8, we also find a lower bound of the singular values $\left|\hat{\mathcal{D}}_{m,l,d}\right|$ of the cone-beam transform $\mathcal{D}$. However, this lower bound is tight.

Theorem 4.10. *Let $d \ge 3$ be an odd integer. There exists a constant $c_d > 0$, which depends only on the dimension d, such that for all $m \in \mathbb{N}_0$ and $l \in \{0, \dots, m\}$ with $m+l$ even, the singular values $\hat{\mathcal{D}}_{m,l,d}$ of the cone-beam transform $\mathcal{D}$ admit the lower bound*

$$\left|\hat{\mathcal{D}}_{m,l,d}\right| \ge c_d \, m^{-1/2}. \tag{4.19}$$

This bound is asymptotically tight, in the sense that the exponent $-1/2$ in (4.19) cannot be replaced by a greater one.

Proof. We extract the smallest value of $\nu_{n,d}^2$ in the following sum

$$\begin{aligned}\left|\hat{\mathcal{D}}_{m,l,d}\right|^2 &= \frac{N_{m+1,d} \, \mu_{m,d}^2 \left|\mathbb{S}^{d-2}\right|}{\left|\mathbb{S}^{d-1}\right|^2} \sum_{n=m+1-l}^{m+1+l}{}' \; \nu_{n,d}^2 \, N_{n,d} \left\langle P_{m+1,d} \, P_{l,d}, P_{n,d}\right\rangle_{w_d} \\ &\ge \frac{N_{m+1,d} \, \mu_{m,d}^2}{\left|\mathbb{S}^{d-1}\right|} \left(\min_{n=m+1-l}^{m+1+l} \nu_{n,d}^2\right) \sum_{n=m+1-l}^{m+1+l}{}' \; \frac{N_{n,d} \left|\mathbb{S}^{d-2}\right|}{\left|\mathbb{S}^{d-1}\right|} \left\langle P_{m+1,d} \, P_{l,d}, P_{n,d}\right\rangle_{w_d}.\end{aligned}$$

Utilizing (4.17) with the roles of $m+1$ and n interchanged, we obtain

$$\left|\hat{\mathcal{D}}_{m,l,d}\right|^2 \ge \frac{N_{m+1,d} \, \mu_{m,d}^2}{\left|\mathbb{S}^{d-1}\right|} \min_{n=m+1-l}^{m+1+l} \nu_{n,d}^2.$$

Since the map

$$n \mapsto \nu_{n,d}^2 = \frac{(n-1)!!^2}{(n+d-3)!!^2}$$

is decreasing, we have

$$\min_{n=m+1-l}^{m+1+l} \nu_{n,d}^2 = \nu_{m+1+l,d}^2.$$

Because $0 \leq l \leq m$ and again $\nu_{m+1+l,d}^2$ decreases with respect to l, we further see that

$$\min_{n=m+1-l}^{m+1+l} \nu_{n,d}^2 \geq \nu_{2m+1,d}^2 = \frac{(2m)!!^2}{(2m+d-2)!!^2}.$$

Hence, we have

$$\begin{aligned}
\left|\hat{\mathcal{D}}_{m,l,d}\right|^2 &\geq \frac{N_{m+1,d}\,\mu_{m,d}^2}{\left|\mathbb{S}^{d-1}\right|} \frac{(2m)!!^2}{(2m+d-2)!!^2} \\
&= \frac{2^{d+1}\,\pi^{d-1}}{(2m+d)} \frac{(d-2)!!}{2^{\frac{d+1}{2}}\pi^{\frac{d-1}{2}}} \frac{(2m+d)\,(m+d-2)!}{(m+1)!\,(d-2)!} \frac{(2m)!!^2}{(2m+d-2)!!^2} \\
&= \frac{2^{\frac{d+1}{2}}\pi^{\frac{d-1}{2}}}{(d-3)!!} \frac{(m+d-2)!}{(m+1)!} \frac{(2m)!!^2}{(2m+d-2)!!^2},
\end{aligned}$$

where we inserted (2.12), (4.12) and (2.10). We are going to apply Stirling's approximation of the factorial (2.71) and the double factorials (2.74), (2.75). We obtain for $m \to \infty$

$$\begin{aligned}
\left|\hat{\mathcal{D}}_{m,l,d}\right|^2 &\geq \frac{2^{\frac{d+1}{2}}\pi^{\frac{d-1}{2}}}{(d-3)!!} \frac{(m+d-2)!}{(m+1)!} \frac{(2m)!!^2}{(2m+d-2)!!^2} \\
&\simeq \frac{2^{\frac{d+1}{2}}\pi^{\frac{d-1}{2}}}{(d-3)!!} \frac{(m+d-2)^{m+d-3/2}\,\mathrm{e}^{-m-d+2}}{(m+1)^{m+3/2}\,\mathrm{e}^{-m-1}} \frac{(2m)^{2m+1}\,\mathrm{e}^{-2m}\pi}{2(2m+d-1)^{2m+d-1}\,\mathrm{e}^{-2m-d-1}} \\
&\simeq \frac{2^{\frac{3-d}{2}}\pi^{\frac{d+1}{2}}}{(d-3)!!}\,\mathrm{e}^4 \frac{(m+d-2)^{m+d-3/2}}{(m+1)^{m+3/2}} \frac{m^{2m+1}}{(m+\frac{d-1}{2})^{2m+d-1}}.
\end{aligned}$$

Hence, there exists a constant $c_d > 0$ such that

$$\left|\hat{\mathcal{D}}_{m,l,d}\right| \geq \sqrt{\frac{2^{\frac{d+1}{2}}\pi^{\frac{d-1}{2}}}{(d-3)!!} \frac{(m+d-2)!}{(m+1)!} \frac{(2m)!!^2}{(2m+d-2)!!^2}} \geq c_d\, m^{-1/2}.$$

In order to show that this bound is tight, we consider the case m even and $l = 0$. We have by (2.17)

$$\begin{aligned}
\left|\hat{\mathcal{D}}_{m,0,d}\right|^2 &= \frac{N_{m+1,d}\,\mu_{m,d}^2\,\left|\mathbb{S}^{d-2}\right|}{\left|\mathbb{S}^{d-1}\right|^2} \nu_{m+1,d}^2\, N_{m+1,d}\left\langle P_{m+1,d}, P_{m+1,d}\right\rangle_{w_d} \\
&= \frac{N_{m+1,d}\,\mu_{m,d}^2}{\left|\mathbb{S}^{d-1}\right|} \nu_{m+1,d}^2.
\end{aligned}$$

By (4.15), we have for $m \to \infty$

$$\left|\hat{\mathcal{D}}_{m,0,d}\right|^2 \simeq \frac{\pi}{(d-2)!} \frac{\mu_{m,d}^2}{|\mathbb{S}^{d-1}|}. \qquad \blacksquare$$

Remark 4.11. While we have seen in Theorem 4.10 that the lower bound $\mathcal{O}(m^{-1/2})$ on the singular values $\hat{\mathcal{D}}_{m,l,d}$ is asymptotically sharp, we have only shown that the singular values can be bounded from above by a constant in Theorem 4.8. However, the degree of ill-posedness of the reconstruction problem depends on the behavior of the smallest singular values, which is here $\mathcal{O}(m^{-1/2})$ and so the same as for the Radon transform in 2D and the Funk–Radon transform on $\mathbb{S}^2$. □

4.1.4 Cone-beam transform in $\mathbb{R}^3$

In this subsection, we state the singular value decomposition of the cone-beam transform $\mathcal{D}^{(odd)}$ from Section 4.1.2 for the dimension $d = 3$, which is the most useful for practical applications. The singular value decomposition in this case was already shown by Kazantsev [Kaz15] using a different approach. Compared with the general result in Theorem 4.4, we obtain a better upper bound on the singular values in this case. The spherical harmonics Y_n^k for $n \in \mathbb{N}_0$ and $k = -n, \ldots, n$ on $\mathbb{S}^2$ are given in (2.39).

Before we state the result, we prove the following small lemma that gives bounds on the quotient of certain double factorials.

Lemma 4.12. *Let $m \in \mathbb{N}$. Then we have*

$$\sqrt{\frac{2}{\pi(2m+1)}} < \frac{(2m-1)!!}{(2m)!!} \le \frac{1}{\sqrt{2m+1}}. \tag{4.20}$$

Proof. We follow the proof of [HPQ18, Lemma 3.2]. With the definition

$$u(m) = \left(\frac{(2m)!!}{(2m-1)!!}\right)^2 \frac{1}{2m+1}, \qquad m \in \mathbb{N}_0,$$

we see that $u(0) = 1$ and u is increasing because of $m \ge 1$ and

$$\frac{u(m)}{u(m-1)} = \frac{(2m)^2}{(2m-1)^2} \frac{2m-1}{2m+1} = \frac{(2m)^2}{(2m)^2-1} > 1.$$

Hence, we have $u(m) \ge 1$ for all $m \in \mathbb{N}_0$, which implies the right inequality of (4.20). Furthermore, Wallis' product states the convergence

$$u(m) = \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7} \cdots \frac{2m}{2m-1}\frac{2m}{2m+1} \longrightarrow \frac{\pi}{2} \tag{4.21}$$

for $m \to \infty$, see also [Bau07]. This and the monotonicity of u show the left inequality of (4.20). $\blacksquare$

Theorem 4.13. *Let $d = 3$. The odd cone-beam transform $\mathcal{D}^{(odd)} \colon L^2(\mathbb{B}^3) \to L^2(\mathbb{S}^2 \times \mathbb{S}^2)$ has the singular value decomposition*

$$\mathcal{D}^{(\mathrm{odd})}\widetilde{V}_{m,l,k} = \hat{\mathcal{D}}_{m,l}\, W_{m,l,k}, \qquad m \in \mathbb{N}_0,\ 0 \le l \le m,\ m+l \text{ even},\ k \in \{-l, \dots, l\}.$$

The polynomials

$$\widetilde{V}_{m,l,k}(s\boldsymbol{\omega}) = \sqrt{2m+3}\, s^l P_{\frac{m-l}{2}}^{\left(0,l+\frac12\right)}(2s^2-1)\, Y_l^k(\boldsymbol{\omega}), \qquad s \in [0,1],\ \boldsymbol{\omega} \in \mathbb{S}^2,$$

form an orthonormal basis of $L^2(\mathbb{B}^3)$. The singular values are given by

$$\hat{\mathcal{D}}_{m,l} := \sqrt{2\pi \sum_{n=m+1-l}^{m+1+l}{}' \frac{(2n+1)(n-1)!!^2}{n!!^2} \left\langle P_{m+1}\, P_n, P_l \right\rangle},$$

where $\sum'$ denotes the summation over odd indices n and

$$\left\langle P_{m+1}\, P_n, P_l \right\rangle = \frac{2(l+m-n)!!\,(l-m+n-2)!!\,(-l+m+n)!!\,(l+m+n+1)!!}{(l+m-n+1)!!\,(l-m+n-1)!!\,(-l+m+n+1)!!\,(l+m+n+2)!!} \tag{4.22}$$

for $n \in \{|m+1-l|\,,\, |m+1-l|+2,\, \dots,\, m+1+l\}$ and zero otherwise. The singular values $\hat{\mathcal{D}}_{m,l}$ satisfy

$$c_1 m^{-1/2} \le \left|\hat{\mathcal{D}}_{m,l}\right| \le c_2 m^{-1/8} \tag{4.23}$$

for some constants $c_1,\ c_2 > 0$ that are independent of m and l. Furthermore, the functions

$$W_{m,l,k}(\boldsymbol{a}, \boldsymbol{\omega}) := \frac{4\pi}{\hat{\mathcal{D}}_{m,l}\sqrt{2m+3}} \sum_{j=-m-1}^{m+1} \overline{Y_{m+1}^j(\boldsymbol{a})} \sum_{n=m+1-l}^{m+1+l}{}' \frac{(-1)^{\frac{n+1}{2}}\,(n-1)!!}{n!!} G_{m+1,j,l,k}^{n,j+k} Y_n^{j+k}(\boldsymbol{\omega})$$

for $\boldsymbol{a}, \boldsymbol{\omega} \in \mathbb{S}^2$ are orthonormal in $L^2(\mathbb{S}^2 \times \mathbb{S}^2)$.

Proof. The singular value decomposition is a special case of the Theorems 4.4 and 4.5. The formula (4.22) of the triple product $\langle P_{m+1}\, P_n, P_l\rangle$ is computed in [Neu78], see also [AS56]. The lower bound of the singular values $\hat{\mathcal{D}}_{m,l}$ in (4.23) is due to Theorem 4.10. It is left to show the upper bound in (4.23), which we do as in [Kaz15]. Changing the roles of m and n in (4.17), we have

$$1 = \sum_{n=m+1-l}^{m+1+l} \frac{2n+1}{2} \left\langle P_{m+1} P_l, P_n \right\rangle . \tag{4.24}$$

Furthermore, since $|P_n(t)| \le 1$ for $|t| \le 1$ by (2.15), we obtain the inequality

$$\left\langle P_{m+1} P_l, P_n \right\rangle \le \int_{-1}^{1} \left|P_{m+1}(t)\, P_n(t)\ P_l(t)\right| \mathrm{d}t \le \int_{-1}^{1} \left|P_{m+1}(t)\, P_n(t)\right| \mathrm{d}t.$$

By the Cauchy–Schwarz inequality and (2.17), we have

$$\langle P_{m+1} P_l, P_n\rangle \le \|P_{m+1}\|_{L^2(-1,1)} \|P_n\|_{L^2(-1,1)} = \frac{2}{\sqrt{(2m+3)(2n+1)}}. \tag{4.25}$$

It follows from (4.20) that for $m \in \mathbb{N}_0$

$$\frac{(2m)!!^2}{(2m+1)!!^2} \le \frac{\pi}{2(2m+1)}.$$

Hence, we have

$$\begin{aligned}\left|\hat{\mathcal{D}}_{m,l}\right|^2 &= 2\pi \sum_{n=1}^{m+1+l}{}' \frac{(2n+1)(n-1)!!^2}{n!!^2} \langle P_{m+1} P_n, P_l\rangle \\ &\le 2\pi^2 \sum_{n=1}^{2m+1}{}' \frac{2n+1}{2n} \langle P_{m+1} P_n, P_l\rangle\end{aligned}$$

because $0 \le l \le m$. By the Cauchy-Schwarz inequality, we have

$$\left|\hat{\mathcal{D}}_{m,l}\right|^2 \le 2\pi^2 \sqrt{\sum_{n=1}^{2m+1}{}' \frac{2n+1}{2} \langle P_{m+1} P_n, P_l\rangle} \sqrt{\sum_{n=1}^{2m+1}{}' \frac{2n+1}{2n^2} \langle P_{m+1} P_n, P_l\rangle}$$

Inserting (4.24) and (4.25), we obtain

$$\begin{aligned}\left|\hat{\mathcal{D}}_{m,l}\right|^2 &= 2\pi^2 \sqrt{\sum_{n=1}^{2m+1}{}' \frac{2n+1}{2n^2} \langle P_{m+1} P_n, P_l\rangle} \\ &\le \frac{2\pi^2}{(2m+3)^{\frac{1}{4}}} \sqrt{\sum_{n=1}^{2m+1}{}' \frac{\sqrt{2n+1}}{n^2}}.\end{aligned}$$

The last sum converges for $m \to \infty$ and thus can be bounded from above by a constant independent of m and l, which implies that $\left|\hat{\mathcal{D}}_{m,l}\right|^2 \in \mathcal{O}(m^{-1/4})$. ■

Remark 4.14. The upper bound on the singular values $\hat{\mathcal{D}}_{m,l} \in \mathcal{O}(m^{-1/8})$ might not be optimal. There is reason to believe that the upper bound can be improved to $\mathcal{O}(m^{-1/2})$ even in general dimension d. This conjecture is backed by numerical computations as well as the following observation, which is not a proof though. We consider Grangeat's formula (4.5)

$$(-1)^d \left(\frac{\partial}{\partial s}\right)^{d-2} \mathcal{R}f(\boldsymbol{\omega}, \boldsymbol{a}^\top \boldsymbol{\omega}) = \mathcal{S}^{(d-2)} \mathcal{D}f(\boldsymbol{a}, \boldsymbol{\omega}).$$

We know that the singular values of the Radon transform $\mathcal{R}$ are $\mathcal{O}(m^{(1-d)/2})$ and those of the $d-2$ differentiations are $\mathcal{O}(m^{d-2})$, so the left side should behave like $\mathcal{O}(m^{(d-3)/2})$. On the right side, $\mathcal{S}^{(d-2)}$ has the singular values $\mathcal{O}(m^{(d-2)/2})$ by Lemma 3.25, so the singular values of the cone-beam transform $\mathcal{D}$ should behave like $\mathcal{O}(m^{-1/2})$. □

4.2 Incomplete great circles

So far, we have considered the integrals along different subspheres or, in the case $d = 3$, circles of the two-sphere $\mathbb{S}^2$. Now, we take a look at a particular situation that is not covered in the previous sections. Contrary to the Funk–Radon transform $\mathcal{F}$, which takes the integrals along full great circles, we consider here the integrals along arcs of great circles on the sphere. In this section, we restrict our considerations to the two-dimensional sphere $\mathbb{S}^2$. Most of the material of this section is published in [HPQ18].

For any two points $\boldsymbol{\xi} \neq \boldsymbol{\zeta}$ on the sphere $\mathbb{S}^2$, there exists a shortest geodesic between $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$. This geodesic is an arc of the great circle that contains $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$. If the points $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ are not antipodal, i. e. $\boldsymbol{\xi} \neq -\boldsymbol{\zeta}$, this geodesic is unique and we denote it by

$$\gamma(\boldsymbol{\xi}, \boldsymbol{\zeta}) \subset \mathbb{S}^2.$$

We aim at the reconstruction of a function $f \colon \mathbb{S}^2 \to \mathbb{C}$ from its arc integrals

$$g(\boldsymbol{\xi}, \boldsymbol{\zeta}) := \int_{\gamma(\boldsymbol{\xi},\boldsymbol{\zeta})} f(\boldsymbol{\eta})\, \mathrm{d}\boldsymbol{\eta}, \quad \boldsymbol{\xi}, \boldsymbol{\zeta} \in \mathbb{S}^2,\ \boldsymbol{\xi} \neq -\boldsymbol{\zeta}. \tag{4.26}$$

The manifold of all great circle arcs on $\mathbb{S}^2$ is four-dimensional since they are determined by the two points $\boldsymbol{\xi}, \boldsymbol{\zeta} \in \mathbb{S}^2$ and only coincide if $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ are interchanged. However, the domain of the function f is only the two-dimensional sphere $\mathbb{S}^2$. So, as for the mean operator $\mathcal{M}$ on the sphere, it will also make sense to consider the reconstruction of f from only a limited set of data $g(\boldsymbol{\xi}, \boldsymbol{\zeta})$.

Remark 4.15. The study of this problem (4.26) of circle arc integrals on the sphere $\mathbb{S}^2$ is motivated by the spherical surface wave tomography. There, one measures the time a seismic wave travels along the Earth's surface from an epicenter to a receiver. Knowing the traveltimes of such waves between many pairs of epicenters and receivers, one wants to recover the local phase velocity. A common approach is the great circle ray approximation, where it is assumed that a wave travels along the arc $\gamma(\boldsymbol{\xi}, \boldsymbol{\zeta})$ of the great circle connecting the epicenter $\boldsymbol{\xi}$ and receiver $\boldsymbol{\zeta}$. Then the traveltime of the wave equals the integral $g(\boldsymbol{\xi}, \boldsymbol{\zeta})$ of the "slowness function" f along the great circle arc connecting the epicenter and the receiver, where the slowness function f is defined as one over the local phase velocity [WD84, TW95, Nol08]. Hence, recovering the local phase velocity $\frac{1}{f}$ as a real-valued spherical function from its mean values along certain arcs of great circles is modeled by (4.26), see [AMS08]. □

4.2.1 The arc transform

The parameterization of a circle arc $\gamma(\boldsymbol{\xi}, \boldsymbol{\zeta})$ by its endpoints seems convenient, but it bears some problems for the integral operator (4.26). This is because the function g cannot be extended continuously to $\mathbb{S}^2 \times \mathbb{S}^2$ because of its behavior near antipodal points. However, a great circle arc $\gamma(\boldsymbol{\xi}, \boldsymbol{\zeta})$ can alternatively be parameterized by its length $2\psi = \arccos(\boldsymbol{\xi}^\top \boldsymbol{\zeta})$ and a rotation $Q \in \mathrm{SO}(3)$ which is defined as follows. Let

$$\boldsymbol{e}_\varphi := (\cos\varphi,\, \sin\varphi,\, 0)^\top \in \mathbb{S}^2 \tag{4.27}$$

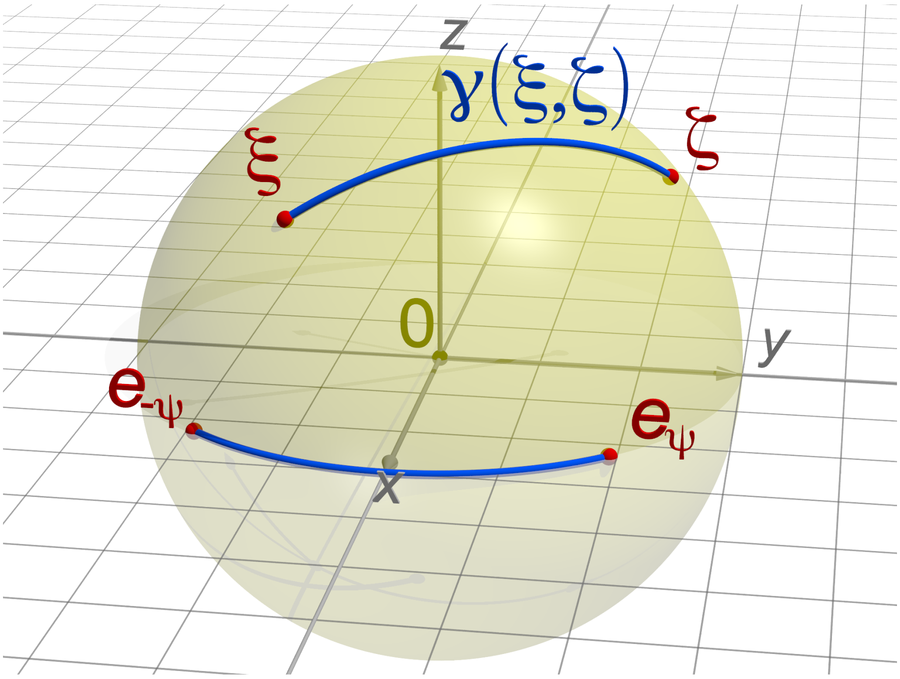

Figure 4.2: Visualization of the circle arc $\gamma(\boldsymbol{\xi},\boldsymbol{\zeta})$ on $\mathbb{S}^2$ and the rotated arc $\gamma(\boldsymbol{e}_{-\psi},\boldsymbol{e}_{\psi})$ of the same length on the equator

denote the point on the equator of $\mathbb{S}^2$ with latitude $\varphi \in \mathbb{R}$. Then there exists a unique rotation $Q \in \mathrm{SO}(3)$ such that $Q(\boldsymbol{\xi}) = \boldsymbol{e}_{-\psi}$ and $Q(\boldsymbol{\zeta}) = \boldsymbol{e}_{\psi}$. Such an arc $\gamma(\boldsymbol{\xi},\boldsymbol{\zeta})$ and its rotation are depicted in Figure 4.2. With this definition, the integral over the arc $\gamma(\boldsymbol{\xi},\boldsymbol{\zeta})$ may be rewritten as

$$\int_{\gamma(\boldsymbol{\xi},\boldsymbol{\zeta})} f(\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta} = \int_{Q\gamma(\boldsymbol{\xi},\boldsymbol{\zeta})} f(Q^{-1}\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta} = \int_{-\psi}^{\psi} f \circ Q^{-1}(\boldsymbol{e}_{\varphi})\,\mathrm{d}\varphi.$$

This motivates the following definition of the *arc transform*

$$\begin{aligned} \mathcal{A} \colon C(\mathbb{S}^2) &\to C(\mathrm{SO}(3) \times [0,\pi]), \\ \mathcal{A}f(Q,\psi) &:= \int_{-\psi}^{\psi} f \circ Q^{-1}(\boldsymbol{e}_{\varphi})\,\mathrm{d}\varphi. \end{aligned} \tag{4.28}$$

The great circle arcs $\gamma(\boldsymbol{\xi},\boldsymbol{\zeta})$ and $\gamma(\boldsymbol{\zeta},\boldsymbol{\xi})$ are identical. This symmetry is transferred to the operator $\mathcal{A}$ as follows.

Corollary 4.16. Let $\alpha,\gamma \in [0,2\pi]$ and $\beta \in [0,\pi]$ be the Euler angles (2.61) of the rotation $Q(\alpha,\beta,\gamma) \in \mathrm{SO}(3)$. Then, for all $\psi \in [0,\pi]$, we have the identity

$$\mathcal{A}f(Q(\alpha,\beta,\gamma),\psi) = \mathcal{A}f(Q(2\pi-\alpha,\pi-\beta,\gamma+\pi),\psi),$$

where we assume the Euler angle γ as 2π-periodic.

Proof. Let $\varphi \in \mathbb{R}$. By the definition of the Euler angle decomposition (2.61) and the observation that the inverse of the rotation $R_3(\alpha)$ is the rotation $R_3(-\alpha)$ about the same axis with negative angle, we have on the one hand

$$\begin{aligned} Q(\alpha,\beta,\gamma)^{-1}\,\boldsymbol{e}_\varphi &= (R_3(\alpha)\,R_2(\beta)\,R_3(\gamma))^{-1}\,\boldsymbol{e}_\varphi \\ &= R_3(-\gamma)\,R_2(-\beta)\,R_3(-\alpha)\,\boldsymbol{e}_\varphi = R_3(-\gamma)\,R_2(-\beta)\,\boldsymbol{e}_{\varphi-\alpha}. \end{aligned}$$

Inserting (2.62) and (4.27), we obtain

$$\begin{aligned} Q(\alpha,\beta,\gamma)^{-1}\,\boldsymbol{e}_\varphi &= R_3(-\gamma)\begin{pmatrix}\cos\beta & 0 & -\sin\beta\\ 0 & 1 & 0\\ \sin\beta & 0 & \cos\beta\end{pmatrix}\begin{pmatrix}\cos(\varphi-\alpha)\\ \sin(\varphi-\alpha)\\ 0\end{pmatrix} \\ &= \begin{pmatrix}\cos\gamma & \sin\gamma & 0\\ -\sin\gamma & \cos\gamma & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos(\beta)\,\cos(\varphi-\alpha)\\ \sin(\varphi-\alpha)\\ \sin(\beta)\,\cos(\varphi-\alpha)\end{pmatrix} \\ &= \begin{pmatrix}\cos(\gamma)\,\cos(\beta)\,\cos(\varphi-\alpha)+\sin(\gamma)\,\sin(\varphi-\alpha)\\ -\sin(\gamma)\,\cos(\beta)\,\cos(\varphi-\alpha)+\cos(\gamma)\,\sin(\varphi-\alpha)\\ \sin(\beta)\,\cos(\varphi-\alpha)\end{pmatrix}. \end{aligned}$$

On the other hand, we have

$$Q(2\pi-\alpha,\pi-\beta,\gamma+\pi)^{-1}\,\boldsymbol{e}_{-\varphi} = R_3(\pi-\gamma)\,R_2(\beta-\pi)\,\boldsymbol{e}_{\alpha-\varphi}.$$

Inserting (2.62) again, we obtain

$$\begin{aligned} Q(2\pi-\alpha,\pi-\beta,\gamma+\pi)^{-1}\,\boldsymbol{e}_{-\varphi} &= R_3(-\gamma-\pi)\begin{pmatrix}-\cos\beta & 0 & \sin\beta\\ 0 & 1 & 0\\ -\sin\beta & 0 & -\cos\beta\end{pmatrix}\begin{pmatrix}\cos(\varphi-\alpha)\\ -\sin(\varphi-\alpha)\\ 0\end{pmatrix} \\ &= \begin{pmatrix}-\cos\gamma & -\sin\gamma & 0\\ \sin\gamma & -\cos\gamma & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}-\cos(\beta)\,\cos(\varphi-\alpha)\\ -\sin(\varphi-\alpha)\\ -\sin(\beta)\,\cos(\varphi-\alpha)\end{pmatrix} \\ &= \begin{pmatrix}\cos(\gamma)\,\cos(\beta)\,\cos(\varphi-\alpha)+\sin(\gamma)\,\sin(\varphi-\alpha)\\ -\sin(\gamma)\,\cos(\beta)\,\cos(\varphi-\alpha)+\cos(\gamma)\,\sin(\varphi-\alpha)\\ \sin(\beta)\,\cos(\varphi-\alpha)\end{pmatrix}. \end{aligned}$$

Hence, we have shown that

$$Q(\alpha,\beta,\gamma)^{-1}\,\boldsymbol{e}_\varphi = Q(\alpha-\pi,\pi-\beta,2\pi-\gamma)^{-1}\,\boldsymbol{e}_{-\varphi}.$$

By the definition of $\mathcal{A}$ in (4.28), we have

$$\begin{aligned} \mathcal{A}f(Q(\alpha,\beta,\gamma),\psi) &= \int_{-\psi}^{\psi} f(Q(\alpha,\beta,\gamma)^{-1}\,\boldsymbol{e}_\varphi)\,\mathrm{d}\varphi \\ &= \int_{-\psi}^{\psi} f(Q(\alpha-\pi,\pi-\beta,2\pi-\gamma)^{-1}\,\boldsymbol{e}_{-\varphi})\,\mathrm{d}\varphi \\ &= \mathcal{A}f(Q(\alpha-\pi,\pi-\beta,2\pi-\gamma),\psi). \end{aligned}$$ ∎

4.2.2 Singular value decomposition of the arc transform

The next theorem shows how the arc transform $\mathcal{A}$ acts on spherical harmonics Y_n^k. The corresponding result for the parameterization (4.26) in terms of the endpoints of an arc is found in [DT98, Appendix C], see also [AMS08]. We are going to make use of the rotational harmonics $D_n^{j,k}$, which were introduced in Section 2.2.

Theorem 4.17. *Let $n \in \mathbb{N}_0$ and $k \in \{-n, \dots, n\}$. Then we have*

$$\mathcal{A}Y_n^k(Q,\psi) = \sum_{j=-n}^{n} \widetilde{P}_n^j(0)\, D_n^{j,k}(Q)\, s_j(\psi), \qquad Q \in \mathrm{SO}(3),\ \psi \in [0,\pi], \tag{4.29}$$

where

$$s_j(\psi) := \begin{cases} 2\psi, & j = 0 \\ \frac{2\sin(j\psi)}{j}, & j \neq 0 \end{cases} \tag{4.30}$$

and

$$\widetilde{P}_n^j(0) = \begin{cases} (-1)^{\frac{n-j}{2}} \sqrt{\frac{2n+1}{4\pi} \frac{(n-j-1)!!(n+j-1)!!}{(n-j)!!(n+j)!!}}, & n+j \textit{ even} \\ 0, & n+j \textit{ odd}. \end{cases} \tag{4.31}$$

Proof. By the relation (2.65) between the spherical harmonics Y_n^k and the rotational harmonics $D_n^{j,k}$, we obtain

$$\mathcal{A}Y_n^k(Q,\psi) = \int_{-\psi}^{\psi} Y_n^k(Q^{-1}(\boldsymbol{e}_\varphi)) = \sum_{j=-n}^{n} D_n^{j,k}(Q) \int_{-\psi}^{\psi} Y_n^j(\boldsymbol{e}_\varphi)\,\mathrm{d}\varphi.$$

By the definition (2.39) of the spherical harmonics Y_n^k, we see that

$$\int_{-\psi}^{\psi} Y_n^j(\boldsymbol{e}_\varphi)\,\mathrm{d}\varphi = \widetilde{P}_n^j(0) \int_{-\psi}^{\psi} \mathrm{e}^{\mathrm{i}j\varphi}\,\mathrm{d}\varphi = \widetilde{P}_n^j(0)\, s_j(\psi).$$

Hence,

$$\mathcal{A}Y_n^k(Q,\psi) = \sum_{j=-n}^{n} D_n^{j,k}(Q)\, \widetilde{P}_n^j(0)\, s_j(\psi).$$

Now, we obtain by (3.60)

$$\begin{aligned} \widetilde{P}_{n,d}^j(0) &= (-1)^{\frac{n-j}{2}} \frac{\sqrt{(2n+1)\,(n+j)!}}{2^{j+\frac{1}{2}}\sqrt{(n-j)!}\,j!} \frac{(n-j-1)!!\,(2j)!!}{(n+j)!!} \\ &= (-1)^{\frac{n-j}{2}} \sqrt{\frac{(n+\frac{1}{2})\,(n+j-1)!!\,(n-j-1)!!}{(n-j)!!\,(n+j)!!}}, \end{aligned}$$

which implies (4.31). ∎

In the following, we compute lower and upper bounds of the $\widetilde{P}_n^j(0)$ from (4.31).

Lemma 4.18. *Let $n \in \mathbb{N}_0$ and $j \in \{-n, \ldots, n\}$. If $n + j$ is odd, then $\widetilde{P}_n^j(0) = 0$. Otherwise, we have*

$$\frac{2n+1}{2\pi^2\sqrt{(n+1)^2 - j^2}} \leq \left|\widetilde{P}_n^j(0)\right|^2 \leq \frac{2n+1}{4\pi\sqrt{(n+1)^2 - j^2}}. \tag{4.32}$$

Furthermore, for $j \in \mathbb{Z}$, we have

$$\lim_{\substack{n\to\infty \\ n+j \text{ even}}} \left|\widetilde{P}_n^j(0)\right| = \frac{1}{\pi}. \tag{4.33}$$

Proof. By (4.31) and (4.20), we obtain the upper bound

$$\left|\widetilde{P}_n^j(0)\right|^2 = \frac{2n+1}{4\pi}\frac{(n-j-1)!!}{(n-j)!!}\frac{(n+j-1)!!}{(n+j)!!} \leq \frac{2n+1}{4\pi}\frac{1}{\sqrt{n-j+1}}\frac{1}{\sqrt{n+j+1}}.$$

The lower bound follows analogously from (4.20), i. e., we have

$$\left|\widetilde{P}_n^j(0)\right|^2 \geq \frac{2n+1}{4\pi}\frac{\sqrt{2}}{\sqrt{\pi\, n-j+1}}\frac{\sqrt{2}}{\sqrt{\pi\,(n+j+1)}}.$$

Moreover, we have for $j \in \mathbb{Z}$ and $m \in \mathbb{N}_0$

$$\left|\widetilde{P}_{j+2m}^j(0)\right|^2 = \frac{2(j+2m)+1}{4\pi}\frac{(2m-1)!!}{(2m)!!}\frac{(2m+2j-1)!!}{(2m+2j)!!}.$$

Hence, Wallis product (4.21) implies that we have

$$\begin{aligned}\lim_{m\to\infty}\left|\widetilde{P}_{j+2m}^j(0)\right|^2 &= \lim_{m\to\infty}\frac{2(j+2m)+1}{4\pi}\frac{2}{\pi}\frac{1}{\sqrt{2m+1}\sqrt{2m+2j+1}} \\ &= \lim_{m\to\infty}\frac{2m+j+\frac{1}{2}}{\pi^2\sqrt{(2m+j+1)^2 - j^2}} = \frac{1}{\pi^2},\end{aligned}$$

which proves the assertion. ∎

Next, we derive the singular value decomposition of the arc transform $\mathcal{A}$. To this end, we define for $n \in \mathbb{N}_0$ and $k = -n, \ldots, n$ the functions $E_n^k \in L^2(\mathrm{SO}(3) \times [0, \pi])$ by

$$E_n^k(Q, \psi) := \sum_{j=-n}^{n} D_n^{j,k}(Q)\,\widetilde{P}_n^j(0)\, s_j(\psi), \qquad Q \in \mathrm{SO}(3),\ \psi \in [0, \pi]. \tag{4.34}$$

Theorem 4.19. *The arc transform $\mathcal{A}\colon L^2(\mathbb{S}^2) \to L^2(\mathrm{SO}(3) \times [0, \pi])$ is a compact operator with the singular value decomposition*

$$\left\{\left(Y_n^k,\ \widetilde{E}_n^k,\ \hat{\mathcal{A}}_n\right); n \in \mathbb{N}_0,\ k \in \{-n, \ldots, n\}\right\},$$

with the singular values

$$\hat{\mathcal{A}}_n := \left\|E_n^k\right\|_{L^2(\mathrm{SO}(3)\times[0,\pi])} = \sqrt{\frac{32\pi^3}{2n+1}}\sqrt{\frac{\pi^2}{3}\left|\widetilde{P}_n^0(0)\right|^2 + \sum_{j=1}^n \frac{1}{j^2}\left|\widetilde{P}_n^j(0)\right|^2} \tag{4.35}$$

satisfying

$$\sqrt{\frac{16}{3}\pi^3} \le \hat{\mathcal{A}}_n\sqrt{n+1} \le \sqrt{\frac{8}{3}\pi^4 + 4\pi^2}, \qquad n \text{ even}, \tag{4.36}$$

$$4\sqrt{\pi} \le \hat{\mathcal{A}}_n\sqrt{n+1} \le 2\pi\sqrt{\frac{4}{\sqrt{3}} + 1}, \qquad n \text{ odd}, \tag{4.37}$$

and the orthonormal function system

$$\widetilde{E}_k^n := \frac{1}{\hat{\mathcal{A}}_n} E_n^k, \qquad n \in \mathbb{N}_0,\ k \in \{-n, \ldots, n\}$$

in $L^2(\mathrm{SO}(3)\times[0,\pi])$.

Proof. By the orthogonality (2.63) of the rotational harmonics $D_n^{j,k}$, we have

$$\begin{aligned}
&\left\langle E_n^k, E_{n'}^{k'}\right\rangle_{L^2(\mathrm{SO}(3)\times[0,\pi])} \\
&= \sum_{j=-n}^{n}\sum_{j'=-n'}^{n'} \widetilde{P}_n^j(0)\,\widetilde{P}_{n'}^{j'}(0)\int_{\mathrm{SO}(3)} D_n^{j,k}(Q)\,\overline{D_{n'}^{j',k'}(Q)}\,\mathrm{d}Q\int_0^\pi s_j(\psi)\,s_{j'}(\psi)\,\mathrm{d}\psi \\
&= \sum_{j=-n}^{n}\sum_{j'=-n'}^{n'} \frac{8\pi^2}{2n+1}\delta_{nn'}\,\delta_{kk'}\,\delta_{jj'}\,\widetilde{P}_n^j(0)\,\widetilde{P}_{n'}^{j'}(0)\int_0^\pi s_j(\psi)\,s_{j'}(\psi)\,\mathrm{d}\psi \\
&= \delta_{nn'}\,\delta_{kk'}\sum_{j=-n}^{n}\frac{8\pi^2}{2n+1}\left|\widetilde{P}_n^j(0)\right|^2\int_0^\pi s_j(\psi)^2\,\mathrm{d}\psi \\
&= \delta_{nn'}\,\delta_{kk'}\frac{8\pi^2}{2n+1}\sum_{j=-n}^{n}\left|\widetilde{P}_n^j(0)\right|^2\cdot\begin{cases}\frac{4\pi^3}{3}, & j=0\\ \frac{2\pi}{j^2}, & j\neq 0.\end{cases}
\end{aligned}$$

This shows that the functions $E_n^k = \mathcal{A}Y_n^k$ are orthogonal in the space $L^2(\mathrm{SO}(3)\times[0,\pi])$ and have the norm

$$\begin{aligned}
\left\|E_n^k\right\|^2_{L^2(\mathrm{SO}(3)\times[0,\pi])} &= \frac{8\pi^2}{2n+1}\sum_{j=-n}^{n}\left|\widetilde{P}_n^j(0)\right|^2\cdot\begin{cases}\frac{4\pi^3}{3}, & j=0\\ \frac{2\pi}{j^2}, & j\neq 0.\end{cases} \\
&= \frac{16\pi^3}{2n+1}\left(\frac{2\pi^2}{3}\left|\widetilde{P}_n^0(0)\right|^2 + 2\sum_{j=1}^n\frac{1}{j^2}\left|\widetilde{P}_n^j(0)\right|^2\right),
\end{aligned}$$

where we used that $\left|\widetilde{P}_n^j(0)\right| = \left|\widetilde{P}_n^{-j}(0)\right|$. In order to prove that $\mathcal{A}$ is compact, we show that the singular values $\hat{\mathcal{A}}_n$ decay for $n \to \infty$. We have by Lemma 4.18 for $n = 2m$ even

$$(\hat{\mathcal{A}}_{2m})^2 \leq 4\pi^2 \left(\frac{2\pi^2}{3} \frac{1}{2m+1} + 2\sum_{j=1}^{m} \frac{1}{(2j)^2} \frac{1}{\sqrt{(2m+1)^2 - (2j)^2}} \right).$$

For n even, we replace the sum by an integral, where we use the convexity of the integrand, and obtain the estimate

$$\begin{aligned}
2\sum_{j=1}^{m} \frac{1}{(2j)^2} \frac{1}{\sqrt{(2m+1)^2 - (2j)^2}} &\leq 2\int_{1/2}^{m+1/2} \frac{1}{(2j)^2} \frac{1}{\sqrt{(2m+1)^2 - (2j)^2}} \,\mathrm{d}j \\
&= 2\left[-\frac{\sqrt{(2m+1)^2 - (2j)^2}}{2j(2m+1)^2} \right]_{1/2}^{m+1/2} \\
&= 2\frac{\sqrt{m^2+m}}{(2m+1)^2} \leq \frac{1}{2m+1},
\end{aligned}$$

Hence, we have

$$(\hat{\mathcal{A}}_{2m})^2 \leq 4\pi^2 \left(\frac{2\pi^2}{3} \frac{1}{2m+1} + \frac{1}{2m+1} \right) = 4\pi^2 \left(\frac{2\pi^2}{3} + 1 \right) \frac{1}{2m+1}.$$

For odd $n = 2m-1$, we proceed analogously. We have

$$(\hat{\mathcal{A}}_{2m-1})^2 \leq 8\pi^2 \sum_{j=1}^{m} \frac{1}{(2j-1)^2} \frac{1}{\sqrt{(2m)^2 - (2j-1)^2}}.$$

Note that, for the estimation of the sum by an integral, we extract the summand for $j = 1$

$$\begin{aligned}
(\hat{\mathcal{A}}_{2m-1})^2 &\leq 8\pi^2 \left(\frac{1}{\sqrt{(2m)^2 - 1}} + \int_{1}^{m+1/2} \frac{1}{(2j-1)^2} \frac{1}{\sqrt{(2m)^2 - (2j-1)^2}} \,\mathrm{d}j \right) \\
&= 8\pi^2 \left(\frac{1}{\sqrt{(2m)^2 - 1}} + \frac{\sqrt{(2m)^2 - 1}}{2(2m)^2} \right) \\
&\leq 8\pi^2 \left(\frac{2}{\sqrt{3}\,2m} + \frac{1}{2(2m)} \right) = 4\pi^2 \left(\frac{4}{\sqrt{3}} + 1 \right) \frac{1}{2m}.
\end{aligned}$$

For the lower bound of the singular values, we also use Lemma 4.18. For even n, we extract the summand $j = 0$ and obtain

$$\begin{aligned}
(\hat{\mathcal{A}}_n)^2 &= \frac{16\pi^3}{2n+1} \left(\frac{2\pi^2}{3} \left|\widetilde{P}_n^0(0)\right|^2 + 2\sum_{j=1}^{n} \frac{1}{j^2} \left|\widetilde{P}_n^j(0)\right|^2 \right) \\
&\geq \frac{32\pi^5}{3(2n+1)} \left|\widetilde{P}_n^0(0)\right|^2 \geq \frac{16\pi^3}{3(n+1)}.
\end{aligned}$$

For odd n, we extract the summand $j = 1$ and obtain

$$(\hat{\mathcal{A}}_n)^2 \geq \frac{32\pi^3}{2n+1} \left| \widetilde{P}_n^1(0) \right|^2 \geq \frac{16\pi}{\sqrt{(n+1)^2 - 1}} \geq \frac{16\pi}{n+1}. \qquad \blacksquare$$

The singular values $\hat{\mathcal{A}}_n$ of the arc transform $\mathcal{A}$ decay with the rate $n^{-1/2}$ for $n \to \infty$. This is the same asymptotic decay rate as of the eigenvalues of the Funk–Radon transform on the two-sphere $\mathbb{S}^2$, cf. Theorem 3.13.

4.2.3 Restrictions of the arc transform

The recovery of a function f from the arc integrals $\mathcal{A}f$ is overdetermined considered that we have full data $\mathcal{A}f(Q, \psi)$ for all $Q \in \mathrm{SO}(3)$ and $\psi \in [0, \pi]$. In the following two paragraphs, we are going to examine simple special cases of restrictions of the arc transform $\mathcal{A}$, where we are able to reconstruct the function f from its integrals only along certain great circle arcs.

Arcs starting in a fixed point

As a simple example, we fix one endpoint of the arcs. Without loss of generality, we assume that this endpoint is the north pole $\boldsymbol{\epsilon}^3$. We write a vector $\boldsymbol{\xi}(\varphi, \vartheta) \in \mathbb{S}^2$ in spherical coordinates $\varphi \in [0, 2\pi)$, $\vartheta \in [0, \pi]$, see (2.36). The arc connecting the north pole $\boldsymbol{\epsilon}^3$ and an arbitrary other point $\boldsymbol{\xi}(\varphi, \vartheta) \in \mathbb{S}^2$ is given by

$$\gamma(\boldsymbol{\epsilon}^3, \boldsymbol{\xi}(\varphi, \vartheta)) = \{\boldsymbol{\eta}(\varphi, \varrho) \in \mathbb{S}^2 \,;\, \varrho \in [0, \vartheta]\}.$$

Since, with $Q = Q\left(\frac{\vartheta}{2}, \frac{\pi}{2}, \frac{3\pi}{2} - \varphi\right) \in \mathrm{SO}(3)$, we have $Q\boldsymbol{\epsilon}^3 = \boldsymbol{e}_{\frac{\vartheta}{2}}$ and $Q\boldsymbol{\xi}(\varphi, \vartheta) = \boldsymbol{e}_{-\frac{\vartheta}{2}}$. The restriction $\mathcal{B} \colon C(\mathbb{S}^2) \to C(\mathbb{S}^2)$ of the operator $\mathcal{A}$ to these arcs satisfies

$$\mathcal{B}f(\boldsymbol{\xi}(\varphi, \vartheta)) := \mathcal{A}f\left(Q\left(\tfrac{\vartheta}{2}, \tfrac{\pi}{2}, \tfrac{3\pi}{2} - \varphi\right), \tfrac{\vartheta}{2}\right) = \int_0^\vartheta f(\boldsymbol{\eta}(\varphi, \varrho))\, \mathrm{d}\varrho.$$

Differentiating the last equation with respect to ϑ, we see that the function f can be recovered from $\mathcal{B}f$ by

$$f(\boldsymbol{\xi}(\varphi, \vartheta)) = \frac{\mathrm{d}}{\mathrm{d}\vartheta} \mathcal{B}f(\boldsymbol{\xi}(\varphi, \vartheta)).$$

The following more general result for the injectivity is due to Amirbekyan [Ami07, Theorem 4.4.1]. Its proof uses a similar idea combined with an extension by density.

Proposition 4.20. *Let S be an open subset of $\mathbb{S}^2$ and $A, B \subset S$ nonempty sets with $\overline{A \cup B} = \overline{S}$. If $f \in C(\mathbb{S}^2)$ and*

$$\int_{\gamma(\boldsymbol{\xi}, \boldsymbol{\zeta})} f(\boldsymbol{\eta})\, \mathrm{d}\boldsymbol{\eta} = 0 \qquad \text{for all } \boldsymbol{\xi} \in A,\ \boldsymbol{\zeta} \in B,$$

then $f \equiv 0$ on S.

For $A = \{\boldsymbol{\epsilon}^3\}$ and $B = \mathbb{S}^2$, we have the arcs starting in the north pole $\boldsymbol{\epsilon}^3$.

Recovery of local functions

We call a subset $\Omega \subset \mathbb{S}^2$ convex if for any two points $\boldsymbol{\xi}, \boldsymbol{\eta} \in \Omega$ every shortest geodesic that connects $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ is contained in Ω. If the two points are antipodal, i. e., $\boldsymbol{\xi} = -\boldsymbol{\eta}$, then all great semicircles that connect $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are shortest geodesics. Otherwise, the shortest geodesic arc $\gamma(\boldsymbol{\xi}, \boldsymbol{\eta})$ is unique. Furthermore, we denote by $\partial\Omega$ the boundary of $\Omega \subset \mathbb{S}^2$ with respect to $\mathbb{S}^2$.

Theorem 4.21. *Let $f \in C(\mathbb{S}^2)$ and Ω be a convex subset of $\mathbb{S}^2$ whose closure $\overline{\Omega}$ is strictly contained in a hemisphere, i. e., there exists a $\boldsymbol{\zeta} \in \mathbb{S}^2$ such that $\langle \boldsymbol{\xi}, \boldsymbol{\zeta} \rangle > 0$ for all $\boldsymbol{\xi} \in \overline{\Omega}$. If*

$$\int_{\gamma(\boldsymbol{\xi},\boldsymbol{\eta})} f(\boldsymbol{\eta}) \, \mathrm{d}\boldsymbol{\eta} = 0 \qquad \text{for all } \boldsymbol{\xi}, \boldsymbol{\eta} \in \partial\Omega, \tag{4.38}$$

then $f = 0$ on Ω.

Proof. Without loss of generality, we assume that $\overline{\Omega}$ is strictly contained in the northern hemisphere, i. e., we have $\xi_3 > 0$ for all $\boldsymbol{\xi} \in \overline{\Omega}$. We define the restriction of f to $\overline{\Omega}$ by

$$f_\Omega(\boldsymbol{\xi}) = \begin{cases} f(\boldsymbol{\xi}), & \boldsymbol{\xi} \in \overline{\Omega} \\ 0, & \boldsymbol{\xi} \in \mathbb{S}^2 \setminus \overline{\Omega}. \end{cases}$$

Since $\gamma(\boldsymbol{\xi}, \boldsymbol{\eta}) \subset \overline{\Omega}$ for all $\boldsymbol{\xi}, \boldsymbol{\eta} \in \partial\Omega$, the function f_Ω also satisfies (4.38).

For $\boldsymbol{\xi} \in \mathbb{S}^2$, we denote by $\boldsymbol{\xi}^\perp = \{\boldsymbol{\eta} \in \mathbb{S}^2 \; ; \; \langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = 0\}$ the great circle that is perpendicular to $\boldsymbol{\xi}$. We show that the Funk–Radon transform

$$\mathcal{F} f_\Omega(\boldsymbol{\xi}) = \int_{\boldsymbol{\xi}^\perp \cap \overline{\Omega}} f_\Omega(\boldsymbol{\eta}) \, \mathrm{d}\boldsymbol{\eta} + \int_{\boldsymbol{\xi}^\perp \setminus \overline{\Omega}} f_\Omega(\boldsymbol{\eta}) \, \mathrm{d}\boldsymbol{\eta} \tag{4.39}$$

vanishes everywhere. The second summand of (4.39) vanishes because f_Ω is zero outside $\overline{\Omega}$ by definition. If $\boldsymbol{\xi}^\perp \cap \overline{\Omega}$ is not empty, there exist two points $\boldsymbol{\eta}^1, \boldsymbol{\eta}^2 \in \partial\Omega$ such that $\gamma(\boldsymbol{\eta}^1, \boldsymbol{\eta}^2) = \boldsymbol{\xi}^\perp \cap \overline{\Omega}$, which shows that also the first summand of (4.39) vanishes. Hence, $\mathcal{F} f_\Omega = 0$ on $\mathbb{S}^2$. Since the Funk–Radon transform $\mathcal{F}$ is injective for even functions, we see that f_Ω must be odd. Since f_Ω is supported strictly inside the northern hemisphere, so f_Ω must be the zero function. By the construction, we see that $f(\boldsymbol{\xi})$ vanishes for all $\boldsymbol{\xi} \in \Omega$. ■

An analogue to Theorem 4.21 for Ω being the northern hemisphere and the arcs being semicircles is shown in [Rub17a].

4.2.4 Arc transform with fixed length

In the following, we consider the integrals along great circle arcs that have the fixed length $\psi \in [0, \pi]$. To this end, we define the restriction of the arc transform

$$\mathcal{A}_\psi(Q) := \mathcal{A}(Q, \psi), \qquad Q \in \mathrm{SO}(3).$$

In the limiting case $\psi = \pi$, the arc transform $\mathcal{A}_\pi$ takes the great circle arcs with length 2π, which are the full great circles. Hence, $\mathcal{A}_\pi$ corresponds to the Funk–Radon transform $\mathcal{F}$, which is injective only for even functions and vanishes on odd functions, see Section 3.2.

In the case $\psi = \pi/2$, the arc transform $\mathcal{A}_{\pi/2}$ takes the integrals along great semicircles. Groemer [Gro98] showed in 1998 that $\mathcal{A}_{\pi/2}$ is injective for all functions $f \in C(\mathbb{S}^{d-1})$.

In the following theorem, we show that the arc transform $\mathcal{A}_\psi$ with any fixed length $\psi \in (0,\pi)$ is injective for all functions $f \in L^2(\mathbb{S}^{d-1})$. This is achieved via a singular value decomposition.

Theorem 4.22. *Let $\psi \in [0,\pi]$ be fixed. The operator $\mathcal{A}_\psi \colon L^2(\mathbb{S}^2) \to L^2(\mathrm{SO}(3))$ has the singular value decomposition*

$$\left\{ \left(Y_n^k, \frac{1}{\hat{\mathcal{A}}_{n,\psi}} \mathcal{A}_\psi Y_n^k, \hat{\mathcal{A}}_{n,\psi} \right); n \in \mathbb{N}_0,\, k \in \{-n,\dots,n\} \right\}, \tag{4.40}$$

with the singular values

$$\hat{\mathcal{A}}_{n,\psi} := \sqrt{\sum_{j=-n}^{n} \frac{8\pi^2}{2n+1} \left| \widetilde{P}_n^j(0) \right|^2 s_j(\psi)^2}, \qquad n \in \mathbb{N}_0, \tag{4.41}$$

and the singular functions

$$\frac{1}{\hat{\mathcal{A}}_{n,\psi}} \mathcal{A}_\psi Y_n^k(Q) = \frac{1}{\hat{\mathcal{A}}_{n,\psi}} \sum_{j=-n}^{n} \widetilde{P}_n^j(0)\, s_j(\psi)\, D_n^{j,k}(Q), \qquad Q \in \mathrm{SO}(3), \tag{4.42}$$

where $s_j(\psi)$ is given in (4.30). In particular, if $\psi \in (0,\pi)$, then $\mathcal{A}_\psi$ is injective.

The singular values $\hat{\mathcal{A}}_{n,\psi}$ satisfy for odd $n = 2m-1$

$$\lim_{m\to\infty} \frac{4m-1}{4} \left(\hat{\mathcal{A}}_{2m-1,\psi}\right)^2 = \begin{cases} 4\pi\psi, & \psi \in [0, \frac{\pi}{2}] \\ 4\pi^2 - 4\pi\psi, & \psi \in [\frac{\pi}{2}, \pi], \end{cases} \tag{4.43}$$

and for even $n = 2m$

$$\lim_{m\to\infty} \frac{4m+1}{4} \left(\hat{\mathcal{A}}_{2m,\psi}\right)^2 = \begin{cases} 4\pi\psi, & \psi \in [0, \frac{\pi}{2}] \\ 12\pi\psi - 4\pi^2, & \psi \in [\frac{\pi}{2}, \pi]. \end{cases} \tag{4.44}$$

Proof. Let $\psi \in [0,\pi]$ be fixed. The exact formula (4.42) of the singular functions $\mathcal{A}_\psi Y_n^k$ follows directly from the general case (4.29). Now we compute the orthogonality and norm of the functions $\mathcal{A}_\psi Y_n^k \in L^2(\mathrm{SO}(3))$. Let $n, n' \in \mathbb{N}_0$, $k \in \{-n,\dots,n\}$ and $k' \in \{-n',\dots,n'\}$. We have by (4.29)

$$\left\langle \mathcal{A}_\psi Y_n^k, \mathcal{A}_\psi Y_{n'}^{k'} \right\rangle_{L^2(\mathrm{SO}(3))} = \sum_{j=-n}^{n} \sum_{j'=-n'}^{n'} \int_{\mathrm{SO}(3)} D_n^{j,k}(Q)\, \overline{D_{n'}^{j',k'}(Q)}\, \widetilde{P}_n^j(0)\, \widetilde{P}_{n'}^{j'}(0)\, s_j(\psi)\, s_{j'}(\psi)\, \mathrm{d}Q.$$

By the orthonormality (2.63) and (2.17) of the rotational harmonics $D_n^{j,k}$ and the normalized associated Legendre functions $\widetilde{P}_n^j$, respectively, we obtain

$$\begin{aligned}\left\langle \mathcal{A}_\psi Y_n^k, \mathcal{A}_\psi Y_{n'}^{k'} \right\rangle_{L^2(\mathrm{SO}(3))} &= \sum_{j=-n}^{n} \sum_{j'=-n'}^{n'} \frac{8\pi^2}{2n+1} \delta_{nn'}\, \delta_{kk'}\, \delta_{jj'}\, \widetilde{P}_n^j(0)\, \widetilde{P}_{n'}^{j'}(0)\, s_j(\psi)\, s_{j'}(\psi) \\ &= \delta_{nn'}\delta_{kk'} \sum_{j=-n}^{n} \frac{8\pi^2}{2n+1} \left| \widetilde{P}_n^j(0) \right|^2 s_j(\psi)^2.\end{aligned}$$

This shows (4.41). Since the span of spherical harmonics is dense in $L^2(\mathbb{S}^{d-1})$, we see that (4.40) is indeed a singular value decomposition.

Now let $\psi \in (0,\pi)$. For the injectivity of $\mathcal{A}_\psi$, we check that the singular values $\hat{\mathcal{A}}_{n,\psi}$ do not vanish for each $n \in \mathbb{N}_0$. We have $\widetilde{P}_n^j(0) = 0$ if and only if $n-j$ is odd. Furthermore, the definition of s_j in (4.30) shows that $s_0(\psi) = 2\psi$ vanishes if and only if $\psi = 0$, and $s_1(\psi) = 2\sin(\psi)$ vanishes if and only if ψ is an integer multiple of π. Hence, the functions $\mathcal{A}_\psi Y_n^k$ are also orthogonal in the space $L^2(\mathrm{SO}(3))$.

In the next step, we come to the asymptotic behavior of the singular values $\hat{\mathcal{A}}_{n,\psi}$. We first show (4.43). Let $m \in \mathbb{N}$. We have by (4.41)

$$\frac{4m-1}{4} \left(\hat{\mathcal{A}}_{2m-1,\psi}\right)^2 = 16\pi^2 \sum_{j=1}^{m} \left| \widetilde{P}_{2m-1}^{2j-1}(0) \right|^2 \frac{\sin^2((2j-1)\psi)}{(2j-1)^2}. \tag{4.45}$$

We denote by $\nu(\psi) := 4\pi\left(\frac{\pi}{2} - \left|\psi - \frac{\pi}{2}\right|\right)$ the right-hand side of (4.43). The Fourier cosine series of ν reads by [GR07, 1.444]

$$16 \sum_{k=1}^{\infty} \frac{\sin((2k-1)\psi)^2}{(2k-1)^2} = 16 \sum_{k=1}^{\infty} \frac{1-\cos((2k-1)2\psi)}{2(2k-1)^2} = \nu(\psi), \qquad \psi \in [0,\pi].$$

We have

$$\begin{aligned}\left\| \frac{4m-1}{4} \left(\hat{\mathcal{A}}_{2m-1,\psi}\right)^2 - \nu(\psi) \right\|_{C([0,\pi])} &= \left\| 16 \sum_{j=1}^{\infty} \frac{\pi^2 \left| \widetilde{P}_{2m-1}^{2j-1}(0) \right|^2 - 1}{(2j-1)^2} \sin^2((2j-1)\psi) \right\|_{C([0,\pi])} \\ &\le \sum_{j=1}^{\infty} \frac{16 \left| \pi^2 \left| \widetilde{P}_{2m-1}^{2j-1}(0) \right|^2 - 1 \right|}{(2j-1)^2}.\end{aligned} \tag{4.46}$$

We show that (4.46) goes to zero for $m \to \infty$, which then implies (4.43). By (4.33), we see that $\pi^2 \left|\widetilde{P}_{2m-1}^{2j-1}(0)\right|^2$ converges to 1 for $m \to \infty$. Using the singular values $\hat{\mathcal{A}}_{2m+1}$ from (4.35) together with their bound (4.37), we obtain the following summable majorant of (4.46):

$$\sum_{j=1}^{\infty} \frac{16\pi^2 \left| \widetilde{P}_{2m-1}^{2j-1}(0) \right|^2}{(2j-1)^2} \le \frac{4m-1}{2\pi} \left(\hat{\mathcal{A}}_{2m-1}\right)^2 \le 2\pi \frac{4m-1}{m} \left(\frac{4}{\sqrt{3}} + 1 \right).$$

Hence, the sum (4.46) converges to 0 for $m \to \infty$ by the dominated convergence theorem of Lebesgue.

In the last part of the proof, we show the estimate (4.44) for the singular values $\hat{\mathcal{A}}_{2m,\psi}$ of even degree. Let $m \in \mathbb{N}$. We have

$$\frac{4m+1}{4}\left(\hat{\mathcal{A}}_{2m,\psi}\right)^2 = 8\pi^2 \left|\widetilde{P}_{2m}^{0}(0)\right|^2 \psi^2 + 4\pi^2 \sum_{k=1}^{m} \left|\widetilde{P}_{2m}^{2k}(0)\right|^2 \frac{\sin^2(2k\psi)}{k^2}. \tag{4.47}$$

We examine both summands on the right side of (4.47). The first summand converges due to (4.33):

$$\lim_{m\to\infty} 8\pi^2 \left|\widetilde{P}_{2m}^{0}(0)\right|^2 \psi^2 = 8\psi^2.$$

We denote the second summand of (4.47) by

$$\lambda_m(\psi) := 4\pi^2 \sum_{k=1}^{m} \left|\widetilde{P}_{2m}^{2k}(0)\right|^2 \frac{\sin^2(2k\psi)}{k^2}$$

and define the function $\lambda\colon [0,\pi) \to \mathbb{R}$ by the following Fourier cosine series, see [GR07, 1.443],

$$\lambda(\psi) := 4\sum_{k=1}^{\infty} \frac{\sin^2(2k\psi)}{k^2} = 4\sum_{k=1}^{\infty} \frac{1-\cos(4k\psi)}{2k^2} = \begin{cases} -8\psi^2 + 4\pi\psi, & \psi \in [0,\frac{\pi}{2}) \\ -8\psi^2 + 12\pi\psi - 4\pi^2, & \psi \in [\frac{\pi}{2},\pi). \end{cases}$$

Then we have

$$\begin{aligned} \|\lambda_m - \lambda\|_{C([0,\pi])} &= \left\| \sum_{k=1}^{\infty} \frac{4\pi^2 \left|\widetilde{P}_{2m}^{2k}(0)\right|^2 - 4}{k^2} \sin^2(2k\psi) \right\|_{C([0,\pi])} \\ &\le 4\sum_{k=1}^{\infty} \frac{\left|\pi^2 \left|\widetilde{P}_{2m}^{2k}(0)\right|^2 - 1\right|}{(2j-1)^2}. \end{aligned}$$

As in the above proof of (4.43), we see with (4.36) that the last sum goes to 0 for $m \to \infty$. Hence, we have $\lim_{m\to\infty} \lambda_m(\psi) = \lambda(\psi)$ for all $\psi \in (0,\pi)$, which proves (4.44). ∎

Remark 4.23. Theorem 4.22 shows that, for all $\psi \in (0,\pi)$, the singular values $\hat{\mathcal{A}}_{n,\psi}$ of $\mathcal{A}_\psi$ decay with the asymptotic rate of $n^{-1/2}$ for $n \to \infty$. This is the same asymptotic rate as for the singular values $\hat{\mathcal{A}}_n$ of the arc transform $\mathcal{A}$ with full data from Theorem 4.19.

The asymptotic behavior of the singular values $\hat{\mathcal{A}}_{n,\psi}$ is illustrated more specifically in Figure 4.3. The upper plot shows the dependence of the singular values $\hat{\mathcal{A}}_{n,\psi}$ on the polynomial degree n for different choices of the arc-length ψ. The lower plot shows the dependency on the arc-length ψ, which illustrates (4.43) and (4.44). If $\psi \le \frac{\pi}{2}$, the arcs are smaller than semicircles. Then, the "normalized" squared singular values

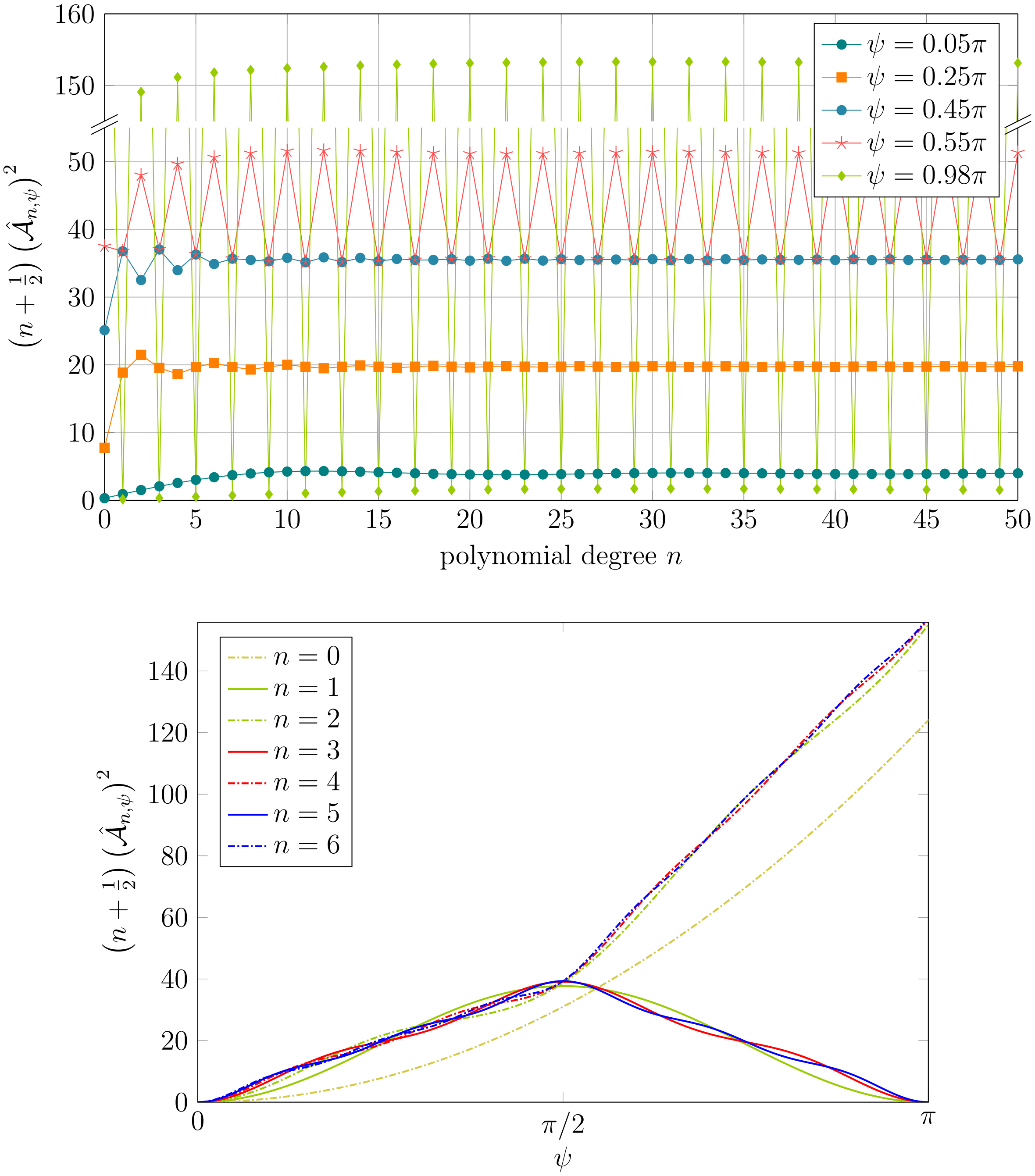

Figure 4.3: The (normalized) squared singular values $\left(n+\frac{1}{2}\right)\left(\hat{\mathcal{A}}_{n,\psi}\right)^2$.
Top: Dependency on the degree n. Note the oscillation for $\psi > \frac{\pi}{2}$.
Bottom: Dependency on the arc-length ψ (dashed lines correspond to even degrees n).

$(n+\frac{1}{2})\,|\hat{\mathcal{A}}_{n,\psi}|^2$ converge for $n \to \infty$ to a constant, which depends linearly on ψ. This is consistent with our theoretical findings. Both (4.43) and (4.44) converge to the same limit if $\psi \in [0, \frac{\pi}{2}]$.

However, the situation becomes a little more difficult in the case $\psi > \frac{\pi}{2}$, which means that the arcs with length 2ψ are longer than semicircles. Then, the "normalized" singular values $(n+\frac{1}{2})\,|\hat{\mathcal{A}}_{n,\psi}|^2$ are larger for even n than for odd n. This might be explained by the fact that for odd n, the spherical harmonics Y_n^k are odd functions and integrating them along such a circle arc, which is longer than a semicircle, yields some cancellation. In the limiting case $\psi = \pi$, we have full circles corresponding to the Funk–Radon transform. There, the singular values $\hat{\mathcal{A}}_{n,\pi}$ vanish for odd n. □

Example 4.24. In the case $\psi = \pi/2$, the arc transform $\mathcal{A}_{\pi/2}$ takes the integrals along great semicircles. Groemer [Gro98] showed in 1998 that $\mathcal{A}_{\pi/2}$ is injective for $f \in C(\mathbb{S}^{d-1})$.

In this situation, the singular values $\hat{\mathcal{A}}_{n,\pi/2}$ admit the following simplified expression. By (4.47) and the fact that the singular values are positive, we see that for even $n = 2m$ with $m \in \mathbb{N}$

$$\hat{\mathcal{A}}_{2m,\pi/2} = \frac{\sqrt{8}\,\pi^2}{\sqrt{4m+1}} \left| \widetilde{P}_{2m}^0(0) \right|.$$

By (4.45), we have for odd $n = 2m-1$

$$\hat{\mathcal{A}}_{2m-1,\pi/2} = \frac{8\pi}{\sqrt{4m-1}} \sqrt{\sum_{j=1}^{m} \left| \widetilde{P}_{2m-1}^{2j-1}(0) \right|^2 \frac{1}{(2j-1)^2}}.$$

Since it can be parameterized by SO(3), the manifold of all great semicircles on $\mathbb{S}^2$ is three-dimensional. In 2017, Rubin [Rub17a] considered only the great semicircles that are subsets of either the upper or the lower hemisphere. It is easy to see that this restriction of $\mathcal{A}_{\pi/2}$ is still injective since every function that is supported in the upper (lower) hemisphere can be uniquely reconstructed by its Funk–Radon transform, which then integrates only over all semicircles in the upper (lower) hemisphere. □

Example 4.25. If the arc-length ψ gets closer to zero, then the respective great circle arcs of the arc transform $\mathcal{A}_\psi$ become shorter. Hence, the integral along such arcs also becomes smaller, e. g., the constant function $f \equiv 1$ is mapped to the constant function $\mathcal{A}_\psi f \equiv 2\psi$. In the extremal case $\psi = 0$, the arc integral along a single point is zero, so we have $\mathcal{A}_0 f \equiv 0$ for all $f\colon \mathbb{S}^2 \to \mathbb{R}$. As ψ approaches zero, the singular values $\hat{\mathcal{A}}_{n,\psi}$ also become smaller. However, the singular values still decay asymptotically with the rate $n^{-1/2}$ for $n \to \infty$. □

Since we have found the asymptotic expressions (4.43) and (4.44) of the singular values $\hat{\mathcal{A}}_{n,\psi}$, we obtain the following result about the arc transform $\mathcal{A}_\psi$ in Sobolev spaces, which follows analogously to Theorem 3.13 for the Funk–Radon transform $\mathcal{F}$. We recall the definition of Sobolev spaces $H^s(\mathrm{SO}(3))$ on the rotation group in (2.66).

Corollary 4.26. Let $\psi \in (0,\pi)$ and $s \in \mathbb{R}$. Then the arc transform $\mathcal{A}_\psi$ with fixed length is a continuous linear operator

$$\mathcal{A}_\psi \colon H^s(\mathbb{S}^2) \to H^{s+\frac{1}{2}}(\mathrm{SO}(3)).$$

Reconstruction

The singular value decomposition of $\mathcal{A}_\psi$ from Theorem 4.22 allows us to reconstruct a function $f \in L^2(\mathbb{S}^2)$ given $g = \mathcal{A}_\psi f$ for some $\psi \in (0,\pi)$ as follows.

Theorem 4.27. *Let $f \in L^2(\mathbb{S}^2)$, $\psi \in (0,\pi)$, and let $g = \mathcal{A}_\psi f \in L^2(\mathrm{SO}(3))$. Then f can be reconstructed from the rotational Fourier coefficients $\hat{g}_n^{j,k}$ of g given in* (2.64) *by*

$$f = \sum_{n=0}^{\infty} \sum_{k=-n}^{n} \frac{\sum_{j=-n}^{n} \widetilde{P}_n^j(0)\, s_j(\psi)\, \hat{g}_n^{j,k}}{\sum_{j=-n}^{n} \widetilde{P}_n^j(0)^2\, s_j(\psi)^2}\, Y_n^k. \tag{4.48}$$

Proof. Let $n \in \mathbb{N}_0$ and $k \in \{-n,\dots,n\}$. By the singular value decomposition in Theorem 4.22, we obtain the spherical Fourier coefficients of f,

$$\hat{f}_n^k = \frac{1}{\left(\hat{\mathcal{A}}_{n,\psi}\right)^2} \left\langle g, \mathcal{A}_\psi Y_n^k \right\rangle_{L^2(\mathrm{SO}(3))}.$$

Inserting the singular functions $\mathcal{A}_\psi Y_n^k$ from (4.42), we obtain

$$\hat{f}_n^k = \frac{1}{\left(\hat{\mathcal{A}}_{n,\psi}\right)^2} \sum_{j=-n}^{n} \widetilde{P}_n^j(0)\, s_j(\psi) \left\langle g, D_n^{j,k} \right\rangle_{L^2(\mathrm{SO}(3))}.$$

By (2.64) and (4.41) for the rotational Fourier coefficients $\hat{g}_n^{j,k}$ and the singular values $\hat{\mathcal{A}}_{n,\psi}$, respectively, we have

$$\hat{f}_n^k = \frac{1}{\sum_{j=-n}^{n} \widetilde{P}_n^j(0)^2 s_j(\psi)^2} \sum_{j=-n}^{n} \widetilde{P}_n^j(0)\, s_j(\psi)\, \hat{g}_n^{j,k},$$

which shows the assertion. ∎

It seems worth mentioning that the reconstruction of a function f from $\mathcal{A}_\psi f$ is still an overdetermined problem. Whereas the function f is defined on the two-dimensional sphere $\mathbb{S}^2$, its arc transform $\mathcal{A}_\psi f$ is defined on the rotation group SO(3), which is a three-dimensional manifold. However, since the singular values of $\mathcal{A}_\psi$ decay with the asymptotic rate $n^{-1/2}$, the reconstruction problem is ill-posed.

An advantage of using the singular value decomposition in Theorem 4.27 for the inversion of the arc transform $\mathcal{A}_\psi$ is that it is straightforward to apply Tikhonov-type regularization or the mollifier method [LM90], which both correspond to a multiplication of the summands in the inversion formula (4.48) with some filter coefficients, cf. Remark 3.16 for the Funk–Radon transform.

Remark 4.28. We have implemented the inversion formula (4.48) of $\mathcal{A}_\psi$ numerically, as described in our paper [HPQ18]. The only part of this formula that depends upon the given data g consists of the rotational Fourier coefficients $\hat{g}_n^{j,k}$. The computation of the coefficients $\hat{g}_n^{j,k}$ is done numerically with the fast SO(3) Fourier transform [PPV09], where we apply a quadrature formula to the integral (2.64). Then the inversion formula (4.48) is evaluated with the fast spherical Fourier transform [PPST18, Chapter 9.6] if the outer sum is truncated to $n \le N \in \mathbb{N}_0$. $\square$

5 Conclusion

5.1 Overview

In this thesis, we have considered the mean operator $\mathcal{M}$ on the sphere $\mathbb{S}^{d-1}$ and its restrictions to various families of hyperplane sections of $\mathbb{S}^{d-1}$. We have managed to extend large parts of the classical theory about the Funk–Radon transform $\mathcal{F}$, which takes the integrals along great circles, to these general classes of hyperplane sections.

For several of these restrictions, we have proven injectivity theorems and range characterizations as well as inversion formulas. If we had to choose one method that was the most important to reach these results, this choice would most certainly fall on the singular value decomposition. For most of the restrictions of the mean operator, we were able to show its singular value decomposition. To the best knowledge of the author, these are new results for the generalized Funk–Radon transform $\mathcal{S}^{(j)}$, the vertical slice transform $\mathcal{V}_z$ and the circular arc transform $\mathcal{A}_\psi$. Another technique is the application of geometric tools in order to establish connections between the different integral operators. By applying the composition of twice the stereographic projection and a properly chosen, uniform scaling, we transferred the results from the Funk–Radon transform $\mathcal{F}$ to the spherical transform $\mathcal{U}_z$, characterizing its nullspace and range. Furthermore, we have seen that Grangeat's formula can be interpreted as a connection between the cone-beam transform $\mathcal{D}$, the Radon transform $\mathcal{R}$ and the new generalized Funk–Radon transform $\mathcal{S}^{(d-2)}$. This observation paved the way to prove the singular value decomposition of the cone-beam transform with sources on the sphere.

An overview about the different restrictions of the mean operator $\mathcal{M}$ we have considered is provided in the following Table 5.1. We explain the meaning of its entries beforehand:

- We start with the name and the definition as a restriction of the mean operator $\mathcal{M}$. If not noted otherwise, we generally assume that $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$ and $t \in [-1, 1]$.
- The third column states whether this operator is injective, if not stated otherwise for $f \in L^2(\mathbb{S}^{d-1})$. We note that some of the operators depend on an additional

parameter. In case it is not injective, we state a condition on the function f that makes this operator injective.

- We state the range for $f \in L^2(\mathbb{S}^{d-1})$ in the forth column. An exact description of the range is denoted by "=". In some cases, we do not know the range exactly, but we can specify a superset of the range.
- We indicate whether there is a singular value decomposition (SVD) for this transform in the fifth column.
- We refer to the section of this thesis about the respective restriction of the mean operator in the last column.

Name	**Definition**	**Injectivity**	**Range**	**SVD**	**Sec.**
mean operator	$\mathcal{M}f(\boldsymbol{\xi}, t)$	✓	$\subset H^{d/2-1,0}_{\mathrm{mix}}$	✓	3.1
Funk–Radon transform	$\mathcal{F}f(\boldsymbol{\xi}) = \mathcal{M}f(\boldsymbol{\xi}, 0)$	$f(\boldsymbol{\xi}) = f(-\boldsymbol{\xi})$	$= H^{\frac{d-2}{2}}_{\mathrm{even}}$	✓	3.2
spherical section transform	$\mathcal{T}_z f(\boldsymbol{\xi}) = \mathcal{M}f(\boldsymbol{\xi}, z)$, $z \in [-1, 1]$ fixed	✓ if $P_{n,d}(z) \neq 0$ $\forall n \in \mathbb{N}_0$	$\subset H^{\frac{d-2}{2}}$	✓	3.3
fixed set of centers	$\mathcal{M}f(\boldsymbol{\xi}, t)$, $\boldsymbol{\xi} \in S$ for $S \subset \mathbb{S}^{d-1}$	✓ if $Y_n(S) \neq \{\mathbf{0}\}$ $\forall Y_n \in \mathscr{Y}_{n,d}$, $n \in \mathbb{N}_0$		✓	3.5.1
centers on a circle	$\mathcal{V}_z f(\boldsymbol{\sigma}, t) = \mathcal{M}f((\begin{smallmatrix}\boldsymbol{\sigma}\\ z\end{smallmatrix}), t)$, $\boldsymbol{\sigma} \in \mathbb{S}^{d-2}$, $z \in (-1, 1)$ fixed	✓ if $P^k_{n,d}(z) \neq 0$ $\forall n \in \mathbb{N}_0, k = 0..n$	$\subset H^{\frac{d-3}{2},0}_{\mathrm{mix}}$	✓	3.5.2
vertical slice transform	$\mathcal{V}f(\boldsymbol{\sigma}, t) = \mathcal{M}f((\begin{smallmatrix}\boldsymbol{\sigma}\\ 0\end{smallmatrix}), t)$, $\boldsymbol{\sigma} \in \mathbb{S}^{d-2}$	$f(\boldsymbol{\xi}', \xi_d) = f(\boldsymbol{\xi}', -\xi_d)$	$\subset H^{0,\frac{d-2}{2}-\frac{1}{4}}_{\mathrm{mix}}$	✓	3.5.3
sections through fixed point	$\mathcal{U}_z f(\boldsymbol{\xi}) = \mathcal{M}f(\boldsymbol{\xi}, z\xi_d)$, $z \in (-1, 1)$ fixed	f even w.r.t. some reflection in $z\boldsymbol{\epsilon}^d$	$= \widetilde{H}_z^{\frac{d-2}{2}}$	✗	3.6
spherical slice transform	$\mathcal{U}_1 f(\boldsymbol{\xi}) = \mathcal{M}f(\boldsymbol{\xi}, \xi_d)$	✓ for $f \in L^\infty(\mathbb{S}^{d-1})$		✗	3.6.5
generalized FRT	$\mathcal{S}^{(j)} f(\boldsymbol{\xi}) = (-\frac{\partial}{\partial t})^j \mathcal{M}f(\boldsymbol{\xi}, 0)$	f even / odd	$\subset H^{\frac{d-2}{2}-j}$	✓	3.4

Table 5.1: Overview about the restrictions of the mean operator $\mathcal{M}$

5.2 Theses

As tradition dictates, we shall present the key results of this dissertation in the form of the following eight short theses.

(1) In the theory of many imaging modalities, the key mathematical task is the reconstruction of a function f that is defined on a subset of $\mathbb{R}^d$ from its mean values along a certain family of submanifolds. In case of a function $f\colon \mathbb{S}^{d-1} \to \mathbb{R}$, its mean values along hyperplane sections of the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ are useful in models of the cone-beam tomography, Compton camera imaging, magnetic resonance imaging, photoacoustic tomography, Radar imaging and seismic imaging. They are described by the mean operator

$$\mathcal{M}f(\boldsymbol{\xi}, t) = \int_{\boldsymbol{\xi}^\top \boldsymbol{\eta} = t} f(\boldsymbol{\eta}) \,\mathrm{d}\mu(\boldsymbol{\eta}), \qquad \boldsymbol{\xi} \in \mathbb{S}^{d-1},\ t \in [-1,1],$$

where $\mathrm{d}\mu$ denotes the surface measure on $C(\boldsymbol{\xi}, t)$ that is normalized to one.

(2) The inversion of the mean operator $\mathcal{M}$ is an overdetermined, ill-posed problem. In particular, any function f on $\mathbb{S}^{d-1}$ can be uniquely reconstructed from the data $\mathcal{M}f(\boldsymbol{\xi}, t)$ on only some subset of $\mathbb{S}^{d-1} \times [-1,1]$. Such a subset is called an injectivity set of $\mathcal{M}$.

Then, a set $D \subset \mathbb{S}^{d-1} \times [-1,1]$ is an injectivity set of the mean operator $\mathcal{M}$ if and only if the partial differential equation

$$\Delta^{\bullet}_{\boldsymbol{\xi}} g(\boldsymbol{\xi}, t) = (1-t^2)^{\frac{3-d}{2}} \frac{\partial}{\partial t} \left((1-t^2)^{\frac{d-1}{2}} \frac{\partial}{\partial t} g(\boldsymbol{\xi}, t) \right)$$

with boundary values on D always has a unique solution g on $\mathbb{S}^{d-1} \times [-1,1]$.

(3) The vertical slice transform $\mathcal{V}$ is the restriction of the mean operator $\mathcal{M}$ to the sections of the sphere with vertical hyperplanes, i. e., the sections are perpendicular to the equatorial hyperplane. It is injective only for functions that are even with respect to the reflection in the equatorial hyperplane. A singular value decomposition of the vertical slice transform $\mathcal{V}$ may be obtained as $\mathcal{V} Y_{n,d}^{m,k}(\boldsymbol{\sigma}, t) = \hat{\mathcal{V}}_{n,d}^{m}\, Y_{m,d-1}^{k}(\boldsymbol{\sigma})\, \widetilde{P}_{n,d}(t)$ with the spherical harmonics $Y_{n,d}^{m,k}$ and the normalized Legendre polynomials $\widetilde{P}_{n,d}$ as singular functions. The singular values $\hat{\mathcal{V}}_{n,d}^{k}$ admit the asymptotically sharp bounds

$$c_1\, n^{\frac{d-2}{2}} \le \left| \hat{\mathcal{V}}_{n,d}^{k} \right| \le c_2\, n^{\frac{d-2}{2} + \frac{1}{4}},$$

for some constants $c_1, c_2 > 0$. In particular, they decay a little slower than those of the Funk–Radon transform.

(4) The situation becomes more difficult for hyperplane sections of $\mathbb{S}^{d-1}$ where the normal vectors $\boldsymbol{\xi} \in \mathbb{S}^{d-1}$ are on the circle of latitude $\xi_d = z$ of the sphere for some $z \in (-1,1)$. This transform is injective for all except countably many values of z. In particular, it is non-injective on all zeros z of the associated Legendre functions $P_{n,d}^{k}(z)$ of dimension d. For a specific value of z, however, it is usually not easy to decide whether this problem is injective.

(5) For the sections of the sphere with hyperplanes containing a common point inside the sphere, there is no singular value decomposition known. However, a more successful approach lies in a connection with the Funk–Radon transform: The composition of twice the stereographic projection with a properly chosen, uniform scaling turns these hyperplane sections into maximal subspheres of $\mathbb{S}^{d-1}$. The nullspace consists of those functions f on $\mathbb{S}^{d-1}$ that are odd with respect to the point reflection in the common point $\boldsymbol{\zeta}$ and the multiplication of some weight. Furthermore, the range is the same Sobolev space $H^{d/2-1}_{\mathrm{even}}(\mathbb{S}^{d-1})$ as for the Funk–Radon transform.

(6) In spherical surface wave tomography, one measures the time a seismic wave travels along the Earth's surface from an epicenter to a receiver. Knowing the traveltimes of such waves between many pairs of epicenters and receivers, one wants to recover the local phase velocity. The simplest but widely used mathematical model asks for the reconstruction of a function defined on the sphere $\mathbb{S}^2$ from its integrals along arcs of great circles. Parameterizing the circle arcs by their length and the rotation that maps them to a reference arc combined with harmonic analysis on the rotation group are the key ingredients of the proof of a singular value decomposition for this arc transform $\mathcal{A}$. Even for the limited data of mean values along great circle arcs with fixed arc-length, the singular value decomposition ensures the injectivity.

(7) The generalized Funk–Radon transform $\mathcal{S}^{(j)}$ takes the j-th order directional derivative of the function f perpendicular to the great circle along which f is integrated. The spherical harmonics are the eigenfunctions of this operator $\mathcal{S}^{(j)}$. The generalized Funk–Radon transform $\mathcal{S}^{(j)}$ is injective for a subset of either the odd or the even functions, depending on the order j. It is a continuous and open operator from the Sobolev space $H^s(\mathbb{S}^{d-1})$ to $H^{s-j+\frac{d-2}{2}}(\mathbb{S}^{d-1})$. Furthermore, $\mathcal{S}^{(-1)}$ is the hemispherical transform.

(8) The cone-beam transform $\mathcal{D}$ integrates a function defined on the Euclidean space $\mathbb{R}^d$ along all rays that start in a certain scanning set. It provides the mathematical background of the three-dimensional X-ray tomography on $\mathbb{R}^3$. Grangeat's formula can be written as a connection of the cone-beam transform $\mathcal{D}$ with the generalized Funk–Radon transform $\mathcal{S}^{(d-2)}$ on the sphere $\mathbb{S}^{d-1}$ and the Radon transform $\mathcal{R}$ on $\mathbb{R}^d$. Hence, the inversion of the cone-beam transform splits up into the inverse Radon transform and the inverse generalized Funk–Radon transform. The singular value decomposition of the transform $\mathcal{S}^{(j)}$ allows to derive a singular value decomposition of the cone-beam transform $\mathcal{D}$ with sources on the sphere.

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List of Figures

List of Symbols

$\mathcal{A}$	arc transform, page 137
$\mathcal{R}$	Radon transform, page 92
$\mathbb{B}^d$	unit ball in the Euclidean space $\mathbb{R}^d$, page 25
$\mathbb{C}$	complex numbers, page 21
$C(\mathbb{S}^{d-1})$	space of continuous functions on the sphere, page 34
$C_n^{(\alpha)}$	Gegenbauer polynomial of degree n, page 28
$C^s(\mathbb{S}^{d-1})$	space of functions on the sphere with continuous derivatives up to order s, page 34
$C^\infty(\mathbb{S}^{d-1})$	space of functions on the sphere with continuous derivatives of arbitrary order, page 34
$\mathcal{D}$	cone-beam transform, page 120
d	dimension of the Euclidean space R^d, $d \geq 2$, page 22
$\Delta^\bullet$	Laplace–Beltrami operator on the sphere, page 34
$D_n^{k,j}$	rotational harmonic (Wigner D-function) of degree n, page 44
$\boldsymbol{\epsilon}^i$	the i-th unit vector in the Euclidean space $\mathbb{R}^d$, page 21
$\mathcal{F}$	the Funk–Radon transform, page 60
$f^\bullet$	extension of a function $f\colon \mathbb{S}^{d-1} \to \mathbb{C}$ to $\mathbb{R}^d$, page 33
$\hat{f}_n^k$	spherical Fourier coefficient, page 30
$G_{n_1,k_1,n_2,k_2}^{n,k}$	Gaunt coefficient on $\mathbb{S}^2$, page 42
$H^s(\mathbb{S}^{d-1})$	Sobolev space of order s, page 35
$H^s_{\text{even}}(\mathbb{S}^{d-1})$	Sobolev space of even functions $f(\boldsymbol{\xi}) = f(-\boldsymbol{\xi})$, page 62

$L^2(\mathbb{S}^{d-1})$	Lebesgue space of square-integrable functions, page 27
$\mathcal{M}$	mean operator on the sphere, page 50
$\mathbb{N}$	positive integers $\mathbb{N} = \{1, 2, \dots\}$, page 21
$\mathbb{N}_0$	nonnegative integers $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, page 21
$N_{n,d}$	the dimension of $\mathscr{Y}_{n,d}(\mathbb{S}^{d-1})$, page 27
$P_n^{(\alpha,\beta)}$	Jacobi polynomial of degree $n \in \mathbb{N}_0$ and orders $\alpha, \beta > -1$, page 29
$P_{n,d}$	Legendre polynomial of degree n in dimension d, page 28
$P_n = P_{n,3}$	Legendre polynomial of degree n, page 28
$P_{n,d}^m$	associated Legendre function of degree n and order m, page 31
$\mathbb{R}$	real numbers, page 21
$\mathcal{R}$	Radon transform, page 11
$\mathbb{S}^2$	two-dimensional sphere, page 32
$\lvert\mathbb{S}^{d-1}\rvert$	volume of the sphere $\mathbb{S}^{d-1}$, page 27
$\mathbb{S}^{d-1}$	the unit sphere in $\mathbb{R}^d$, page 22
$\mathcal{S}^{(j)}$	generalized Funk–Radon transform, page 69
SO(3)	rotation group, page 43
$\mathcal{T}_z$	spherical section transform, page 66
$\mathcal{U}_z$	spherical transform (hyperplane sections through the common point $z\boldsymbol{\epsilon}^d$), page 97
$\mathcal{V}$	vertical slice transform, page 85
$\boldsymbol{\xi}(\varphi, \vartheta)$	spherical coordinates on $\mathbb{S}^2$, page 32
$\mathscr{Y}_{n,d}(\mathbb{S}^{d-1})$	the space of spherical harmonics of degree n, page 27
Y_n^k	spherical harmonic of degree n, page 33