Optimal mollifiers for spherical deconvolution

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(joint work with Ralf Hielscher)

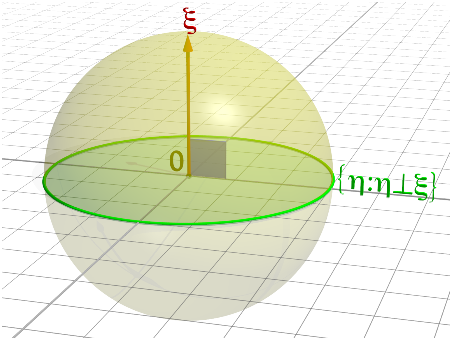
Content

Funk–Radon transform

- Sphere $\mathbb{S}^2 = \{\boldsymbol{\xi} \in \mathbb{R}^3 : \|\boldsymbol{\xi}\| = 1\}$
- Function $f : \mathbb{S}^2 \to \mathbb{C}$ on the sphere
- **Funk–Radon transform** computes the integrals along all great circles

$$\mathcal{R}f(\boldsymbol{\xi}) = \int_{\boldsymbol{\xi}\cdot\boldsymbol{\eta}=0} f(\boldsymbol{\eta})\,\mathrm{d}s(\boldsymbol{\eta})$$

$$= \int_{\mathbb{S}^2} f(\boldsymbol{\eta})\delta(\boldsymbol{\xi}\cdot\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta}$$



What we want to do

Compute f from the given values $\mathcal{R}f$ (inverse problem)

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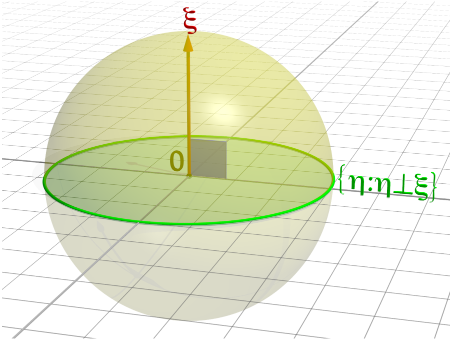

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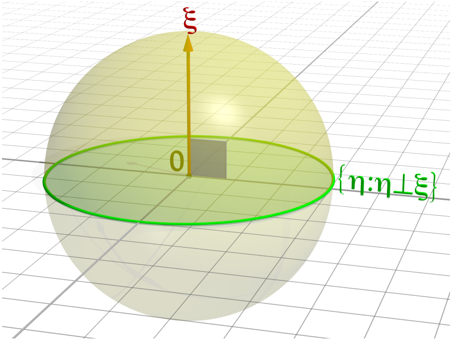

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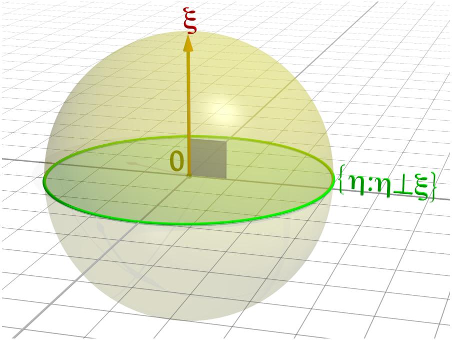

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Paul Funk.
Über Flächen mit lauter geschlossenen geodätischen Linien.
Math. Ann., 74(2):278 – 300, June 1913.

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The Radon Transform.
Birkhäuser, 2nd edition, 1999.

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Inverse Problems, 27(3):035015, March 2011.

Martin Riplinger and Malte Spiess.
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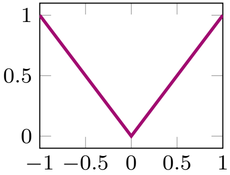
J. Inverse Ill-Posed Probl., 22(4):497 – 536, December 2013.

Ralf Hielscher and Michael Quellmalz.
Optimal mollifiers for spherical deconvolution.
Preprint 2015-04, Faculty of Mathematics, Technische Universität Chemnitz, 2015.

Spherical convolution

- Function $f : \mathbb{S}^2 \to \mathbb{C}$ on the sphere
- Kernel function $h : [-1, 1] \to \mathbb{C}$ on the interval



$h(t) = |t|$

Definition (convolution operator)

The operator $\mathcal{M}$ of convolution with h is defined as

$$\mathcal{M}f(\xi) = h \star f(\xi) = \int_{\mathbb{S}^2} f(\eta) h(\xi \cdot \eta) \, \mathrm{d}\eta, \quad \xi \in \mathbb{S}^2.$$

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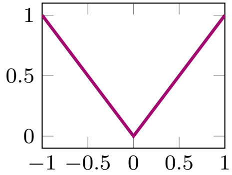

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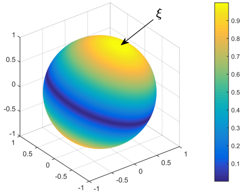

$h(\boldsymbol{\xi} \cdot \circ) = |\boldsymbol{\xi} \cdot \circ|$

- Every function $f \in L^2(\mathbb{S}^2)$ can be written as Fourier series

$$f = \sum_{n=0}^{\infty} \sum_{k=-n}^{n} \hat{f}(n,k) Y_n^k$$

- Fourier coefficients $\hat{f}(n,k) := \int_{\mathbb{S}^2} f(\boldsymbol{\xi}) \overline{Y_n^k(\boldsymbol{\xi})} \, \mathrm{d}\boldsymbol{\xi}$
- Y_n^k – spherical harmonics of degree n

Funk–Hecke formula (for convolution operators)

$$\mathcal{M}f = \sum_{n=0}^{\infty} \sum_{k=-n}^{n} \hat{\mathcal{M}}(n) \hat{f}(n,k) Y_n^k$$

with

$$\hat{\mathcal{M}}(n) = 2\pi \int_{-1}^{1} h(t) P_n(t) \, \mathrm{d}t$$

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Let $s \geq 0$. The **Sobolev space** $H^s(\mathbb{S}^2)$ is the completion of the space of polynomials $f : \mathbb{S}^2 \to \mathbb{C}$ with the norm

$$\|f\|_s^2 := \sum_{n=0}^{\infty} \sum_{k=-n}^{n} \left|\hat{f}(n,k)\right|^2 \left(n + \frac{1}{2}\right)^{2s} .$$

Assumption on $\mathcal{M}$

For $s > 0$ and $\beta > 0$, let the convolution operator

$$\mathcal{M} : H^s(\mathbb{S}^2) \to H^{s+\beta}(\mathbb{S}^2)$$

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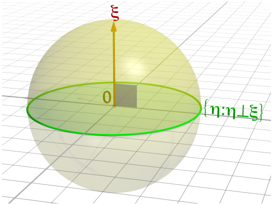
Some notable examples of convolution operators



- **Funk–Radon transform** is the convolution with the delta distribution $h(t) = \delta(t)$

$$\mathcal{R} : H^s_{\mathrm{e}}(\mathbb{S}^2) \to H^{s+\frac{1}{2}}_{\mathrm{e}}(\mathbb{S}^2)$$

- **Hemispherical transform** is the convolution with $h(t) = \mathbf{1}_{t\geq 0}(t)$ (Funk, 1915)

$$\mathcal{H} : H^s_{\mathrm{o}}(\mathbb{S}^2) \to H^{s+\frac{3}{2}}_{\mathrm{o}}(\mathbb{S}^2)$$

- **spherical cosine transform** is the convolution with $h(t) = |t|$ (Petty, 1961)

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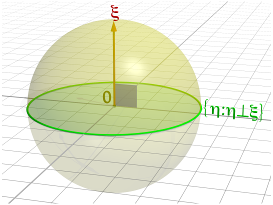

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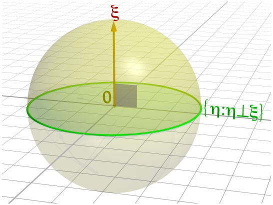

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What are these transforms good for?

- Various applications in stereology and geometric tomography

- **Funk–Radon transform**
- Intersection bodies
- Q–ball imaging in medicine (Tuch, 2004)
- Surface wave models for earthquakes (Amirbekyan & Michel & Simons, 2008)
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- **Hemispherical transfrom**
- Discrete choice models in economy (Gautier & Kitamura, 2013)

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The inverse problem

We have $g = \mathcal{M}f \in H^{s+\beta}(\mathbb{S}^2)$

We want $f \in H^{s}(\mathbb{S}^2)$

- The decomposition in eigenfunctions and eigenvalues yields

$$f = \mathcal{M}^{-1}g = \sum_{n=0}^{\infty} \sum_{k=-n}^{n} \frac{\hat{g}(n,k)}{\hat{\mathcal{M}}(n)} Y_n^k \tag{1}$$

- Small deviation ε (white noise)
- Idea: multiply the Fourier coefficients in (1) with suitable filter coefficients $\hat{\psi}(n) \in [0,1]$
- This is a convolution with ψ: mollifier method (Louis & Maas, 1990)

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- We have **discrete and noisy data**

$$g(\boldsymbol{\xi}_m) = \mathcal{M}f(\boldsymbol{\xi}_m) + \varepsilon(\boldsymbol{\xi}_m), \quad m = 1, \ldots, M.$$

- Idea: Use a **quadrature formula** to calculate

$$\hat{g}(n,k) = \int_{\mathbb{S}^2} g(\xi)\, \overline{Y_n^k(\xi)}\, \mathrm{d}\xi \approx \sum_{m=1}^{M} \omega_m g(\xi_m) \overline{Y_n^k(\xi_m)}$$

- Define **hyperinterpolation** of degree N (Sloan, 1995) (Hesse & Sloan, 2006)

$$\mathcal{L}_N g = \sum_{n=0}^{N} \sum_{k=-n}^{n} \left(\sum_{m=1}^{M} \omega_m g(\xi_m) \overline{Y_n^k(\xi_m)} \right) Y_n^k$$

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How to measure the error

- Mean integrated squared error **MISE**

$$\mathbb{E}\left\|f - \mathcal{E}_{N,\psi}(\mathcal{M}f + \varepsilon)\right\|_{L^2}^2 = \mathbb{E}\int_{\mathbb{S}^2} \left|f(\boldsymbol{\xi}) - \mathcal{E}_{N,\psi}(\mathcal{M}f + \varepsilon)(\boldsymbol{\xi})\right|^2 \,\mathrm{d}\boldsymbol{\xi}$$

- For $s, S \geq 0$, define the class of functions

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Theorem

- *Let $s > 1.62$ and $S > 0$*
- *Convolution operator $\mathcal{M} : H^s(\mathbb{S}^2) \to H^{s+\beta}(\mathbb{S}^2)$ be bijective and continuous*
- *For every $N \in \mathbb{N}$ let the hyperinterpolation $\mathcal{L}_N$ be exact (i.e. a projector) with $M \sim N^2$ nodes and almost constant weights* cf. (Filbir & Mhaskar, 2010)
- *White noise $\varepsilon\,(\xi_m)$*

There exists an asymptotically optimal family of mollifiers $\{\psi_L^s \mid L \in \mathbb{R}_+\}$ for the class $\mathscr{F}(s, S)$. For $N \to \infty$ there are parameters $L(N)$ such that

$$\sup_{f \in \mathscr{F}(s,S)} \mathbb{E} \left\| f - \mathcal{E}_{N,\psi^s_{L(N)}}(\mathcal{M}f + \varepsilon) \right\|_{L^2}^2 \simeq \inf_{\psi} \sup_{f \in \mathscr{F}(s,S)} \mathbb{E} \left\| f - \mathcal{E}_{N,\psi}(\mathcal{M}f + \varepsilon) \right\|_{L^2}^2 \simeq \mathrm{const} \cdot M^{\frac{-s}{s+\beta+1}}.$$

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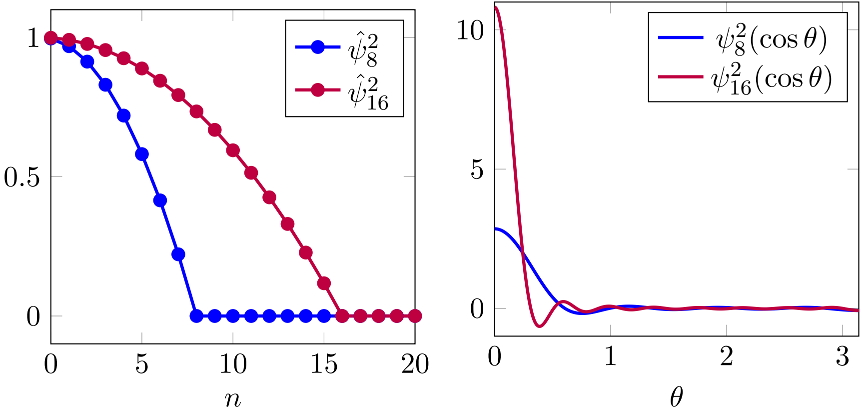

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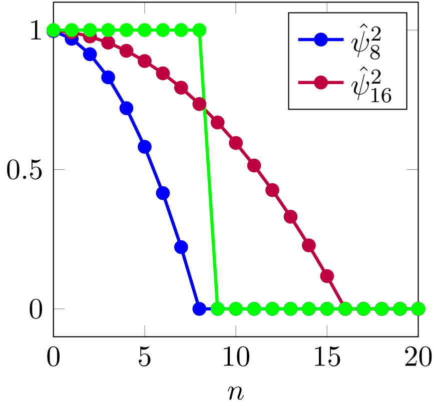

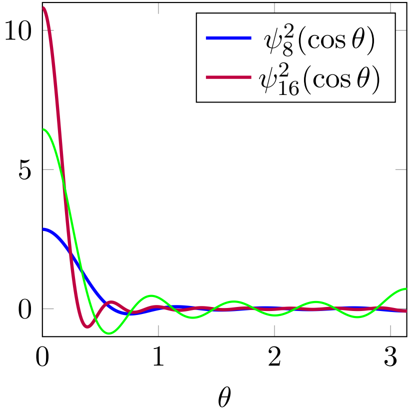

Algorithm to compute the estimator

We have $g(\boldsymbol{\xi}_m)$ and quadrature weights $\omega_m,\ m = 1, \dots, M$

1. Compute the Fourier coefficients $\widehat{\mathcal{L}_N g}(n,k) = \sum_{m=1}^{M} \omega_m g(\boldsymbol{\xi}_m) \overline{Y_n^k(\boldsymbol{\xi}_m)}$
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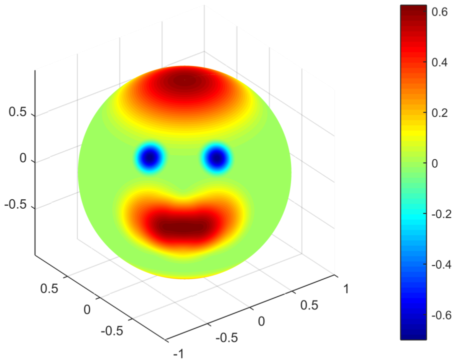
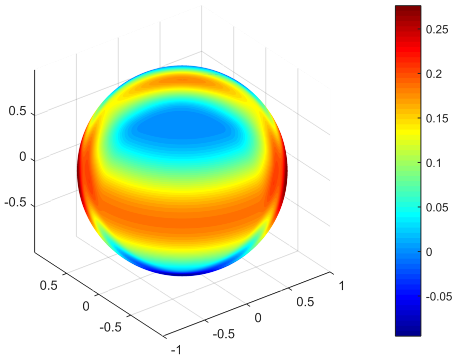
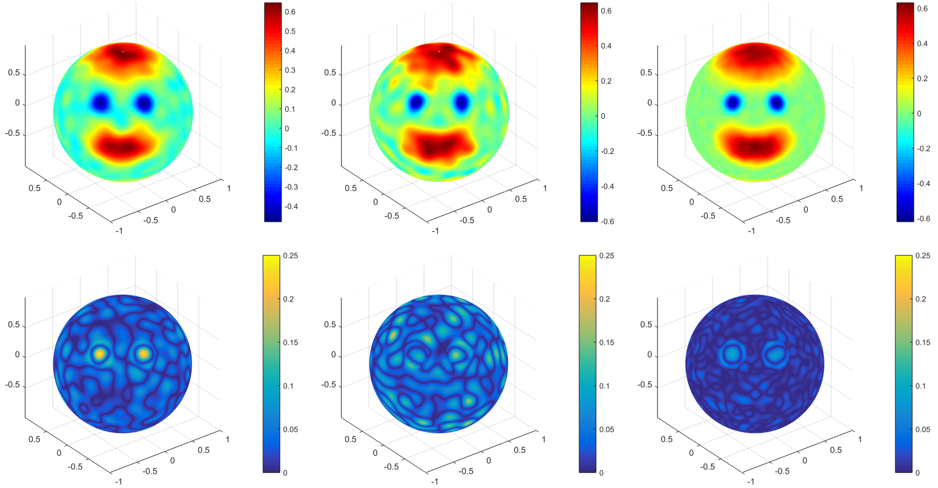




Test function f (quadratic spline)

Funk–Radon transform of f

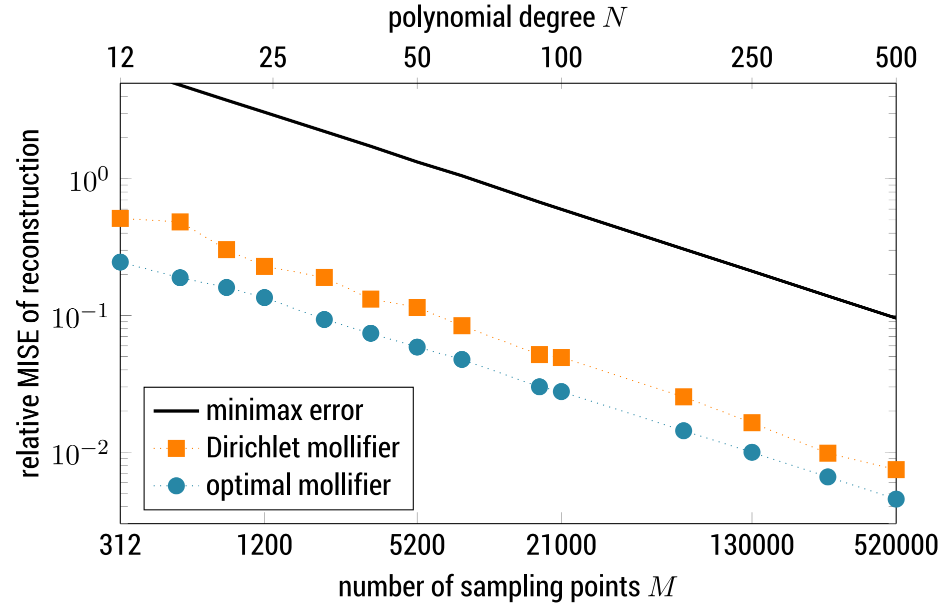
$$\mathcal{R}f(\boldsymbol{\xi}) = \int_{\boldsymbol{\xi}\cdot\boldsymbol{\eta}=0} f(\boldsymbol{\eta})\,\mathrm{d}s(\boldsymbol{\eta})$$



optimal mollifier
N=100

Dirichlet mollifier
N=100

optimal mollifier
N=500


polynomial degree N
12
25
50
100
250
500
relative MISE of reconstruction
10^0
10^{-1}
10^{-2}
minimax error
Dirichlet mollifier
optimal mollifier
312
1200
5200
21000
130000
520000
number of sampling points M

\endinput