# A generalization of the Funk–Radon transform

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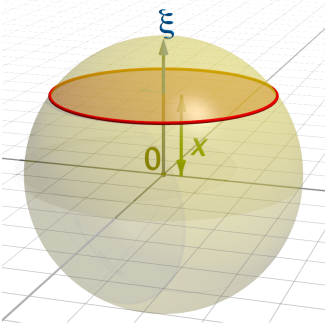
- Sphere $\mathbb{S}^2 = \{\boldsymbol{\xi} \in \mathbb{R}^3 : \|\boldsymbol{\xi}\| = 1\}$
- Function $f\colon \mathbb{S}^2 \to \mathbb{C}$

- Circles on the sphere are intersections of the sphere with planes:

$$\{\boldsymbol{\eta} \in \mathbb{S}^2 : \langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = x\},$$

$(\boldsymbol{\xi}, x) \in \mathbb{S}^2 \times [-1, 1]$

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$$\mathcal{S}\colon C(\mathbb{S}^2) \to C(\mathbb{S}^2 \times [-1,1]),$$

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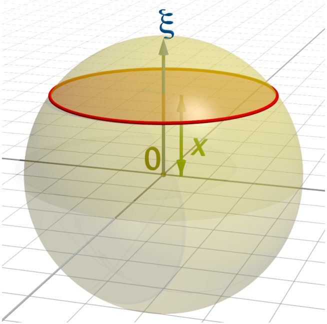


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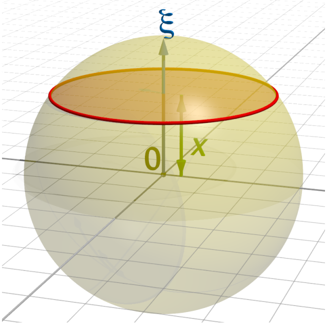


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# Funk–Radon transform

[Funk, 1913]

- Restriction to all great circles
- **Funk–Radon transform** (a.k.a. Funk transform or spherical Radon transform)

$$\mathcal{F}\colon C(\mathbb{S}^2) \to C(\mathbb{S}^2),$$

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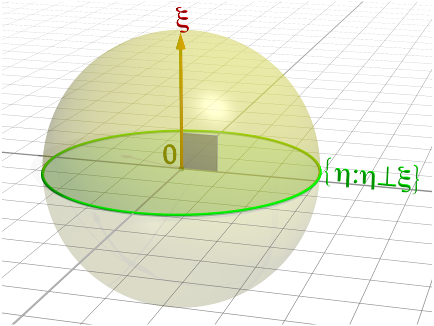


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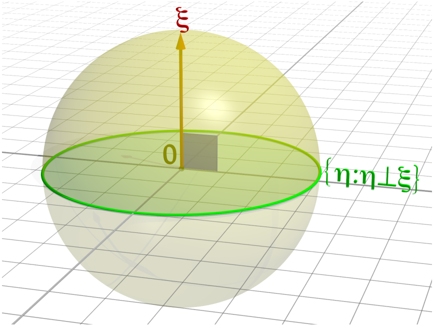


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# Fourier series

Write $f \in L^2(\mathbb{S}^2)$ in terms of the **spherical harmonics** $Y_n^k$ of degree $n$

$$f = \sum_{n=0}^{\infty} \sum_{k=-n}^{n} \hat{f}(n,k) Y_n^k.$$

## Eingenvalue decomposition [Minkowski, 1904]

The Funk–Radon transform is given by

$$\mathcal{F} Y_n^k(\xi) = P_n(0) Y_n^k(\xi), \qquad P_n(0) = \begin{cases} \frac{(n-1)!!}{n!!}, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

$P_n$ – Legendre polynomial of degree $n$

$\Longrightarrow$ The Funk–Radon transform is injective for even functions $f(\xi) = f(-\xi)$.

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# Sobolev spaces

For $s \geq 0$, the **Sobolev space** $H^s(\mathbb{S}^2)$ is the completion of the space of polynomials $f \colon \mathbb{S}^2 \to \mathbb{C}$ with the norm

$$\|f\|_s^2 = \sum_{n=0}^{\infty} \sum_{k=-n}^{n} \left|\hat{f}(n,k)\right|^2 \left(n + \frac{1}{2}\right)^{2s}.$$

**Theorem** [Strichartz, 1981]

The Funk–Radon transform is bijective

$$\mathcal{F} \colon L^2_{\text{even}}(\mathbb{S}^2) \to H^{\frac{1}{2}}_{\text{even}}(\mathbb{S}^2).$$

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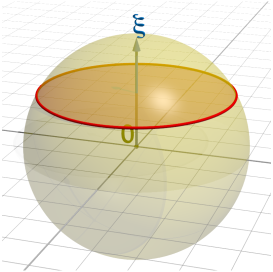
# Circles with fixed radius

- For fixed $x_0 \in [-1, 1]$, compute

$$\mathcal{S}_{x_0} f(\boldsymbol{\xi}) = \int_{\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = x_0} f(\boldsymbol{\eta}) \, \mathrm{d}\boldsymbol{\eta}$$

- Eigenvalue decomposition

$$\mathcal{S}_{x_0} Y_n^k = P_n(x_0) Y_n^k$$




**"Freak theorem"** [Schneider, 1969]

The set of values $x_0$ for which $\mathcal{S}_{x_0}$ is **not** injective is countable and dense in $[-1, 1]$.

Explicit algorithm to determine if $\mathcal{S}_{x_0}$ is injective for given $x_0$ [Rubin, 2000]

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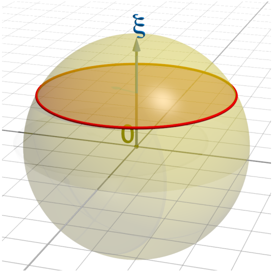


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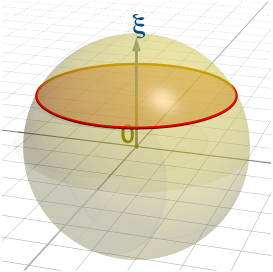


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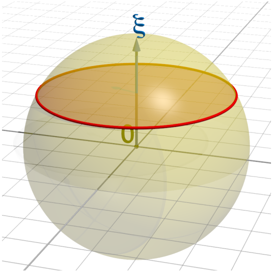


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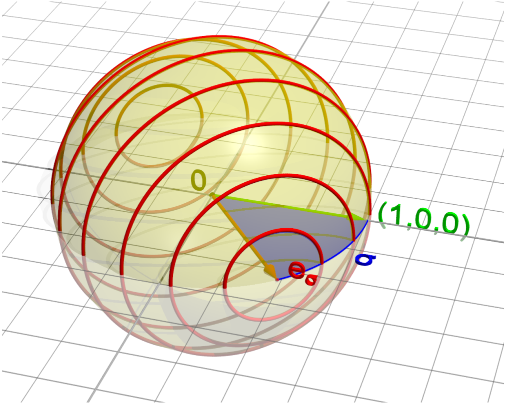
# Vertical slices

$$\mathcal{S}(\boldsymbol{\xi}, x) = \int_{\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = x} f(\boldsymbol{\eta}) \, \mathrm{d}s(\boldsymbol{\eta}), \qquad \xi_3 = 0$$




- **Circles perpendicular to the equator**
- Injective for symmetric functions $f(\xi_1, \xi_2, \xi_3) = f(\xi_1, \xi_2, -\xi_3)$
- Proof1: Orthogonal projection onto equatorial plane [Gindikin, Reeds & Shepp, 1994]
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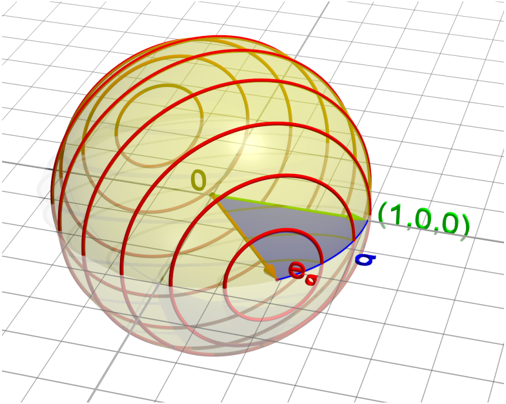


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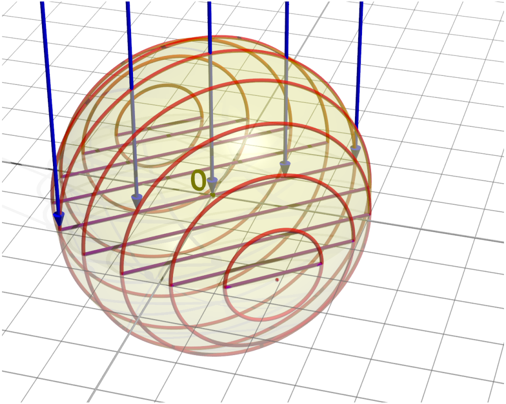


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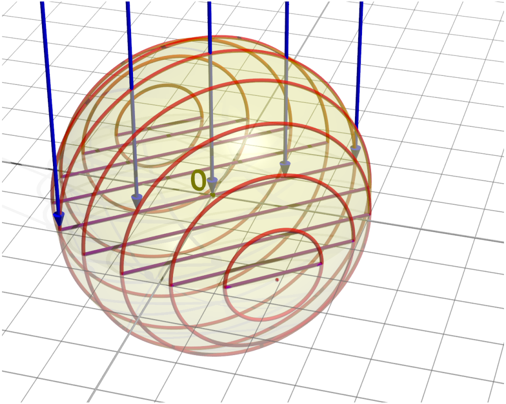


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# Planes through a fixed point

[Salman, 2015]

Replace $0$ by an arbitrary point

$$(0,0,z), \qquad 0 \leq z < 1$$

inside the sphere.

Circle through $(0,0,z)$ is

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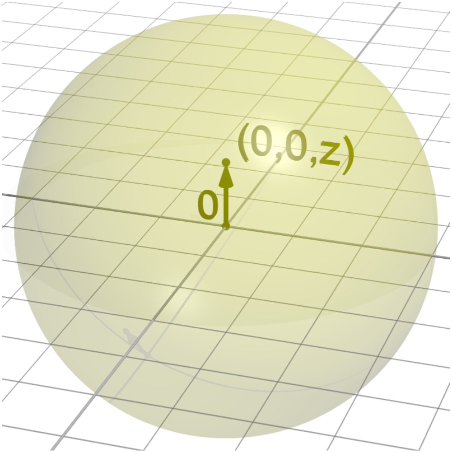


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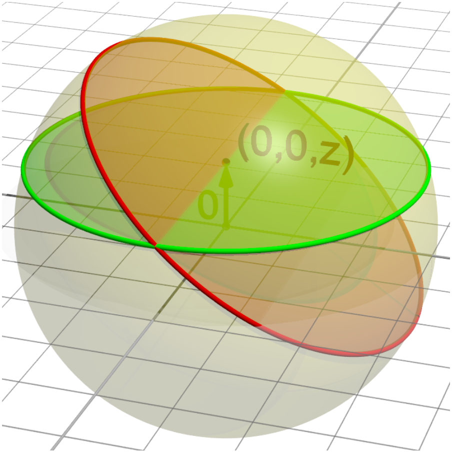


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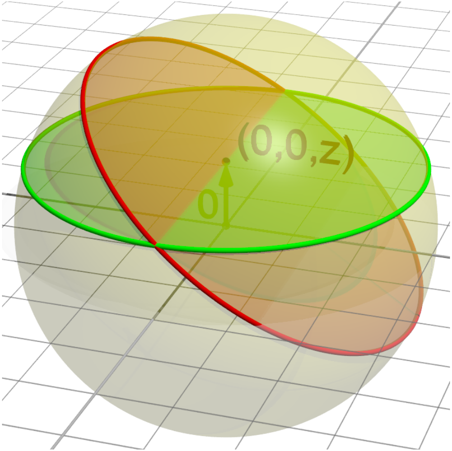


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# From great circles to small circles

## Definition

Define the map

$$h\colon \mathbb{S}^2 \to \mathbb{S}^2, \quad h = \pi^{-1} \circ \sigma \circ \pi$$

consisting of

1. Stereographic projection $\pi\colon \mathbb{S}^2 \to \mathbb{R}^2$
2. Scaling in the plane $\sigma\colon \mathbb{R}^2 \to \mathbb{R}^2,\ \boldsymbol{x} \mapsto \sqrt{\frac{1+z}{1-z}}\,\boldsymbol{x}$
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## We show that

$h$ maps great circles to small circles through $(0, 0, z)$.

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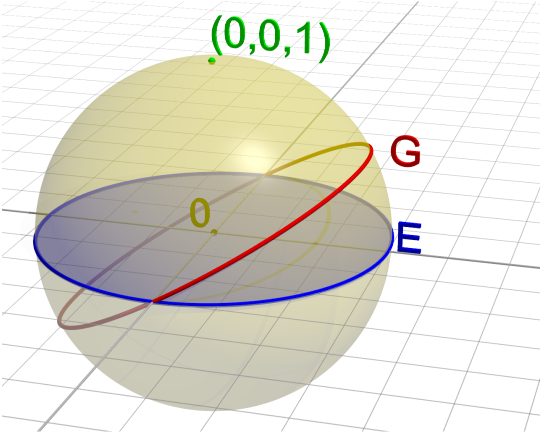
# 1) Stereographic projection $\pi$

- $G$ ... Great circle of $\mathbb{S}^2$
- $E$ ... Equator of $\mathbb{S}^2$

- $G$ intersects $E$ in two antipodal points (or is identical to $E$)

- $\pi(E) = E$
- $\pi(G)$ is a circle or line in $\mathbb{R}^2$ that intersects $\pi(E)$ in two antipodal points



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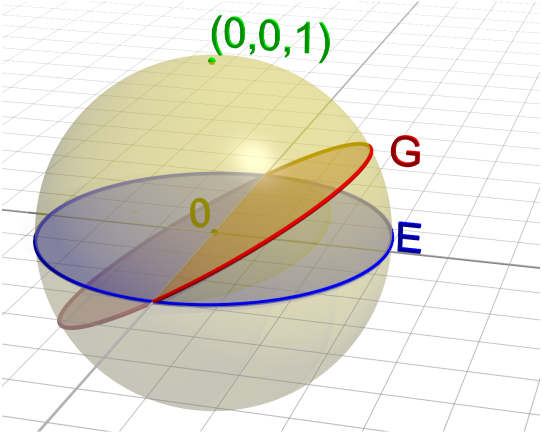

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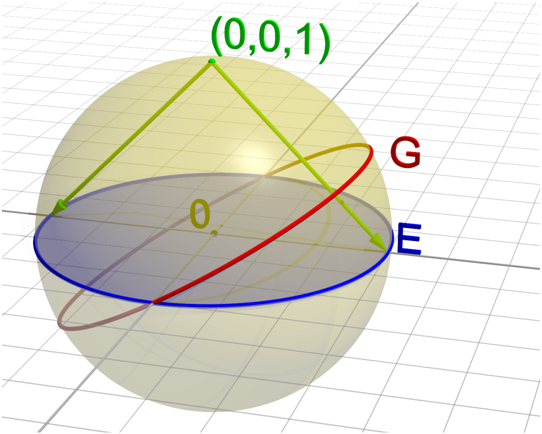

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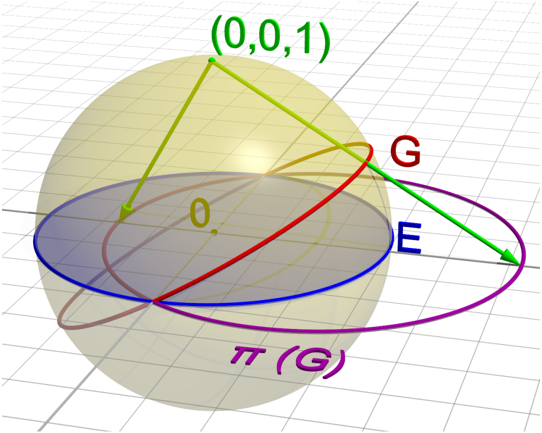

# 2) Scaling $\sigma$ in the plane

- Uniform scaling with scale factor $s = \sqrt{\frac{1+z}{1-z}}$
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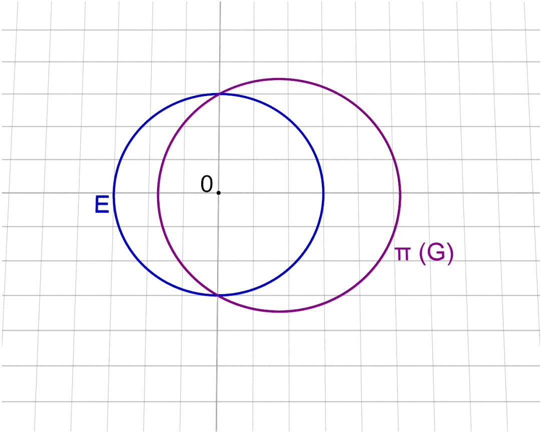

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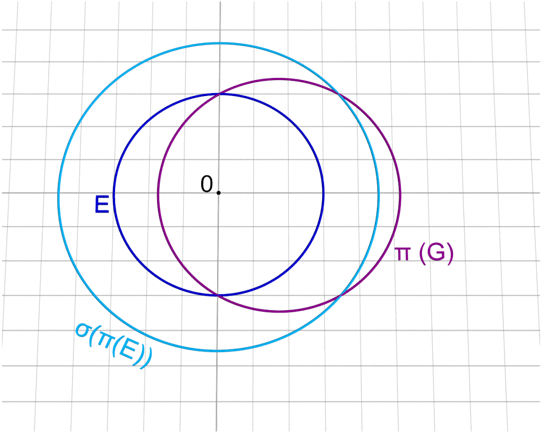

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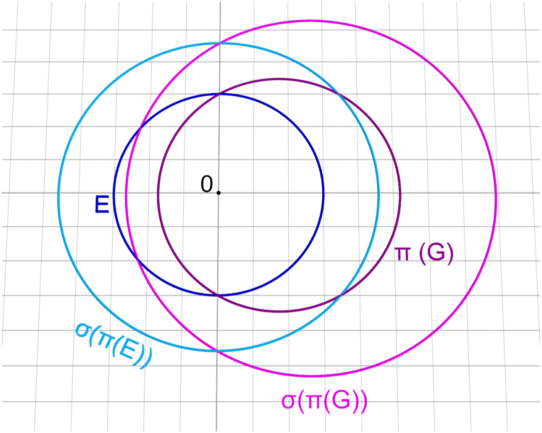
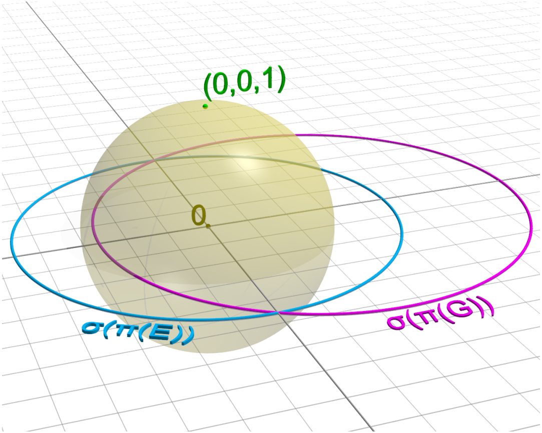


# 3) Inverse stereographic projection $\pi^{-1}$

- The circle with radius $s$ is mapped to the circle of latitude $z$; $h(E)$
- $h(G) = \pi^{-1}(\sigma(\pi(G)))$ intersects $h(E)$ in two antipodal points
- $h(G)$ is a small circle through $(0, 0, z)$



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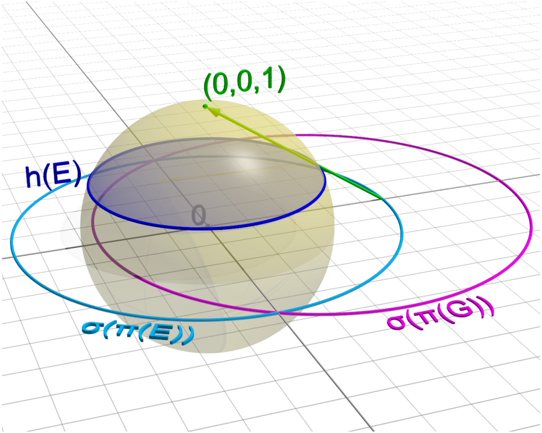

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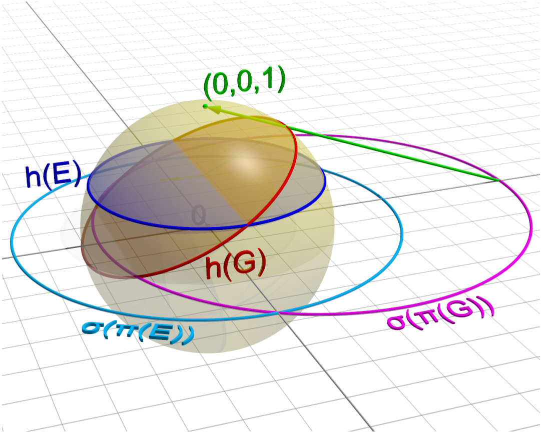

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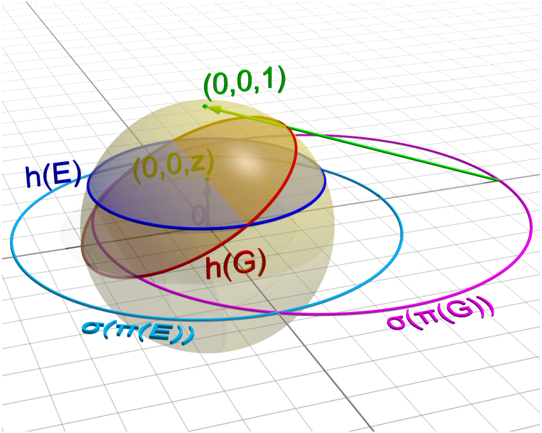
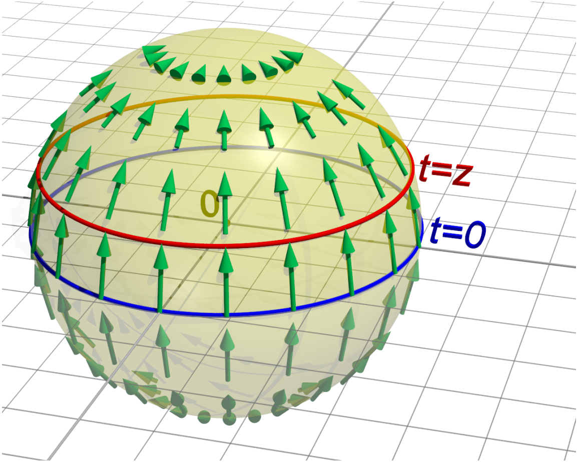


# The resulting map $h$




$$h(\boldsymbol{\eta}) = \begin{pmatrix} \frac{\sqrt{1-z^2}}{1-z\eta_3}\eta_1 \\ \frac{\sqrt{1-z^2}}{1-z\eta_3}\eta_2 \\ \frac{z+\eta_3}{1-z\eta_3} \end{pmatrix}$$

$h\colon \mathbb{S}^2 \to \mathbb{S}^2$ is conformal

## Theorem

*Let $z \in [0,1)$. The generalized Radon transform $\mathcal{U}_z$ can be represented with the operators $\mathcal{M}_z, \mathcal{F}, \mathcal{N}_z \colon L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)$ via*

$$\mathcal{U}_z = \mathcal{N}_z \mathcal{F} \mathcal{M}_z.$$

These operators are defined for $f \in C(\mathbb{S}^2)$ by

- $\mathcal{M}_z f(\boldsymbol{\xi}) = \dfrac{\sqrt{1-z^2}}{1+z\xi_3} [f \circ h](\boldsymbol{\xi})$
- $\mathcal{F}$ ... Funk–Radon transform
- $\mathcal{N}_z f(\boldsymbol{\xi}) = f\left(\dfrac{1}{\sqrt{1-z^2\xi_3^2}} \left(\xi_1, \xi_2, \sqrt{1-z^2}\xi_3\right)\right)$

# Nullspace of $\mathcal{U}_z$

## Theorem

$\mathbf{R}$ ... Reflection of the sphere about the point $(0, 0, z)$

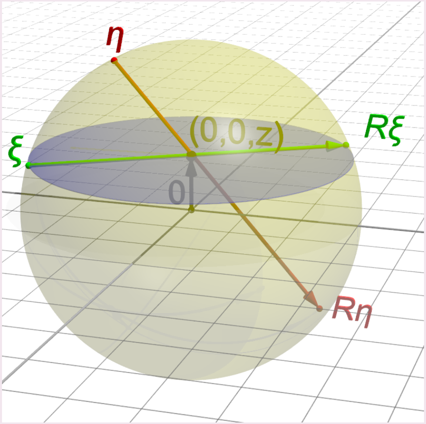
$f \in L^2(\mathbb{S}^2)$

We have

$$\mathcal{U}_z f = 0$$

if and only if for almost all $\eta \in \mathbb{S}^2$

$$f(\eta) = -f(\mathbf{R}\eta)\frac{1 - z^2}{1 + z^2 - 2z\eta_3}.$$



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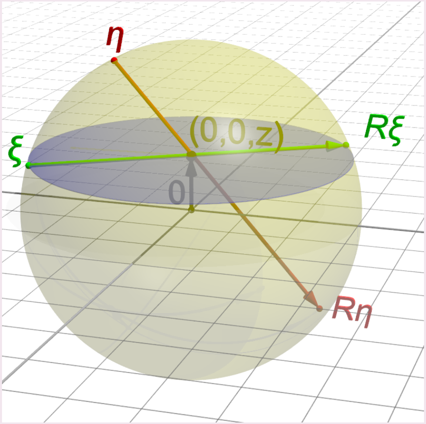

# Range of $\mathcal{U}_z$

## Theorem

The generalized Radon transform

$$\mathcal{U}_z \colon \widetilde{L}^2_{\mathrm{e}}(\mathbb{S}^2) \to H^{1/2}_{\mathrm{e}}(\mathbb{S}^2)$$

is continuous and bijective.

- $\widetilde{L}^2_{\mathrm{e}}(\mathbb{S}^2) = \left\{ f \in L^2(\mathbb{S}^2) \mid f(\boldsymbol{\eta}) = f(\mathbf{R}\boldsymbol{\eta}) \dfrac{1-z^2}{1+z^2-2z\eta_3} \right\}$
- $H^{1/2}_{\mathrm{e}}(\mathbb{S}^2)$ ... Sobolev space of smoothness $1/2$ that contains only even functions

# Inversion via Fourier expansion

- Use the factorization

$$\mathcal{U}_z^{-1} = \mathcal{M}_z^{-1} \mathcal{F}^{-1} \mathcal{N}_z^{-1}$$

- $\mathcal{M}_z^{-1}$ and $\mathcal{N}_z^{-1}$ can be computed explicitly
- For $\mathcal{F}^{-1}$: **Fourier expansion of the Funk–Radon transform** combined with the mollifier method (as regularization)

  [Louis et al., 2011] [Hielscher & Q., 2015]
- Efficient implementation of the spherical Fourier transform available

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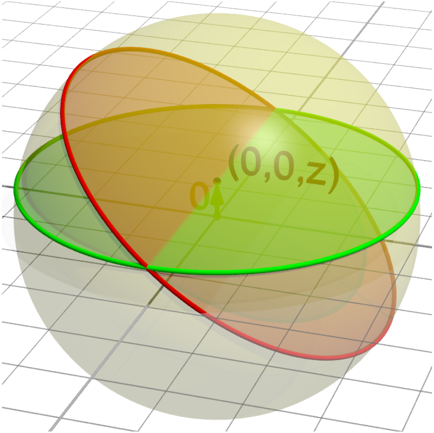
# Continuity in $z$




Circles of integration of $\mathcal{U}_z$ depend "smoothly" on $z$

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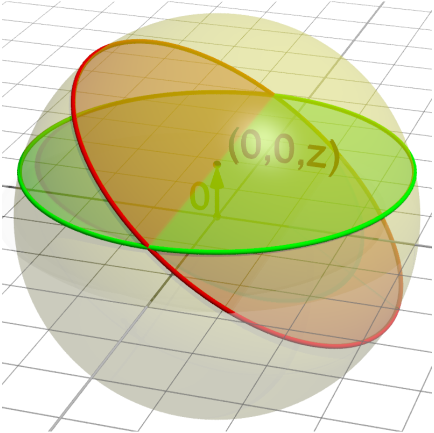


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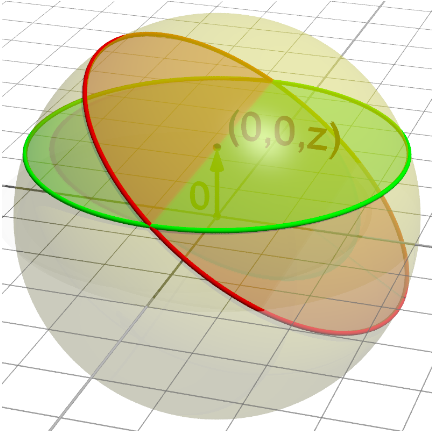


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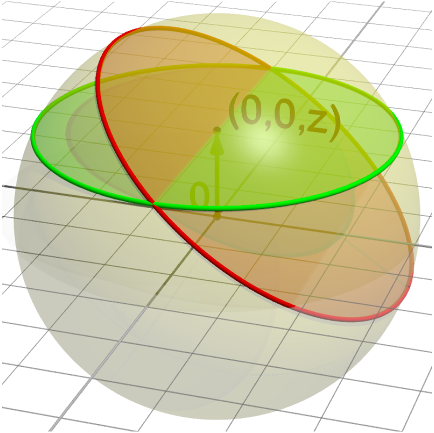


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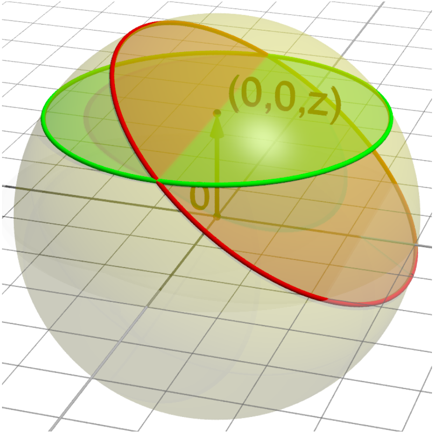


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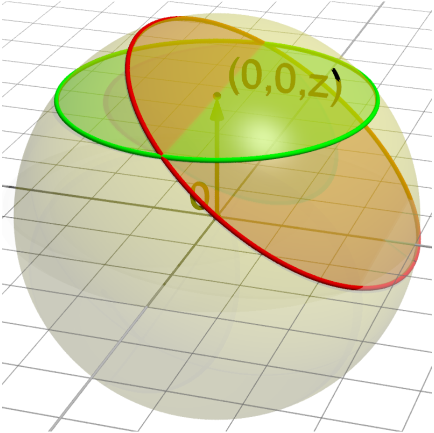


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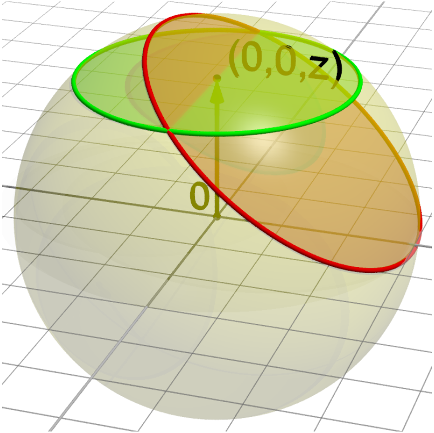


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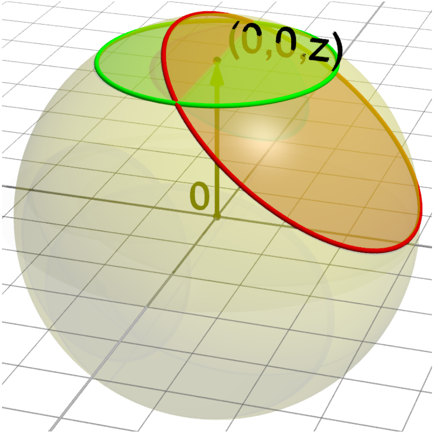


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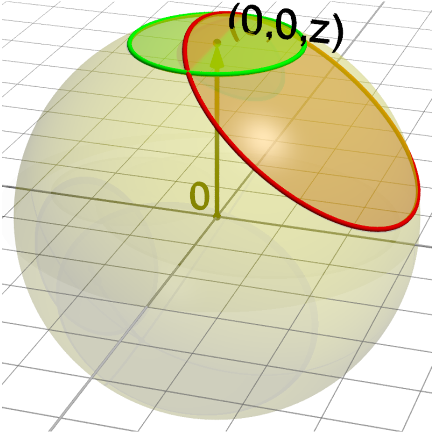


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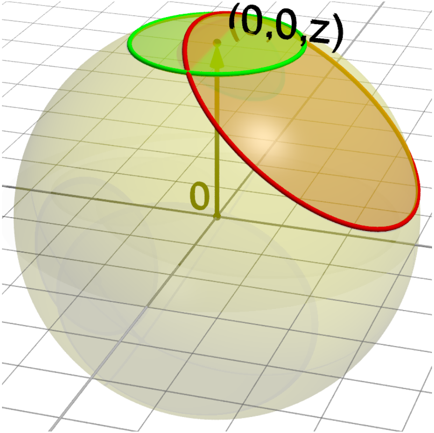


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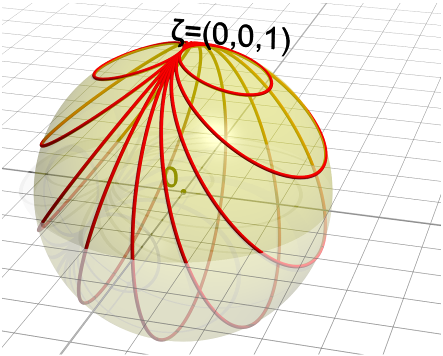
## Theorem

Let $f \in C(\mathbb{S}^2)$ and $z \in [0,1]$. Then

$$\lim_{y \to z} \|\mathcal{U}_y f - \mathcal{U}_z f\|_{L^\infty(\mathbb{S}^2)} = 0.$$

# $z = 1$: Spherical slice transform $\mathcal{U}_1$ [Abouelaz & Daher, 1993]




Circles through the north pole

## We already know

1. $\mathcal{U}_z$ is continuous with respect to $z$
2. $\mathcal{U}_z$ is injective for functions vanishing for $\eta_3 > z$ for all $z < 1$

## Injectivity of $\mathcal{U}_1$

The spherical slice transform $\mathcal{U}_1$ is injective for Lipschitz functions vanishing around the north pole.

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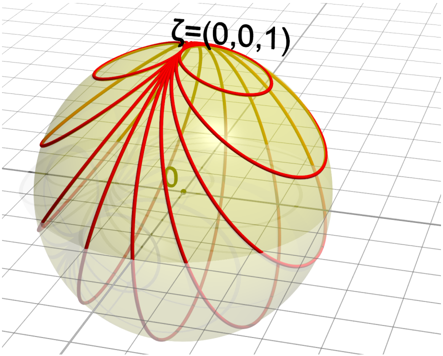


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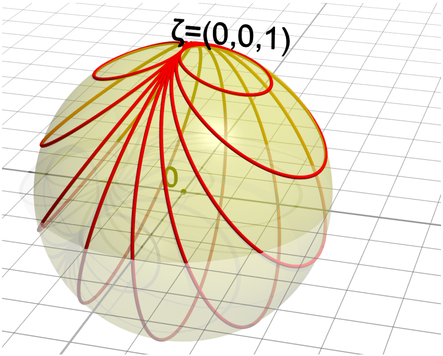


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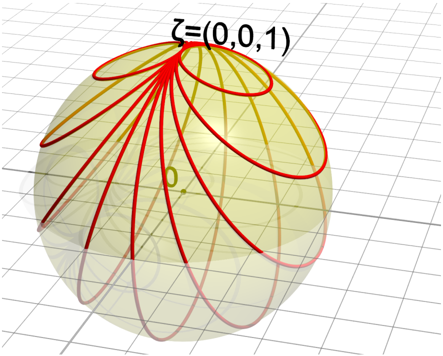


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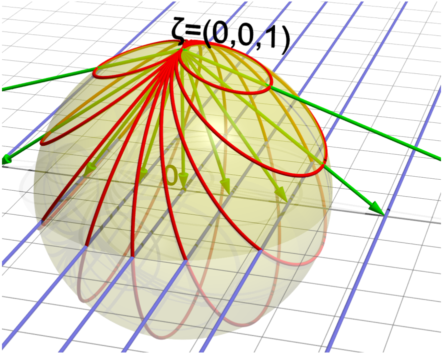


Circles through the north pole

- Stereographic projection turns circles into lines in the plane
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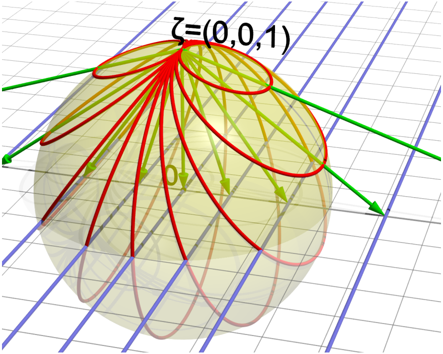


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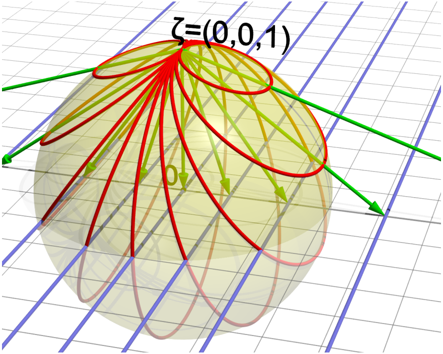


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# Selected bibliography

P. Funk.
Über Flächen mit lauter geschlossenen geodätischen Linien.
*Math. Ann.*, 74(2): 278 – 300, 1913.

Y. Salman.
An inversion formula for the spherical transform in $S^2$ for a special family of circles of integration.
*Anal. Math. Phys.*, 6(1): 43 – 58, 2016.

M. Quellmalz.
A generalization of the Funk–Radon transform to circles passing through a fixed point.
Preprint 2015-17, Faculty of Mathematics, TU Chemnitz, 2015.

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