



A generalization of the Funk–Radon transform

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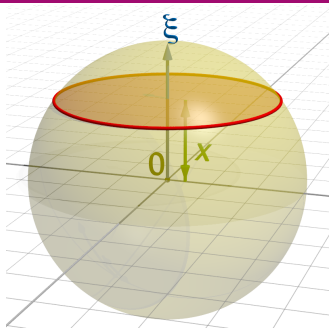
Corollaries of the factorization

Continuity in z

- ▶ Sphere $\mathbb{S}^2 = \{\xi \in \mathbb{R}^3 : \|\xi\| = 1\}$
- ▶ Function $f: \mathbb{S}^2 \rightarrow \mathbb{C}$
- ▶ Circles on the sphere are intersections of the sphere with planes:

$$\{\eta \in \mathbb{S}^2 : \langle \xi, \eta \rangle = x\},$$

$$\xi \in \mathbb{S}^2, x \in [-1, 1]$$



Spherical mean operator

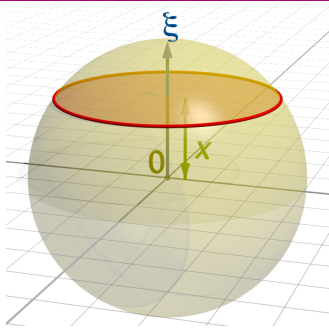
$$\mathcal{S}: C(\mathbb{S}^2) \rightarrow C(\mathbb{S}^2 \times [-1, 1]),$$

$$\mathcal{S}f(\xi, x) = \int_{\langle \xi, \eta \rangle = x} f(\eta) d\lambda(\eta)$$

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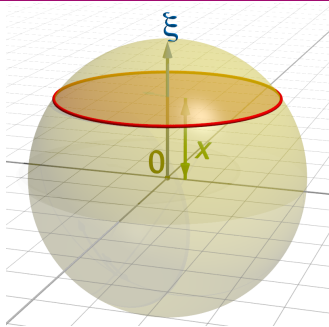
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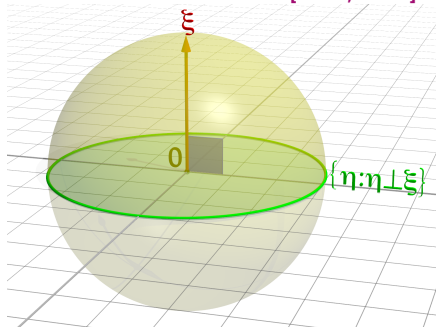
Funk–Radon transform

- ▶ Restriction to all great circles
- ▶ **Funk–Radon transform** (a.k.a. Funk transform or spherical Radon transform)

$$\mathcal{F}: C(\mathbb{S}^2) \rightarrow C(\mathbb{S}^2),$$

$$\mathcal{F}f(\xi) = \int_{\langle \xi, \eta \rangle = 0} f(\eta) \, d\lambda(\eta)$$

[Funk, 1913]



Questions

1. Injectivity
(Knowing the mean values of f on great circles, can we reconstruct f ?)
2. Range

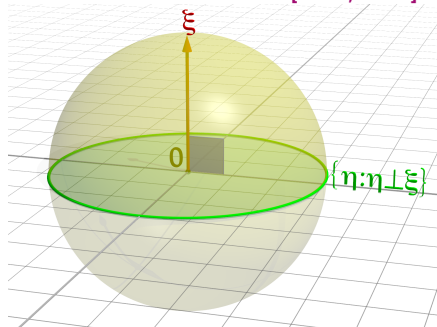
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Fourier series

Write $f \in L^2(\mathbb{S}^2)$ in terms of the **spherical harmonics** Y_n^k of degree n

$$f = \sum_{n=0}^{\infty} \sum_{k=-n}^n \hat{f}(n, k) Y_n^k.$$

Eigenvalue decomposition

[Minkowski, 1904]

The Funk–Radon transform is given by

$$\mathcal{F}Y_n^k(\xi) = P_n(0)Y_n^k(\xi), \quad P_n(0) = \begin{cases} \frac{(n-1)!!}{n!!}, & n \text{ even,} \\ 0, & n \text{ odd.} \end{cases}$$

P_n – Legendre polynomial of degree n

\implies The Funk–Radon transform is injective for even functions

$$f(\xi) = f(-\xi).$$

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Sobolev spaces

For $s \geq 0$, the **Sobolev space** $H^s(\mathbb{S}^2)$ is the completion of the space of polynomials $f : \mathbb{S}^2 \rightarrow \mathbb{C}$ with the norm

$$\|f\|_s^2 = \sum_{n=0}^{\infty} \sum_{k=-n}^n \left| \hat{f}(n, k) \right|^2 \left(n + \frac{1}{2} \right)^{2s}.$$

Theorem

[Strichartz, 1981]

The Funk–Radon transform is bijective

$$\mathcal{F} : L_{\text{even}}^2(\mathbb{S}^2) \rightarrow H_{\text{even}}^{\frac{1}{2}}(\mathbb{S}^2).$$

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Q: What else can we do?

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A: We consider other circles than the great circles.

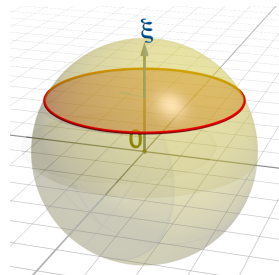
Circles with fixed radius

- ▶ For fixed $x_0 \in [-1, 1]$, compute

$$\mathcal{S}_{x_0} f(\xi) = \int_{\langle \xi, \eta \rangle = x_0} f(\eta) \, d\eta$$

- ▶ Eigenvalue decomposition yields

$$\mathcal{S}_{x_0} Y_n^k = P_n(x_0) Y_n^k$$



“Freak theorem”

[Schneider, 1969]

The set of values x_0 for which \mathcal{S}_{x_0} is **not** injective is countable and dense in $[-1, 1]$.

Explicit algorithm to determine if \mathcal{S}_{x_0} is injective for given x_0

[Rubin, 2000]

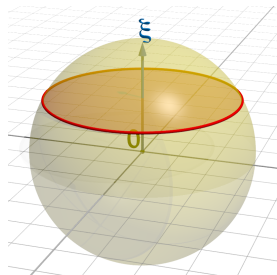
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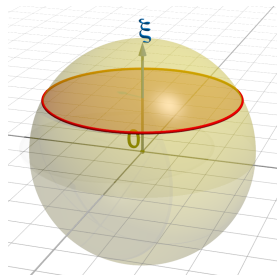
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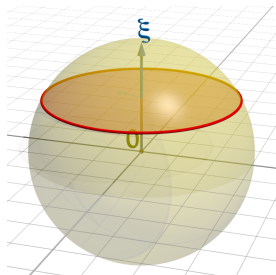
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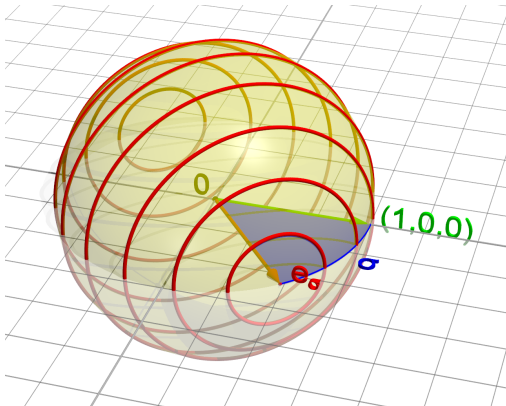
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Vertical slices

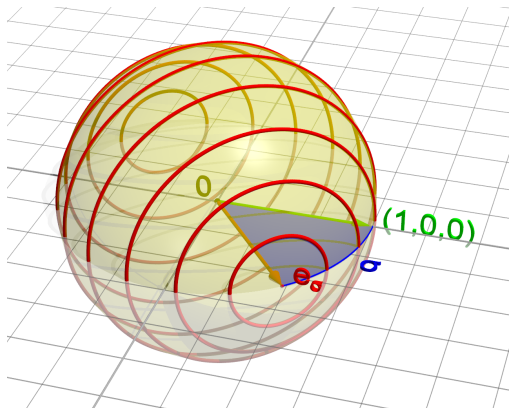
$$\mathcal{S}(\xi, x) = \int_{\langle \xi, \eta \rangle = x} f(\eta) \, ds(\eta), \quad \xi_3 = 0$$



- ▶ **Circles perpendicular to the equator**
- ▶ Injective for symmetric functions $f(\xi_1, \xi_2, \xi_3) = f(\xi_1, \xi_2, -\xi_3)$
- ▶ Proof1: Orthogonal projection onto equatorial plane
[Gindikin, Reeds & Shepp, 1994]
- ▶ Proof2: Spherical harmonics
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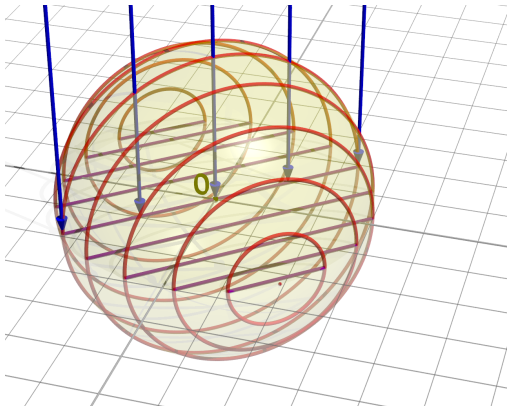
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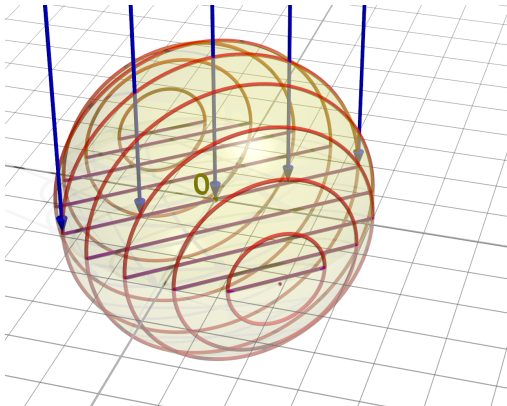
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[Salman, 2015]

Replace 0 by an arbitrary point

$$(0, 0, z), \quad 0 \leq z < 1$$

inside the sphere.

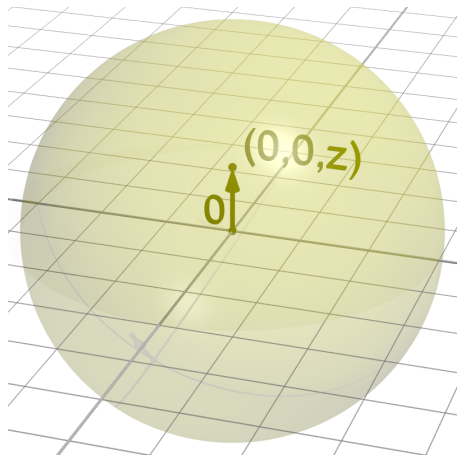
Circle through $(0, 0, z)$ is

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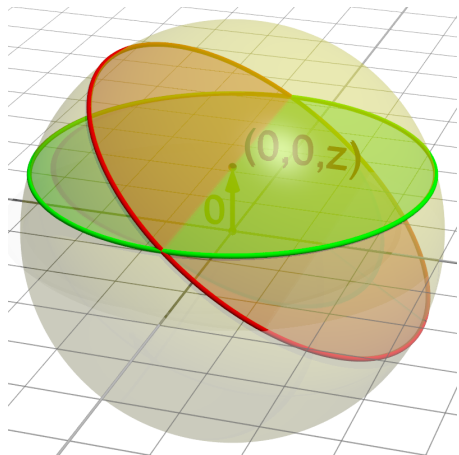
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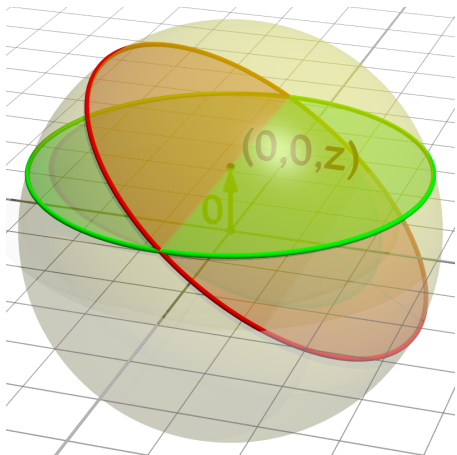
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From great circles to small circles

Definition

Define the map

$$h: \mathbb{S}^2 \rightarrow \mathbb{S}^2, \quad h = \pi^{-1} \circ \sigma \circ \pi$$

consisting of

1. Stereographic projection $\pi: \mathbb{S}^2 \rightarrow \mathbb{R}^2$
2. Scaling in the plane $\sigma: \mathbb{R}^2 \rightarrow \mathbb{R}^2, \mathbf{x} \mapsto \sqrt{\frac{1+z}{1-z}} \mathbf{x}$
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h maps great circles to small circles through $(0, 0, z)$.

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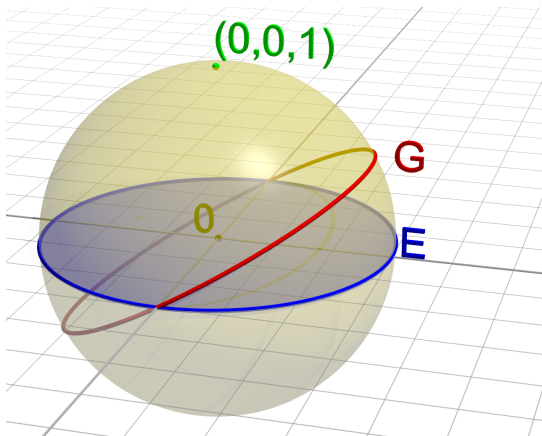
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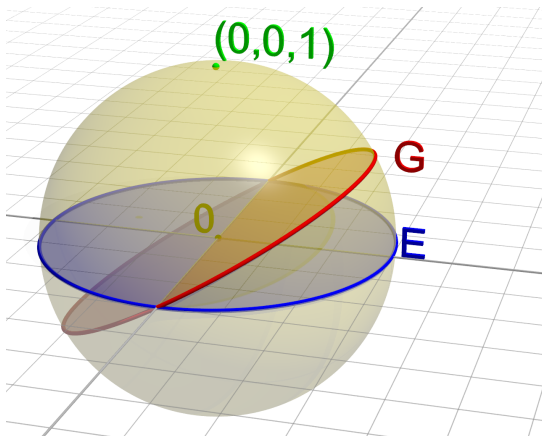
1) Stereographic projection π

- ▶ G ... Great circle of \mathbb{S}^2
- ▶ E ... Equator of \mathbb{S}^2
- ▶ G intersects E in two antipodal points (or is identical to E)
- ▶ $\pi(E) = E$
- ▶ $\pi(G)$ is a circle or line in \mathbb{R}^2 that intersects $\pi(E)$ in two antipodal points



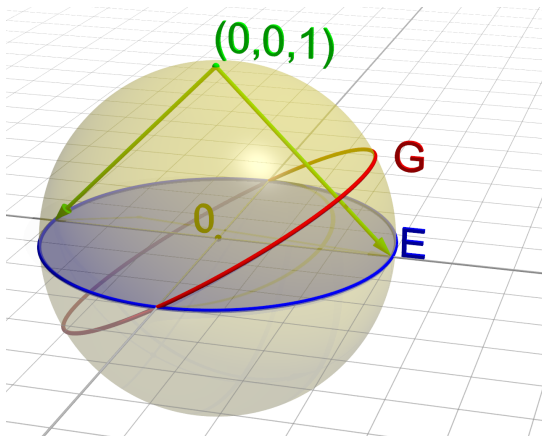
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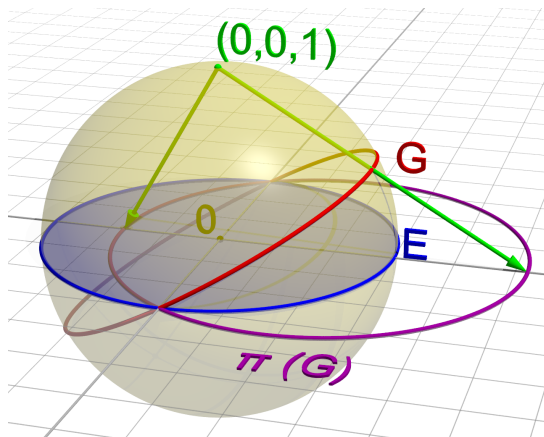
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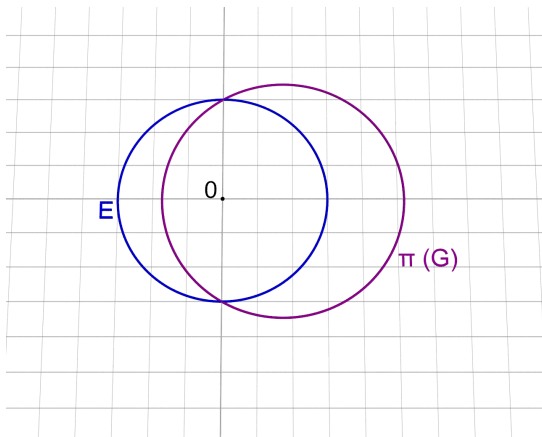
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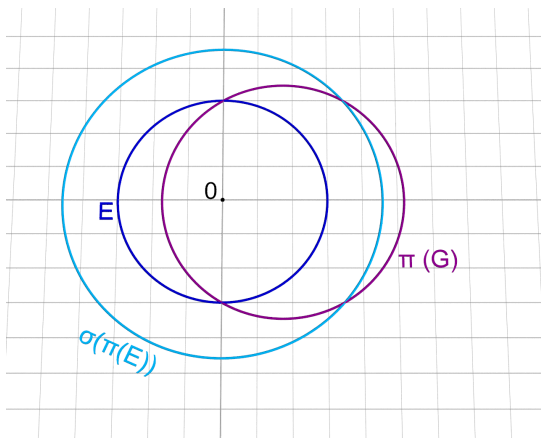
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- ▶ Unit circle E is mapped to the circle $\sigma(\pi(E))$ with radius s
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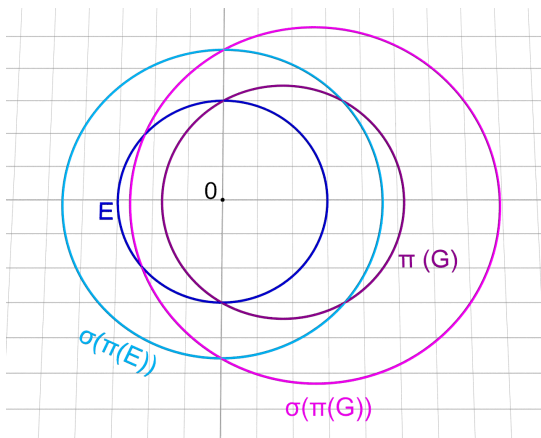
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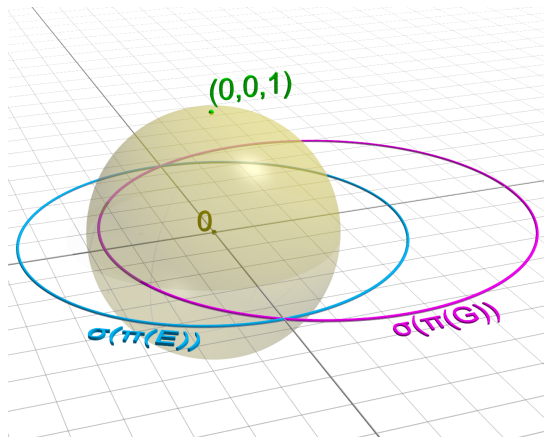
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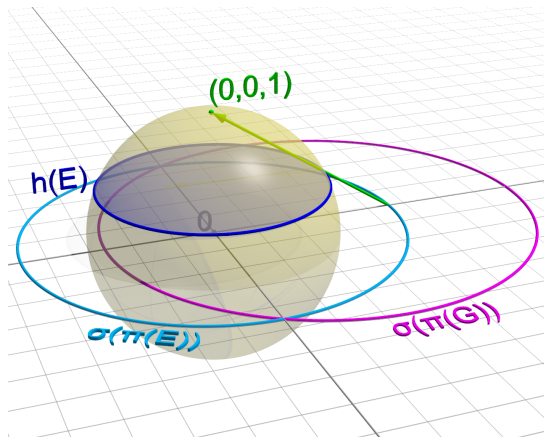
3) Inverse stereographic projection π^{-1}

- ▶ The circle with radius s is mapped to the circle of latitude z ; $h(E)$
- ▶ $h(G) = \pi^{-1}(\sigma(\pi(G)))$ intersects $h(E)$ in two antipodal points
- ▶ $h(G)$ is a small circle through $(0, 0, z)$



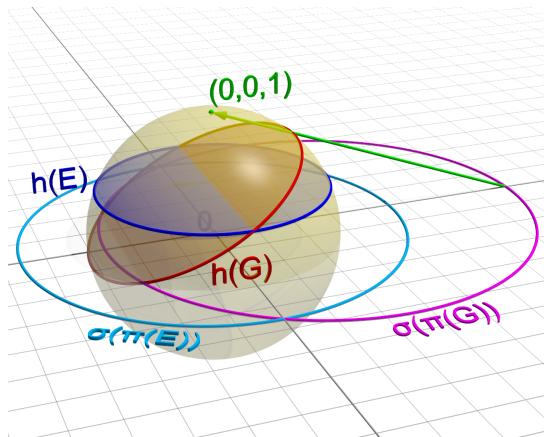
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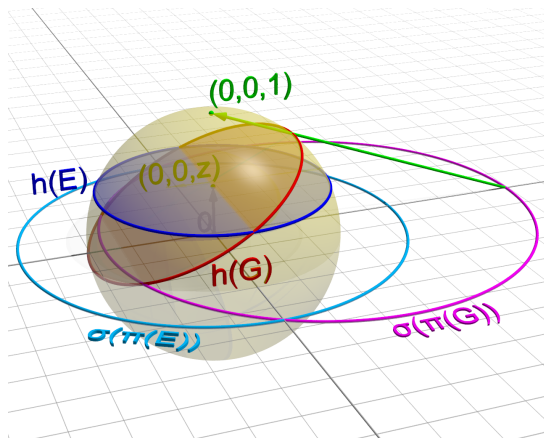
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- ▶ The circle with radius s is mapped to the circle of latitude z ; $h(E)$
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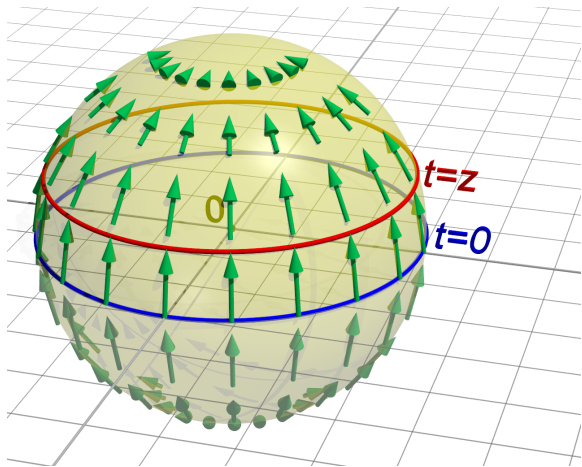


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The resulting map h



$$h(\eta) = \begin{pmatrix} \frac{\sqrt{1-z^2}}{1-z\eta_3} \eta_1 \\ \frac{\sqrt{1-z^2}}{1-z\eta_3} \eta_2 \\ \frac{z+\eta_3}{1-z\eta_3} \end{pmatrix}$$

$h: \mathbb{S}^2 \rightarrow \mathbb{S}^2$ is conformal

Theorem

Let $z \in [0, 1)$. The generalized Radon transform \mathcal{U} can be represented with the operators $\mathcal{M}, \mathcal{F}, \mathcal{N}: L^2(\mathbb{S}^2) \rightarrow L^2(\mathbb{S}^2)$ via

$$\mathcal{U} = \mathcal{N}\mathcal{F}\mathcal{M}.$$

These operators are defined for $f \in C(\mathbb{S}^2)$ by

- ▶ $\mathcal{M}f(\boldsymbol{\xi}) = \frac{\sqrt{1-z^2}}{1+z\xi_3} [f \circ h](\boldsymbol{\xi})$
- ▶ \mathcal{F} ... Funk–Radon transform
- ▶ $\mathcal{N}f(\boldsymbol{\xi}) = f\left(\frac{1}{\sqrt{1-z^2\xi_3^2}}\left(\xi_1, \xi_2, \sqrt{1-z^2}\xi_3\right)\right)$

Nullspace of \mathcal{U}

Theorem

$\mathbf{R} \dots$ Reflection of the sphere about the point $(0, 0, z)$

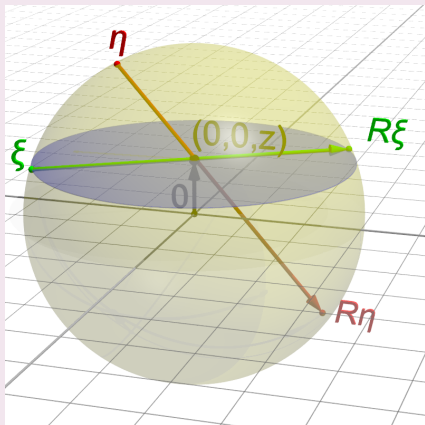
$$f \in L^2(\mathbb{S}^2)$$

We have

$$\mathcal{U}f = 0$$

if and only if for almost all $\eta \in \mathbb{S}^2$

$$f(\eta) = -f(\mathbf{R}\eta) \frac{1 - z^2}{1 + z^2 - 2z\eta_3}.$$



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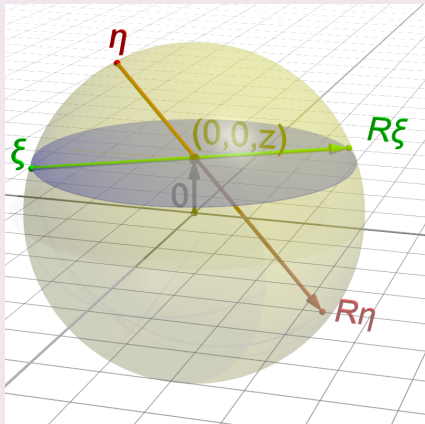
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Range of \mathcal{U}

Theorem

The generalized Radon transform

$$\mathcal{U}: \tilde{L}_e^2(\mathbb{S}^2) \rightarrow H_e^{1/2}(\mathbb{S}^2)$$

is continuous and bijective.

- ▶ $\tilde{L}_e^2(\mathbb{S}^2) = \left\{ f \in L^2(\mathbb{S}^2) \mid f(\boldsymbol{\eta}) = f(\mathbf{R}\boldsymbol{\eta}) \frac{1 - z^2}{1 + z^2 - 2z\eta_3} \right\}$
- ▶ $H_e^{1/2}(\mathbb{S}^2)$... Sobolev space of smoothness 1/2 that contains only even functions

Inversion formula

Based on the inversion of the Funk–Radon transform by [Helgason, 1980]

$$f(\eta) = \frac{1}{2\pi} \frac{d}{du} \int_0^u \int_{\langle \xi, \eta \rangle^2 = 1 - w^2} \mathcal{F}f(\xi) \, d\ell(\xi) \frac{1}{\sqrt{u^2 - w^2}} \, dw \Big|_{u=1}$$

Theorem

Let $z \in [0, 1)$ and $f \in \tilde{L}_e^2(\mathbb{S}^2)$. For $\eta \in \mathbb{S}^2$,

$$f(\eta) = \frac{1 - z^2}{2\pi(1 - zv)} \frac{d}{du} \int_0^u \int_{\mathcal{S}_z(\eta, w)} \mathcal{U}f \left(\frac{(\sqrt{1 - z^2}(\xi_1, \xi_2), \xi_3)}{\sqrt{1 - z^2 + z^2\xi_3^2}} \right) \, ds(\xi) \frac{dw}{\sqrt{u^2 - w^2}} \Big|_{u=1}$$

where ds is the arc-length on the circle

$$\mathcal{S}_z(\eta, w) = \left\{ \xi \in \mathbb{S}^2 \mid \left\langle \xi, \left(\sqrt{1 - z^2}(\eta_1, \eta_2), \eta_3 - z \right) \right\rangle = (1 - z\eta_3)\sqrt{1 - w^2} \right\}.$$

Inversion via Fourier expansion

- ▶ The last inversion formula is numerically unstable

- ▶ Use the factorization

$$U = \mathcal{N}\mathcal{F}\mathcal{M}$$

- ▶ \mathcal{M}^{-1} and \mathcal{N}^{-1} can be computed explicitly

- ▶ For \mathcal{F}^{-1} : **Fourier expansion of the Funk–Radon transform** combined with the mollifier method (as regularization)

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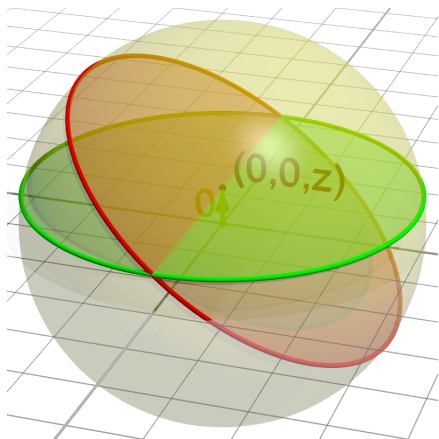
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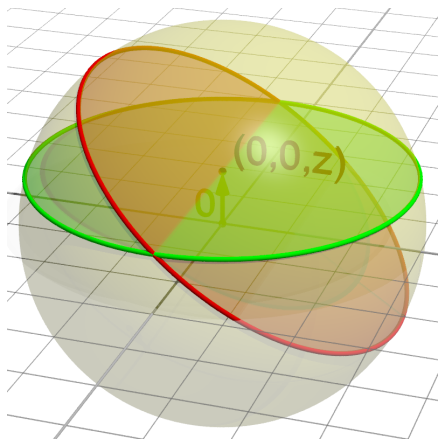
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Continuity in z



Circles of integration of \mathcal{U}_z depend
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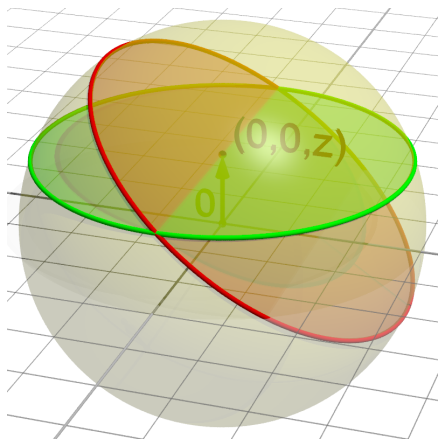
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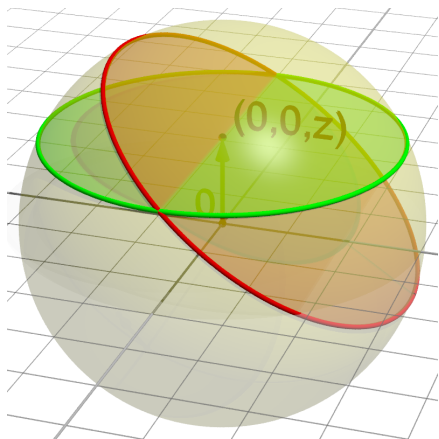
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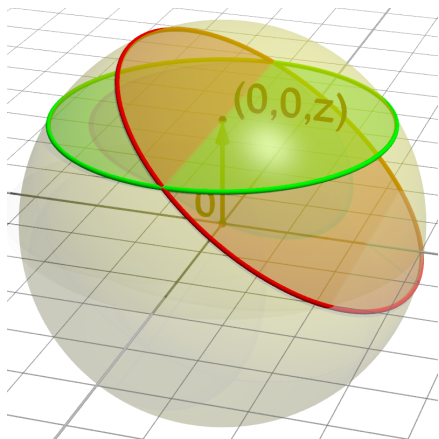
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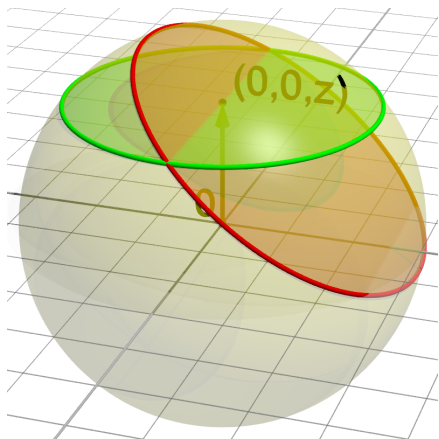
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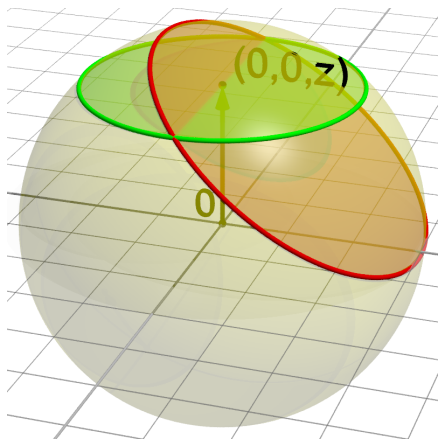
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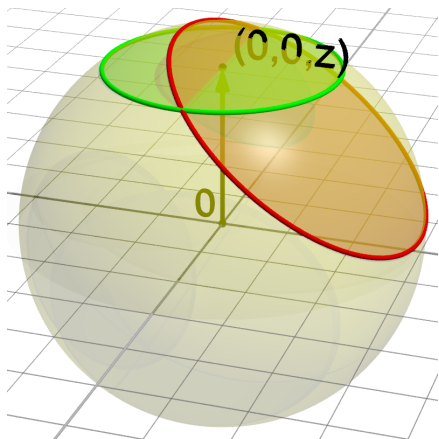
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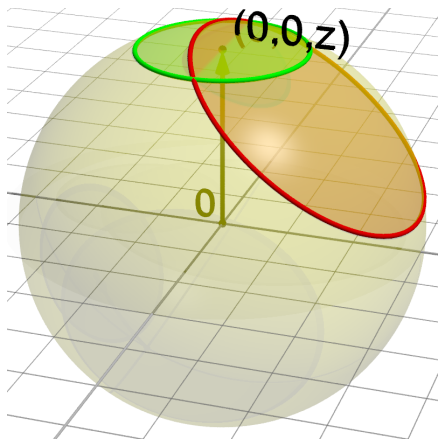
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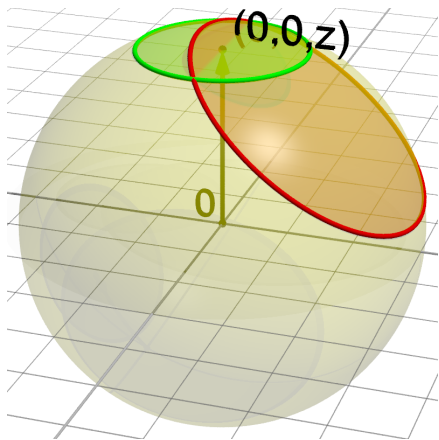
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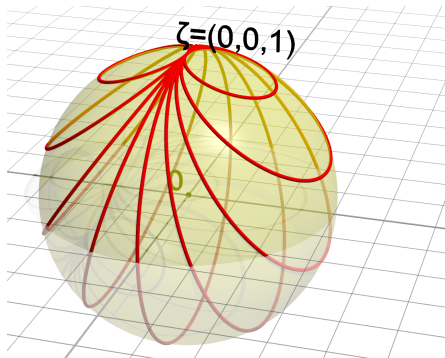
Theorem

Let $f \in C(\mathbb{S}^2)$ and $z \in [0, 1]$. Then

$$\lim_{y \rightarrow z} \|\mathcal{U}_y f - \mathcal{U}_z f\|_{L^\infty(\mathbb{S}^2)} = 0.$$

Spherical slice transform \mathcal{U}_1

[Abouelaz & Daher, 1993]



Circles through the north pole

We already know

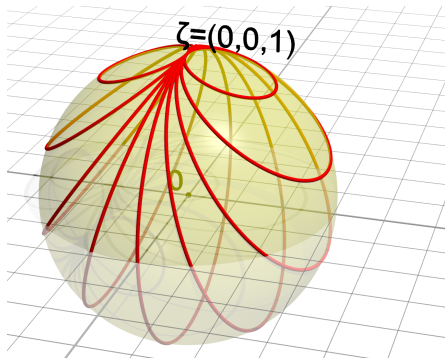
1. \mathcal{U}_z is continuous with respect to z
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Injectivity of \mathcal{U}_1

The spherical slice transform \mathcal{U}_1 is injective for Lipschitz functions vanishing around the north pole.

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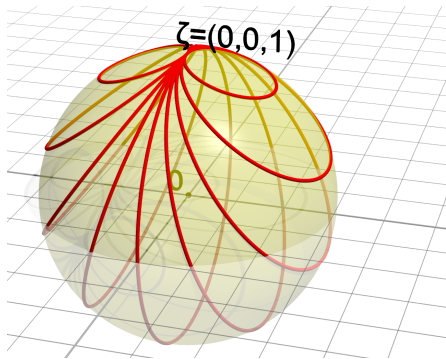
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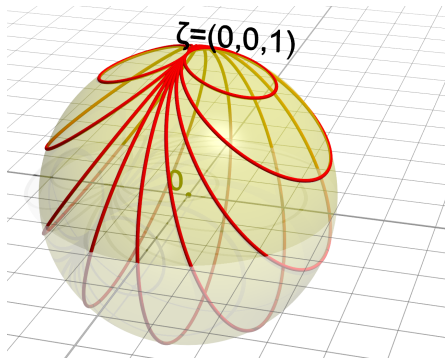
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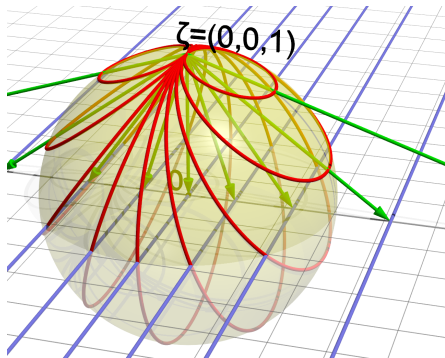
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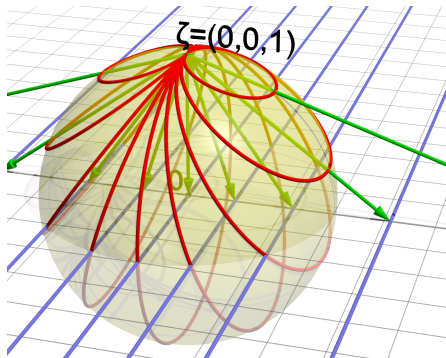


Circles through the north pole

- ▶ Stereographic projection turns circles into lines in the plane
↗ Radon transform in \mathbb{R}^2
- ▶ \mathcal{U}_1 is injective if f is differentiable and vanishes at $(0, 0, 1)$ [Helgason, 1999]
- ▶ \mathcal{U}_1 is injective for all bounded functions [Rubin, 2015]

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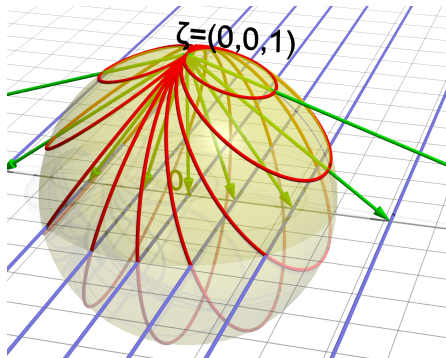


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