# An SVD in Spherical Surface Wave Tomography

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(joint work with Ralf Hielscher and Daniel Potts)

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Geomathematics Meets Medical Imaging (GEMMI)
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# Content

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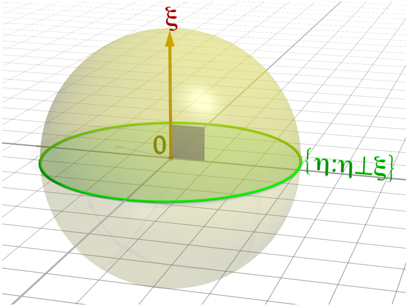
# Funk–Radon transform

- Sphere $\mathbb{S}^2 = \{\boldsymbol{\xi} \in \mathbb{R}^3 : \|\boldsymbol{\xi}\| = 1\}$
- Function $f\colon \mathbb{S}^2 \to \mathbb{C}$
- **Funk–Radon transform** (a.k.a. Funk transform or spherical Radon transform)

$$\mathcal{F}\colon C(\mathbb{S}^2) \to C(\mathbb{S}^2),$$

$$\mathcal{F}f(\boldsymbol{\xi}) = \int_{\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = 0} f(\boldsymbol{\eta}) \, \mathrm{d}\lambda(\boldsymbol{\eta})$$




**Theorem** [Funk 1911]

Any even function $f$ can be reconstructed from $\mathcal{F}f$.

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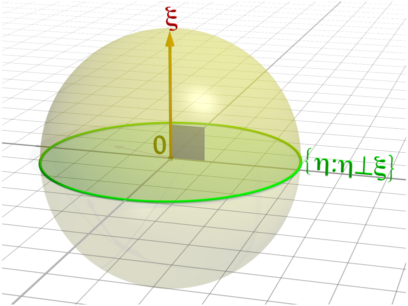


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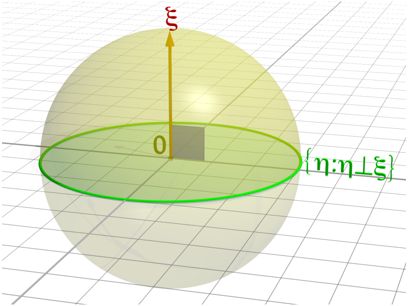


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# Spherical surface wave tomography

- Seismic waves propagate along the surface
- Speed of propagation depends on the position on $\mathbb{S}^2$

## Method

- Measure the traveltimes of surface waves between many pairs of epicenter and detector
- Reconstruct the speed of propagation

## Assumption

A wave propagates along the arc of a great circle.

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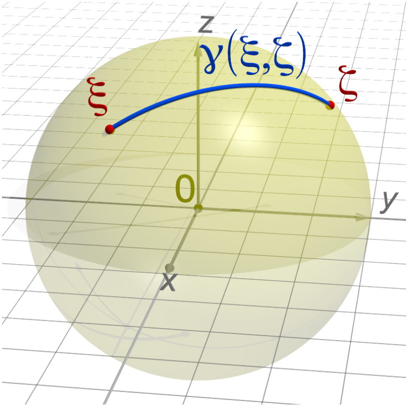
# The arc transform

- Function $f : \mathbb{S}^2 \to \mathbb{R}$
    - Surface waves: $f = \frac{1}{c}$ ($c$ ... speed of sound)
- $\xi,\ \zeta \in \mathbb{S}^2$ not antipodal
- $\gamma(\xi,\ \zeta)$ geodesic arc

## Definition

$$t(\xi, \zeta) = \int_{\gamma(\xi,\zeta)} f \, \mathrm{d}\gamma$$



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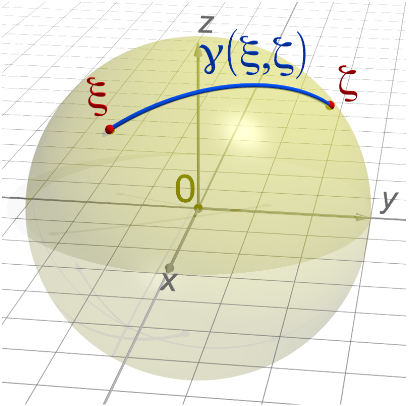


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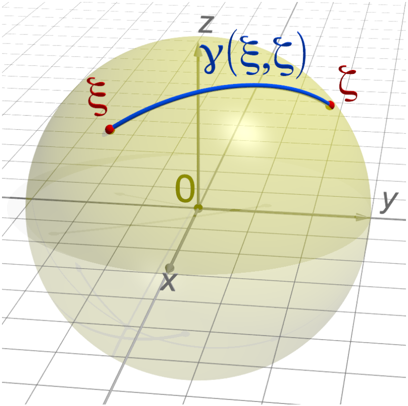

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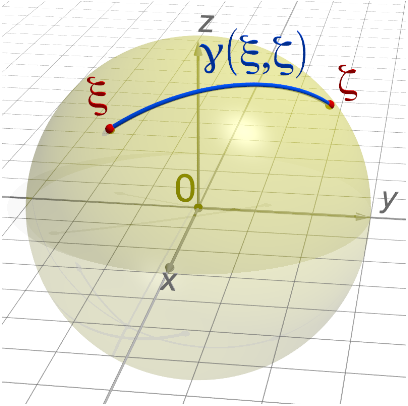


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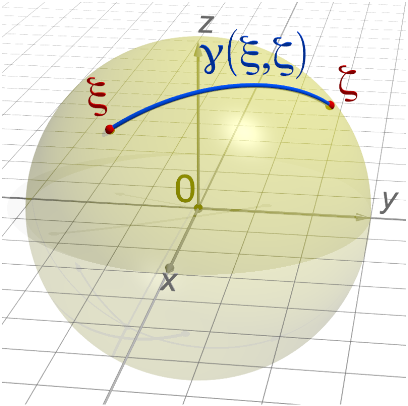

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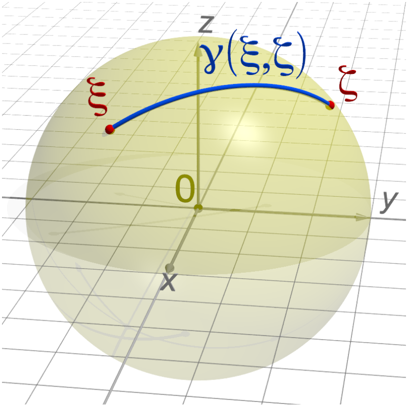

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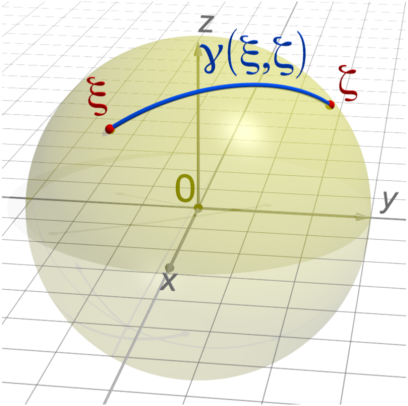


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**We choose a different parameterization**

# The arc transform: alternative parameterization

- $\psi = \arccos(\boldsymbol{\xi}^\top \boldsymbol{\zeta})$ ... length of $\gamma$
- $Q \in \mathrm{SO}(3)$ such that
  - $Q\boldsymbol{\xi} = \boldsymbol{e}_{-\psi/2}$ and
  - $Q\boldsymbol{\zeta} = \boldsymbol{e}_{\psi/2}$,

  where $\boldsymbol{e}_\psi = (\sin\psi, \cos\psi, 0)$

**Definition**

$$\mathcal{A}\colon C(\mathbb{S}^2) \to C(\mathrm{SO}(3) \times [0,\pi]),$$

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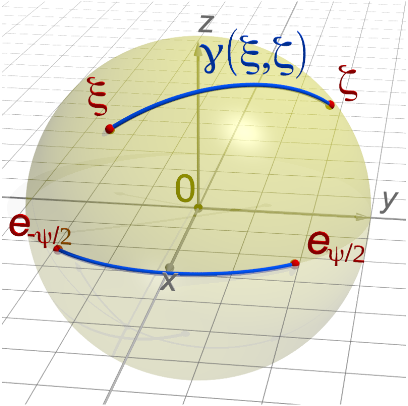

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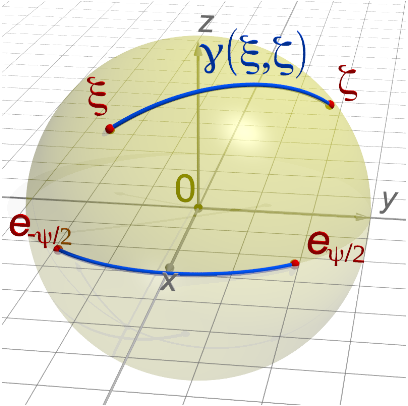

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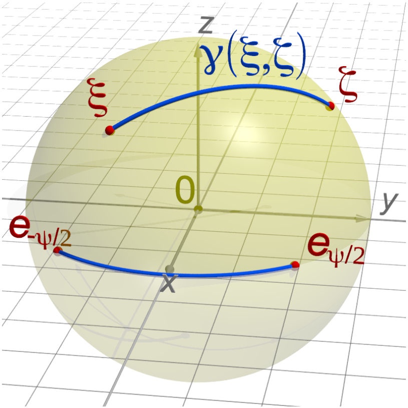


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# On the sphere $\mathbb{S}^2$

- Polar coordinates

$$\boldsymbol{\xi}(\varphi, t) = \sqrt{1 - t^2}\, \boldsymbol{e}_\varphi + t\boldsymbol{e}_3$$

- Orthonormal basis on $L^2(\mathbb{S}^2)$: **spherical harmonics** of degree $n$

$$Y_n^k(\varphi, t) = \sqrt{\frac{2n+1}{4\pi} \frac{(n-k)!}{(n+k)!}}\, P_n^k(t)\, \mathrm{e}^{\mathrm{i}k\varphi}$$

- $P_n^k$ ... associated Legendre function

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# On the rotation group $\mathrm{SO}(3)$

- Rotation group

$$\mathrm{SO}(3) = \{Q \in \mathbb{R}^{3\times 3} : Q^{-1} = Q^*,\ \det(Q) = 1\}$$

- Orthogonal basis on $L^2(\mathrm{SO}(3))$: **rotational harmonics** (Wigner D-functions)

$$D_n^{j,k}(Q) = \int_{\mathbb{S}^2} Y_n^k(Q^{-1}\xi)\,\overline{Y_n^j(\xi)}\,\mathrm{d}\xi$$

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## Theorem

[Dahlen & Tromp 1998]

Let $n \in \mathbb{N}$ and $k \in \{-n, \dots, n\}$. Then

$$\mathcal{A}Y_n^k(Q,\psi) = \sum_{j=-n}^{n} \widetilde{P}_n^j(0)\, D_n^{j,k}(Q)\, s_j(\psi),$$

where

$$s_j(\psi) = \begin{cases} \psi, & j = 0 \\ \frac{2\sin(j\psi/2)}{j}, & j \neq 0 \end{cases}$$

and

$$\widetilde{P}_n^j(0) = \begin{cases} (-1)^{\frac{n+j}{2}} \sqrt{\frac{2n+1}{4\pi} \frac{(n-j-1)!!(n+j-1)!!}{(n-j)!!(n+j)!!}}, & n+j \text{ even} \\ 0, & n+j \text{ odd.} \end{cases}$$

## Singular value decomposition

[Hielscher, Potts, Q. 2017]

The operator $\mathcal{A}\colon L^2(\mathbb{S}^2) \to L^2(\mathrm{SO}(3) \times [0,\pi])$ is compact with the singular value decomposition

$$\{(Y_n^k,\ E_n^k,\ \sigma_n) : n \in \mathbb{N},\ k \in \{-n,\dots,n\}\},$$

with singular values

$$\sigma_n = \sqrt{\frac{32\pi^3}{2n+1}} \sqrt{\frac{\pi^2}{3} \left| \widetilde{P}_n^0(0) \right|^2 + \sum_{j=1}^{n} \frac{1}{j^2} \left| \widetilde{P}_n^j(0) \right|^2} \in \mathcal{O}(\sqrt{n})$$

and the orthonormal functions in $L^2(\mathrm{SO}(3) \times [0,\pi])$

$$E_k^n = \sigma_n^{-1} \sum_{j=-n}^{n} \widetilde{P}_n^j(0)\, D_n^{j,k}(Q)\, s_j(\psi).$$

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# Arcs from the north pole

- Fix one endpoint of the arcs as the north pole $e^3$:

$$\mathcal{B}f(\boldsymbol{\xi}(\varphi,\vartheta)) = \int_{\gamma(\boldsymbol{e}^3,\boldsymbol{\xi}(\varphi,\vartheta))} f \,\mathrm{d}\gamma$$

- If $f$ is differentiable, it can be recovered from $\mathcal{B}f$ by

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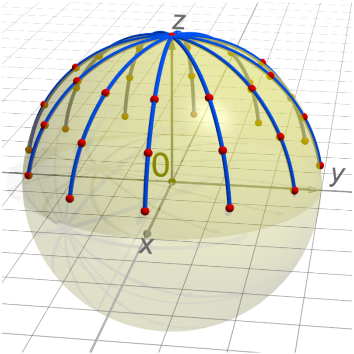

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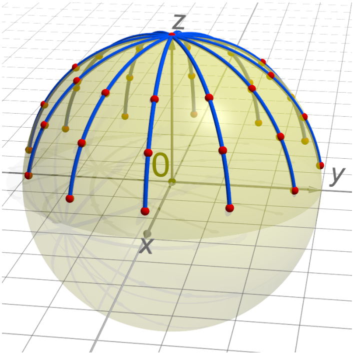


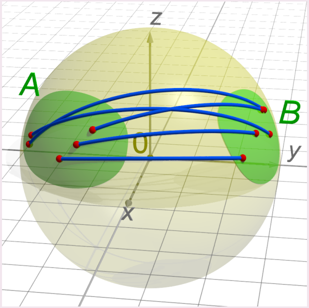
# Arcs between two sets

## More general Theorem [Amirbekyan 2007]

Let $S$ be an open subset of $\mathbb{S}^2$ and $A, B \subset S$ nonempty sets with $\overline{A \cup B} = \overline{S}$. If $f \in C(\mathbb{S}^2)$ and

$$\int_{\gamma(\boldsymbol{\xi},\boldsymbol{\zeta})} f \,\mathrm{d}\gamma = 0 \qquad \forall \boldsymbol{\xi} \in A,\ \boldsymbol{\zeta} \in B,$$

then $f \equiv 0$ on $S$.



## Theorem

[Hielscher, Potts, Q. 2017]

Let $f \in C(\mathbb{S}^2)$ and $\Omega$ be a convex subset of $\mathbb{S}^2$ whose closure $\overline{\Omega}$ is strictly contained in a hemisphere, i.e., there exists a $\boldsymbol{\zeta} \in \mathbb{S}^2$ such that $\langle \boldsymbol{\xi}, \boldsymbol{\zeta} \rangle > 0$ for all $\boldsymbol{\xi} \in \overline{\Omega}$. If

$$\int_{\gamma(\boldsymbol{\xi},\boldsymbol{\eta})} f \,\mathrm{d}\gamma = 0 \qquad \text{for all } \boldsymbol{\xi}, \boldsymbol{\eta} \in \partial\Omega, \tag{1}$$

then $f = 0$ on $\Omega$.

## Proof

- Extend $f$ to zero outside $\Omega$
- (??) implies that the Funk–Radon transform of $f$ must vanish
- $f$ must be odd
- Because $\operatorname{supp} f \subset \bar{\Omega}$ is contained in a hemisphere, $f$ must vanish

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- Because $\operatorname{supp} f \subset \bar{\Omega}$ is contained in a hemisphere, $f$ must vanish

## Theorem [Hielscher, Potts, Q. 2017]

Let $f \in C(\mathbb{S}^2)$ and $\Omega$ be a convex subset of $\mathbb{S}^2$ whose closure $\overline{\Omega}$ is strictly contained in a hemisphere, i.e., there exists a $\zeta \in \mathbb{S}^2$ such that $\langle \boldsymbol{\xi}, \zeta \rangle > 0$ for all $\boldsymbol{\xi} \in \overline{\Omega}$. If

$$\int_{\gamma(\boldsymbol{\xi},\boldsymbol{\eta})} f \,\mathrm{d}\gamma = 0 \qquad \text{for all } \boldsymbol{\xi}, \boldsymbol{\eta} \in \partial\Omega, \tag{1}$$

then $f = 0$ on $\Omega$.

## Proof

- Extend $f$ to zero outside $\Omega$
- (??) implies that the Funk–Radon transform of $f$ must vanish
- $f$ must be odd
- Because $\operatorname{supp} f \subset \bar{\Omega}$ is contained in a hemisphere, $f$ must vanish

# Arcs with fixed length

We fix the arclength $\psi \in [0, 2\pi]$ and define

$$\mathcal{A}_\psi = \mathcal{A}(\cdot, \psi) \colon L^2(\mathbb{S}^2) \to L^2(\mathrm{SO}(3)).$$

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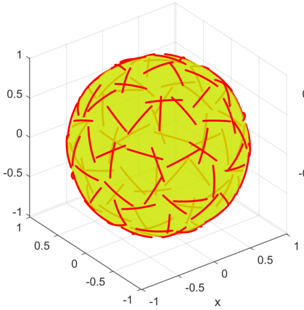
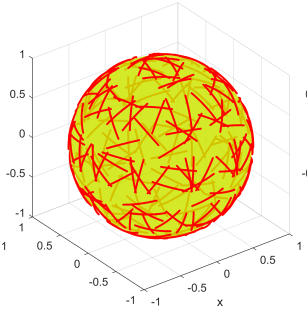
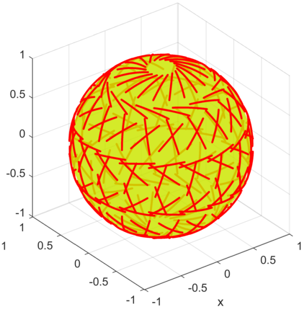






## Singular Value Decomposition [Hielscher, Potts, Q. 2017]

Let $\psi \in (0, 2\pi)$ be fixed. The operator $\mathcal{A}_\psi \colon L^2(\mathbb{S}^2) \to L^2(\mathrm{SO}(3))$ has the SVD

$$\left\{ \left( Y_n^k, Z_{n,\psi}^k, \mu_n(\psi) \right) : n \in \mathbb{N},\, k \in \{-n, \ldots, n\} \right\},$$

with singular values

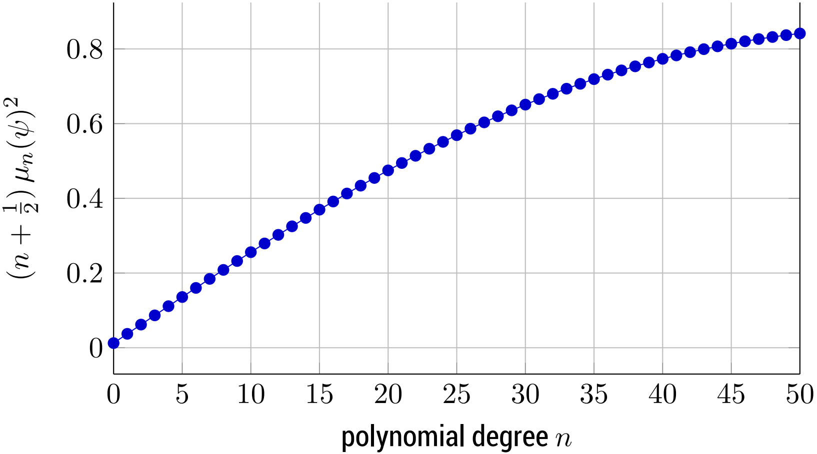
$$\mu_n(\psi) = \sqrt{\sum_{j=-n}^{n} \frac{8\pi^2}{2n+1} \left| \widetilde{P}_n^j(0) \right|^2 s_j(\psi)^2}$$

and singular functions

$$Z_{n,\psi}^k = \frac{1}{\mu_n(\psi)} \sum_{j=-n}^{n} \widetilde{P}_n^j(0)\, s_j(\psi)\, D_n^{j,k} \in L^2(\mathrm{SO}(3)).$$

# Singular values $\mu_n(\psi)$: dependency on $n$

$\psi = 0.02\,\pi$



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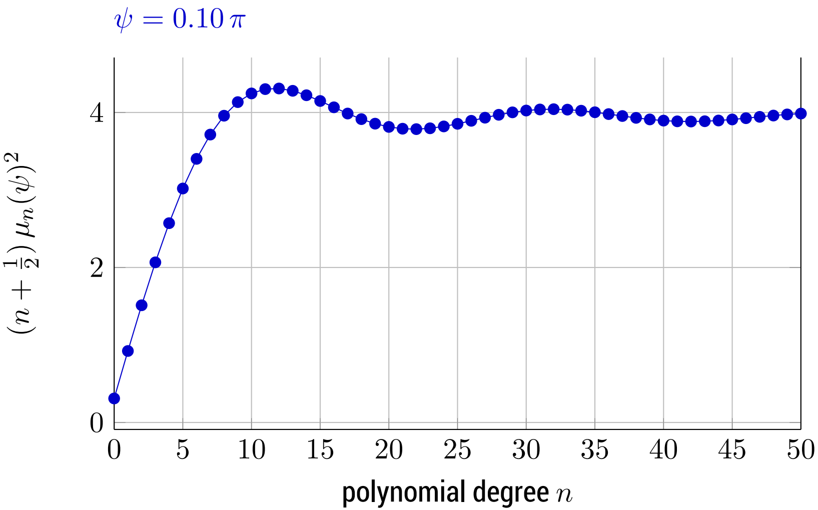

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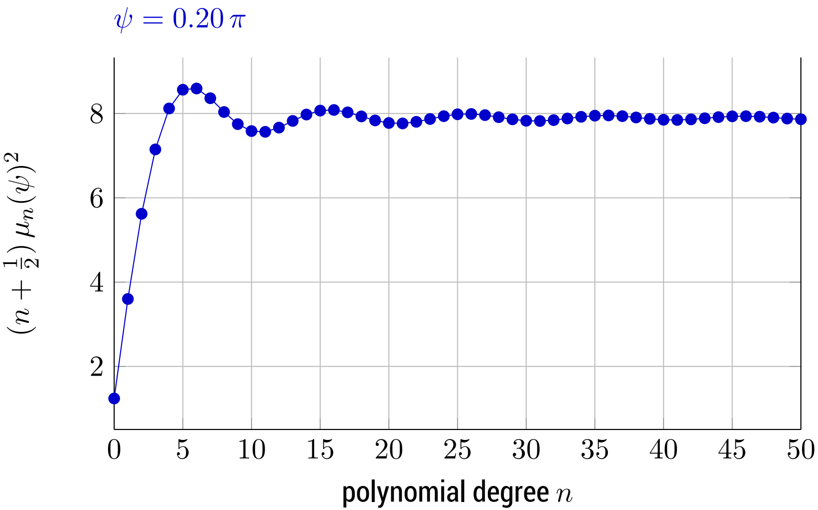

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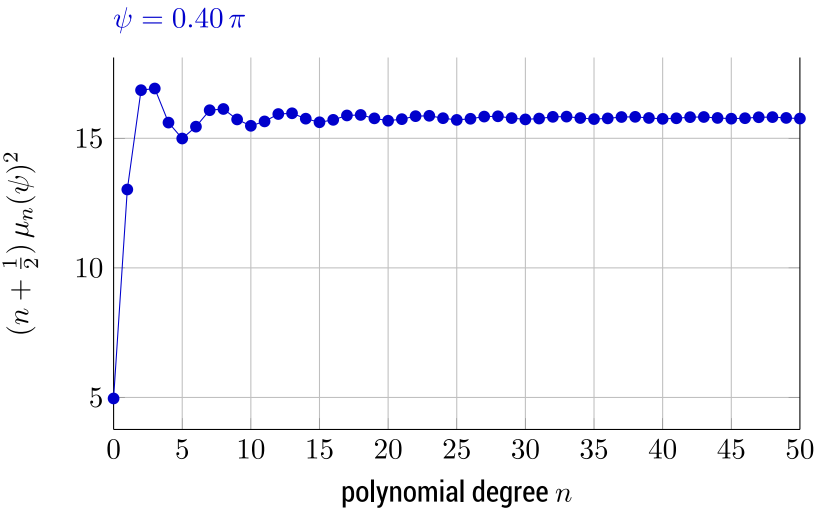
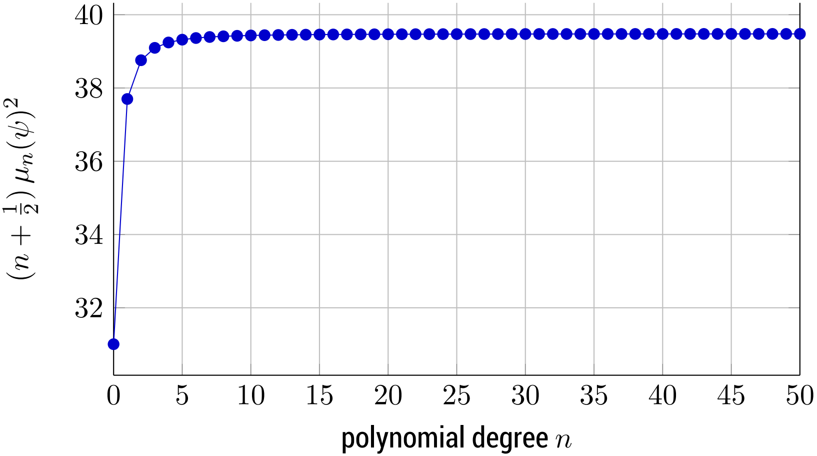


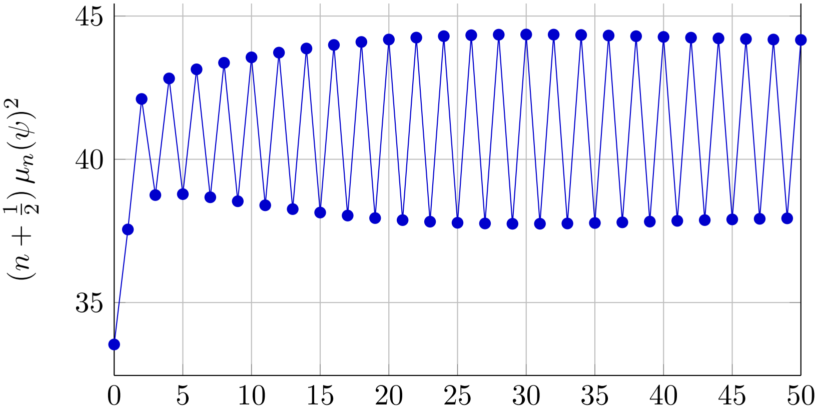
# Singular values $\mu_n(\psi)$: dependency on $n$

$\psi = 1.00\,\pi$ **(half circle)**



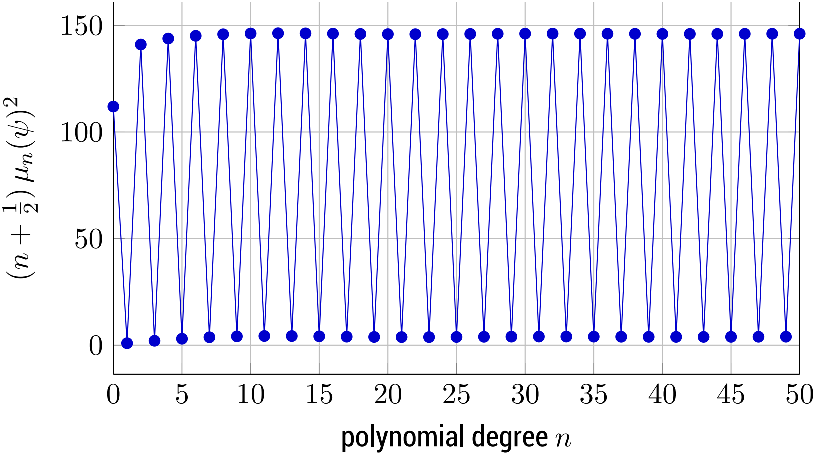
# Singular values $\mu_n(\psi)$: dependency on $n$

$\psi = 1.04\,\pi$



# Singular values $\mu_n(\psi)$: dependency on $n$

$\psi = 1.90\,\pi$



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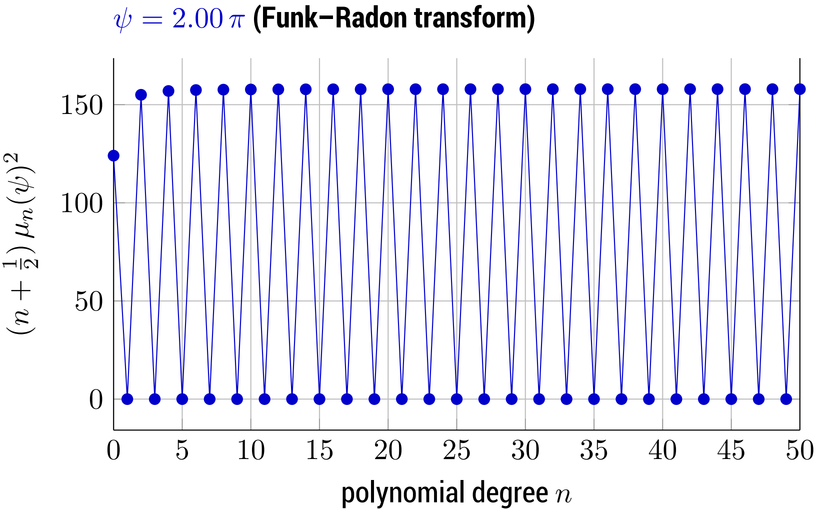

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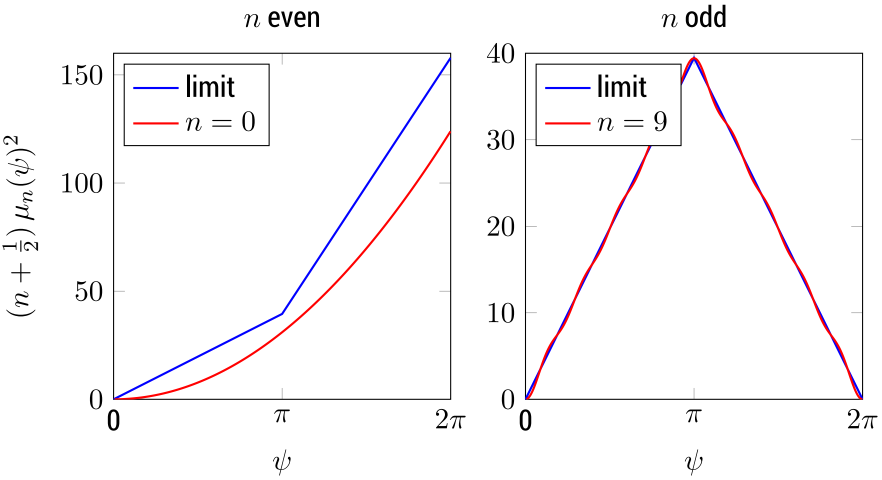

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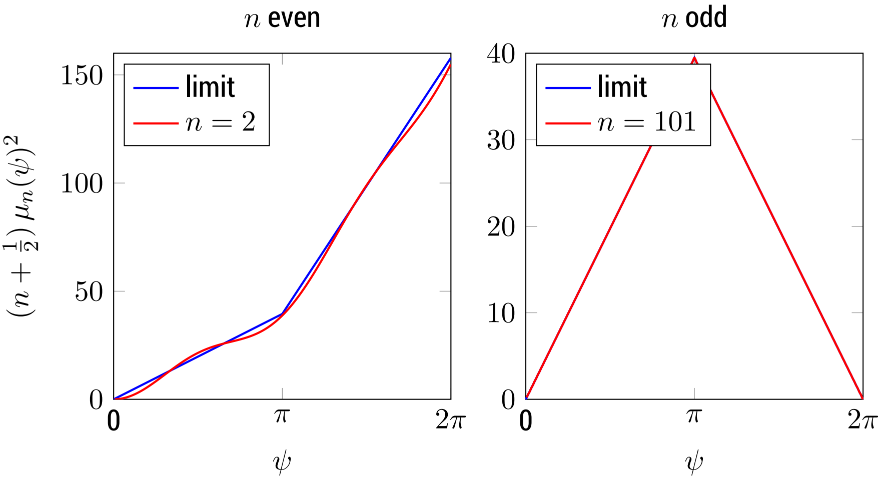

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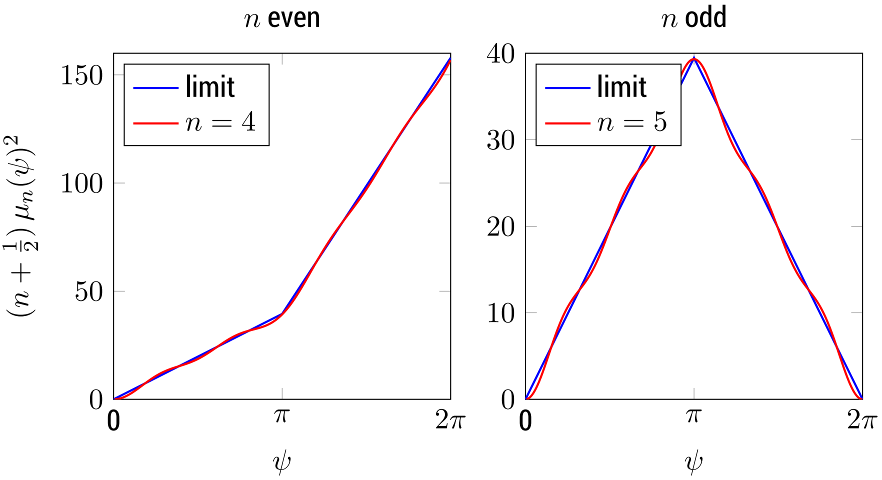

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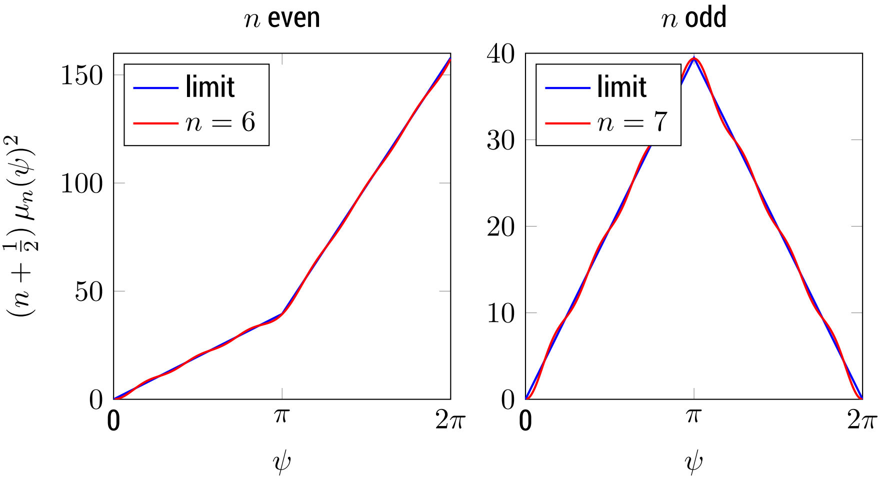

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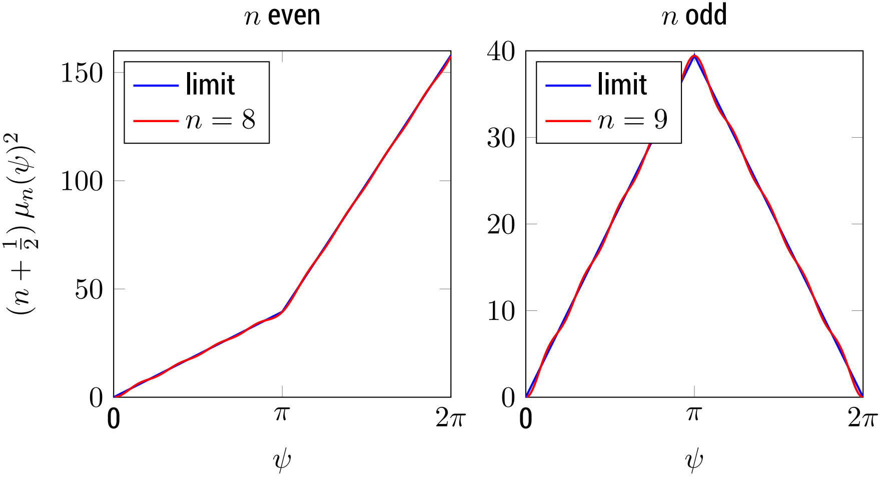

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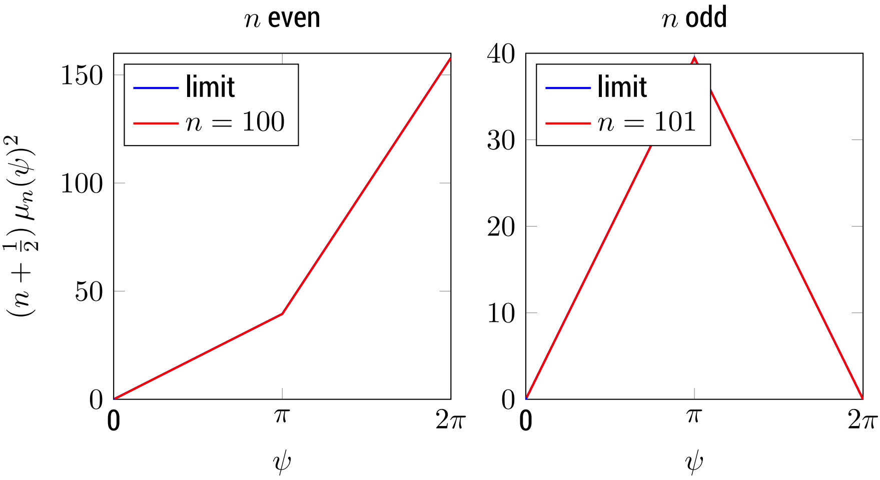


# Singular values: asymptotic behavior

## Theorem [Hielscher, Potts, Q. 2017]

The singular values $\mu_n(\psi)$ of $\mathcal{A}_\psi$ satisfy for odd $n = 2m-1$

$$\lim_{m\to\infty} \frac{4m-1}{4}\,\mu_{2m-1}(\psi)^2 = \begin{cases} 2\pi\psi, & \psi \in [0,\pi] \\ 4\pi^2 - 2\pi\psi, & \psi \in [\pi, 2\pi], \end{cases}$$

and for even $n = 2m$

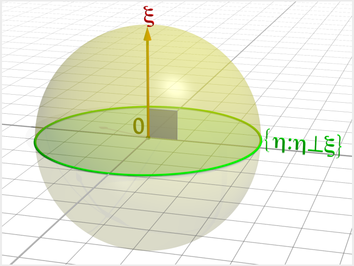
$$\lim_{m\to\infty} \frac{4m+1}{4}\,\mu_{2m}(\psi)^2 = \begin{cases} 2\pi\psi, & \psi \in [0,\pi] \\ 12\pi\psi - 2\pi^2, & \psi \in [\pi, 2\pi]. \end{cases}$$

## $\psi = \pi$: Great circles [Funk 1911]

- Funk–Radon transform

$$\mathcal{F}\colon C(\mathbb{S}^2) \to C(\mathbb{S}^2),$$
$$\mathcal{F}f(\boldsymbol{\xi}) = \int_{\langle \boldsymbol{\xi},\boldsymbol{\eta}\rangle=0} f(\boldsymbol{\eta})\,\mathrm{d}\gamma(\boldsymbol{\eta})$$

- Injective only for even functions




## $\psi = \frac{\pi}{2}$: Half-circle transform

- Injective for all functions [Groemer 1998], [Goodey & Weil 2006]
- [Rubin 2017] half circles in one hemisphere

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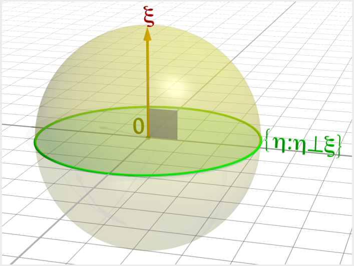


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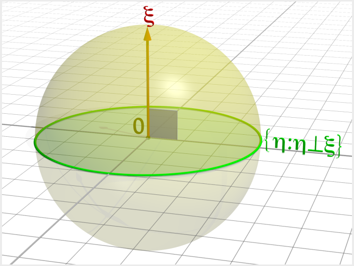


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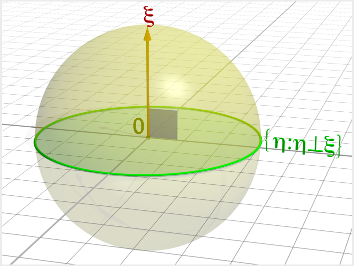


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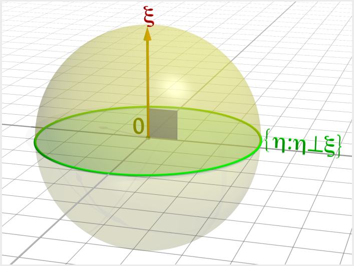


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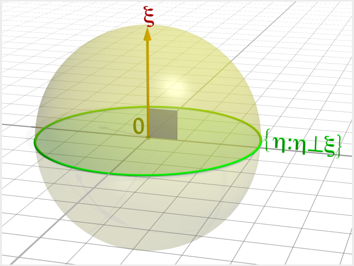


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