A generalization of the Funk–Radon transform

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Radon-type transforms: Basis for Emerging Imaging
100 Years of the Radon Transform
Linz, March 30, 2017

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- Sphere $\mathbb{S}^2 = \{\boldsymbol{\xi} \in \mathbb{R}^3 : \|\boldsymbol{\xi}\| = 1\}$
- Function $f\colon \mathbb{S}^2 \to \mathbb{C}$

- Circle is the intersection of $\mathbb{S}^2$ with a plane:

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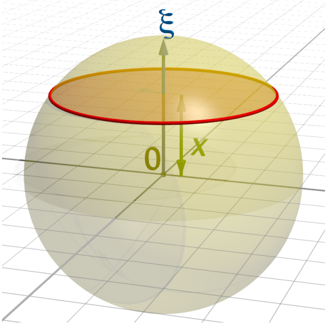
$\boldsymbol{\xi} \in \mathbb{S}^2,\ x \in [-1, 1]$



Spherical means

$$\mathcal{S}\colon C(\mathbb{S}^2) \to C(\mathbb{S}^2 \times [-1,1]),$$

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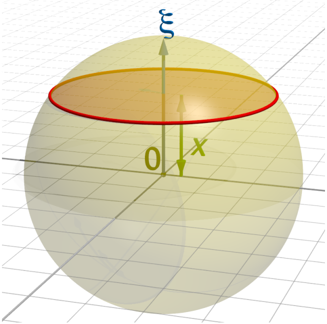

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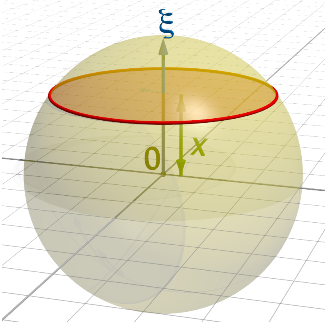

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Funk–Radon transform

[Funk, 1911]

- Restriction to great circles
- **Funk–Radon transform** (a.k.a. Funk transform or spherical Radon transform)

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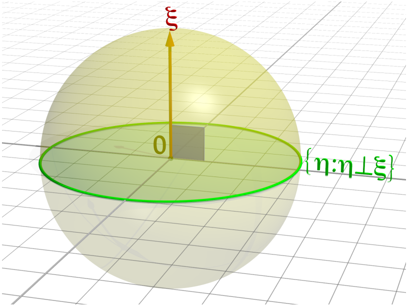

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   (Can we reconstruct f from its means along all great circles?)
2. Range of $\mathcal{F}$

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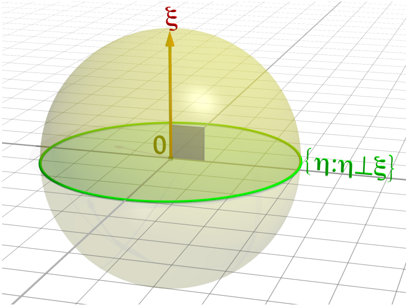

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Fourier series

Write $f \in L^2(\mathbb{S}^2)$ with respect to **spherical harmonics** Y_n^k of degree n

$$f = \sum_{n=0}^{\infty} \sum_{k=-n}^{n} \hat{f}(n,k) Y_n^k.$$

Eigenvalue decomposition [Minkowski, 1904]

The Funk–Radon transform

$$\mathcal{F} Y_n^k(\xi) = P_n(0) Y_n^k(\xi), \qquad P_n(0) = \begin{cases} \frac{(n-1)!!}{n!!}, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

P_n – Legendre polynomial of degree n

Funk–Radon transform is injective for even functions $f(\xi) = f(-\xi)$.

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Sobolev spaces

Let $s \geq 0$. The **Sobolev space** $H^s(\mathbb{S}^2)$ consists of functions $f : \mathbb{S}^2 \to \mathbb{C}$ with norm

$$\|f\|_s^2 = \sum_{n=0}^{\infty} \sum_{k=-n}^{n} \left|\hat{f}(n,k)\right|^2 \left(n + \frac{1}{2}\right)^{2s}.$$

Theorem [Strichartz, 1981]

The Funk–Radon transform is continuous and bijective

$$\mathcal{F} \colon L^2_{\text{even}}(\mathbb{S}^2) \to H^{\frac{1}{2}}_{\text{even}}(\mathbb{S}^2).$$

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Circles with fixed radius

- For $x_0 \in [-1,1]$ fixed, we define

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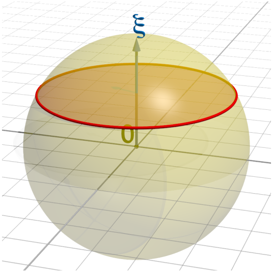

- Eigenvalue decomposition

$$\mathcal{S}_{x_0} Y_n^k = P_n(x_0) Y_n^k$$

“Freak theorem” [Schneider, 1969]

The set of values x_0 for which $\mathcal{S}_{x_0}$ is **not** injective is countable and dense in $[-1,1]$.

$\mathcal{S}_{x_0} : L^2(\mathbb{S}^2) \to H^{\frac{1}{2}}(\mathbb{S}^2)$ is continuous [Rubin, 2000]

Explicit algorithm to determine if $\mathcal{S}_{x_0}$ is injective for given x_0

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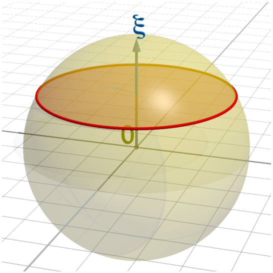

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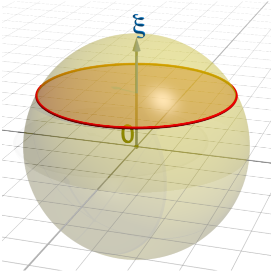

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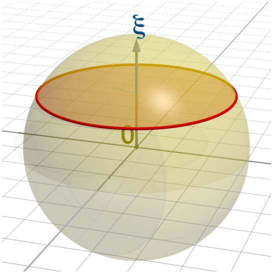

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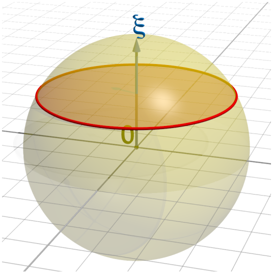

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Vertical slices

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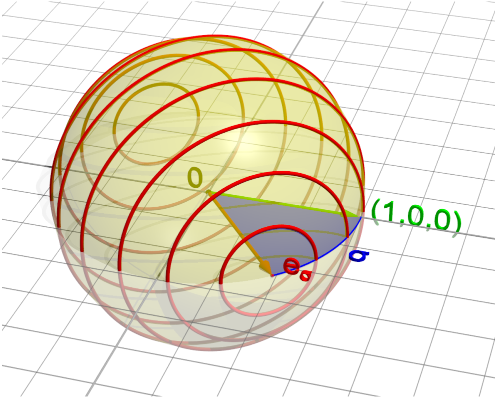

- **Circles perpendicular to the equator**
- Injective for symmetric functions $f(\xi_1, \xi_2, \xi_3) = f(\xi_1, \xi_2, -\xi_3)$
- Proof1: Orthogonal projection onto the equatorial plane [Gindikin, Reeds & Shepp, 1994]
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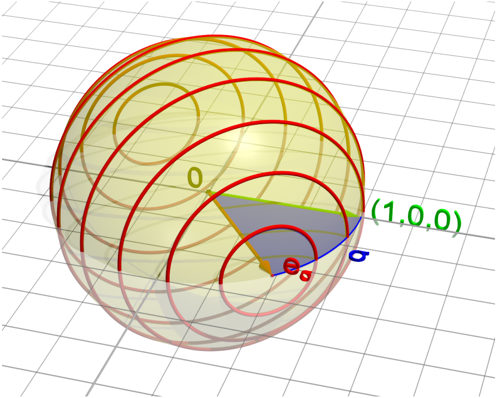

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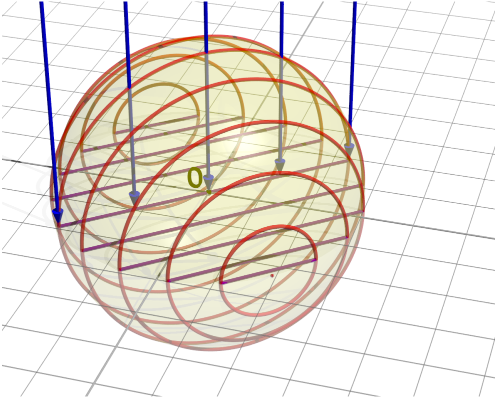

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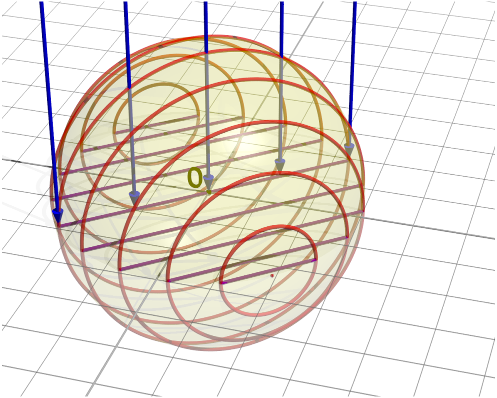

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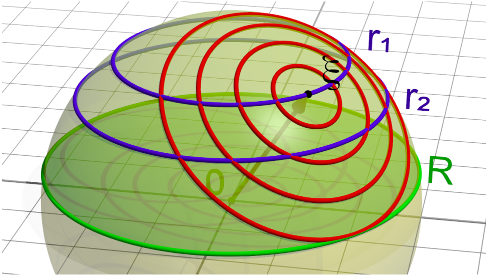
Local two-radii problem

- $B_R = \{\boldsymbol{\xi} \in \mathbb{S}^2 : \xi_3 > \cos R\}$ ... spherical cap north of the circle of latitude $R \in (0, \pi]$
- $r_1, r_2 < R$ ... latitudes of centers



Theorem [Volchkov & Volchkov, 2014]

Then

$$\mathcal{S}f(\boldsymbol{\xi}, t) = 0, \quad \xi_{d+1} = \cos r_j,\ t > \cos(R - r_j),\ j \in \{1, 2\}$$

implies $f = 0$ for all $f \in L^1_{\mathrm{loc}}(B_R)$ if and only if $R \geq r_1 + r_2$ and

$$P^k_\nu(\cos r_1)^2 + P^k_\nu(\cos r_2)^2 > 0 \qquad \forall k \in \mathbb{N}_0,\ \nu > k.$$

Higher dimensions

- $\mathbb{S}^d$ ... d-dimensional sphere
- $\mathbb{R}^{d+1}$ ... $(d+1)$-dimensional ambient space

- Great circle becomes $(d-1)$-dimensional subsphere

- For $f \in C(\mathbb{S}^d)$, we set

$$\mathcal{F}f(\boldsymbol{\xi}) = \int_{\mathbb{S}^d \cap \{\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = 0\}} f(\boldsymbol{\eta}) \, \mathrm{d}\mu$$

Theorem [Strichartz, 1981]

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[Salman, 2016]

Replace $\mathbf{0}$ by

$$z\boldsymbol{e}^{d+1}, \qquad 0 \le z < 1$$

inside the sphere.

Intersection with hyperplane through $z\boldsymbol{e}^{d+1}$ is given by

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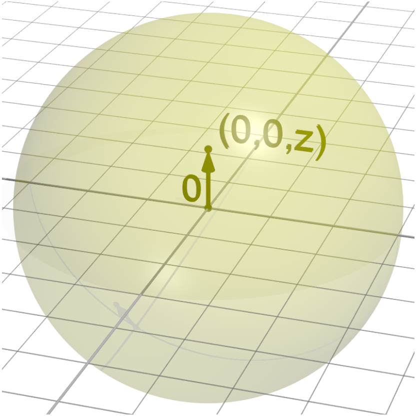

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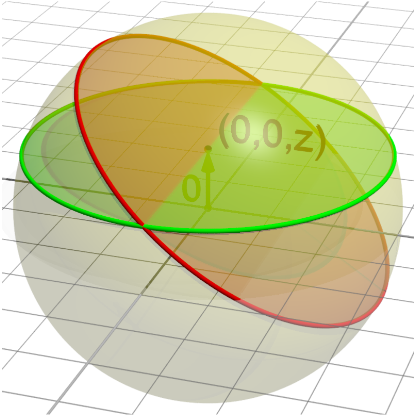

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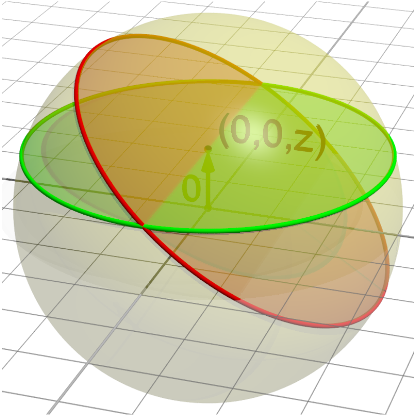

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Theorem [Salman, 2016]

Let $z \in [0,1)$, $\sigma = \sqrt{\frac{1+z}{1-z}}$ and $f \in C^1(\mathbb{S}^d)$ be supported in the interior of $\{\boldsymbol{\xi} \in \mathbb{S}^d : -1 \leq \xi_{d+1} \leq z\}$. Then f can be reconstructed by

$$(f \circ \boldsymbol{\pi}^{-1}) \left(\frac{2\sigma \boldsymbol{x}}{1+\sqrt{1+4|\boldsymbol{x}|^2}} \right) =$$

$$\begin{cases} \frac{(-1)^{\frac{d-2}{2}}(1-z)\sqrt{1+4|\boldsymbol{x}|^2}}{2^{3d-2}\pi^d \sigma^{d-5}} \left(\frac{\left(1+\sqrt{1+4|\boldsymbol{x}|^2}\right)^2 + 4\sigma^2|\boldsymbol{x}|^2}{1+\sqrt{1+4|\boldsymbol{x}|^2}} \right)^{d-1} \\ \triangle_{\boldsymbol{x}}^{\frac{d}{2}} \int_{\mathbb{S}^{d-1}} \int_0^{\frac{\pi}{2}} (\mathcal{U}_z f)((\cos\theta)\boldsymbol{\phi} + (\sin\theta)\boldsymbol{\epsilon}^{d+1}) \log \left| \boldsymbol{x}\cdot\boldsymbol{\phi} - \frac{1}{2}\sqrt{1-z^2}\tan\theta \right| \frac{\mathrm{d}\theta\mathrm{d}\boldsymbol{\phi}}{\sqrt{1-z^2\sin^2\theta}\cos\theta}, & d \geq 2 \text{ even} \\ \frac{(-1)^{\frac{d-1}{2}}(1-z)\sqrt{1+4|\boldsymbol{x}|^2}}{2^{3d-1}\pi^{d-1} \sigma^{d-2}} \left(\frac{\left(1+\sqrt{1+4|\boldsymbol{x}|^2}\right)^2 + 4\sigma^2|\boldsymbol{x}|^2}{1+\sqrt{1+4|\boldsymbol{x}|^2}} \right)^{d-1} \\ \triangle_{\boldsymbol{x}}^{\frac{d+1}{2}} \int_{\mathbb{S}^{d-1}} \int_0^{\frac{\pi}{2}} (\mathcal{U}_z f)((\cos\theta)\boldsymbol{\phi} + (\sin\theta)\boldsymbol{\epsilon}^{d+1}) \left| \boldsymbol{x}\cdot\boldsymbol{\phi} - \frac{1}{2}\sqrt{1-z^2}\tan\theta \right| \frac{\mathrm{d}\theta\mathrm{d}\boldsymbol{\phi}}{\sqrt{1-z^2\sin^2\theta}\cos\theta}, & d \geq 3 \text{ odd.} \end{cases}$$

Theorem [Q., 2017]

Let $z \in [0,1)$. The generalized Radon transform $\mathcal{U}_z$ can be represented through

$$\mathcal{U}_z = \mathcal{N}_z \mathcal{F} \mathcal{M}_z.$$

- $$\mathcal{M}_z f(\boldsymbol{\xi}) = \left(\frac{\sqrt{1-z^2}}{1+z\xi_{d+1}} \right)^{d-1} f \circ h(\boldsymbol{\xi}), \qquad \boldsymbol{\xi} \in \mathbb{S}^d$$

  $$h(\boldsymbol{\xi}) = \sum_{i=1}^{d} \frac{\sqrt{1-z^2}}{1+z\xi_{d+1}} \xi_i \boldsymbol{e}^i + \frac{z+\xi_{d+1}}{1+z\xi_{d+1}} \boldsymbol{e}^{d+1}$$
- $\mathcal{F}$ ... Funk–Radon transform
- $$\mathcal{N}_z f(\boldsymbol{\xi}) = (1-z^2\xi_{d+1}^2)^{-\frac{d-1}{2}} f \circ g(\boldsymbol{\xi}), \qquad \boldsymbol{\xi} \in \mathbb{S}^d$$

  $$g(\boldsymbol{\xi}) = \frac{1}{\sqrt{1-z^2\xi_{d+1}^2}} \left(\sum_{i=1}^{d} \xi_i \boldsymbol{e}^i + \sqrt{1-z^2}\xi_{d+1} \boldsymbol{e}^{d+1} \right)$$

Geometric interpretation of h (for $\mathbb{S}^2$)

Theorem

The map

$$h(\boldsymbol{\xi}) = \pi^{-1}\left(\sqrt{\frac{1+z}{1-z}}\,\pi(\boldsymbol{\xi})\right)$$

is conformal. It consists of

1. Stereographic projection $\pi\colon \mathbb{S}^d \to \mathbb{R}^d$
2. Uniform scaling $\mathbb{R}^d \to \mathbb{R}^d$, $\boldsymbol{x} \mapsto \sqrt{\frac{1+z}{1-z}}\,\boldsymbol{x}$
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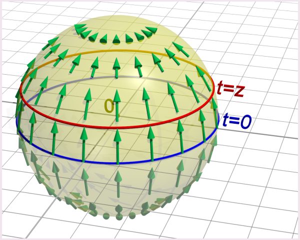

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h maps great circles to small circles through $z\boldsymbol{e}^{d+1}$.

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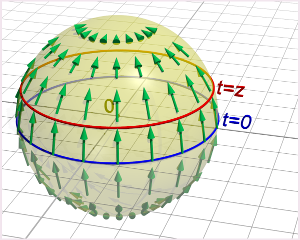

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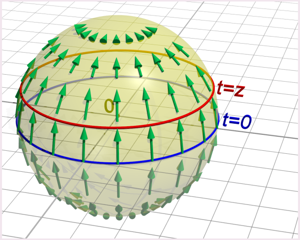

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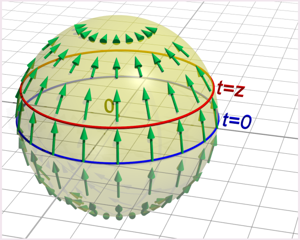

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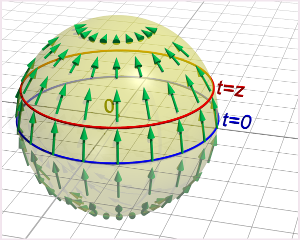

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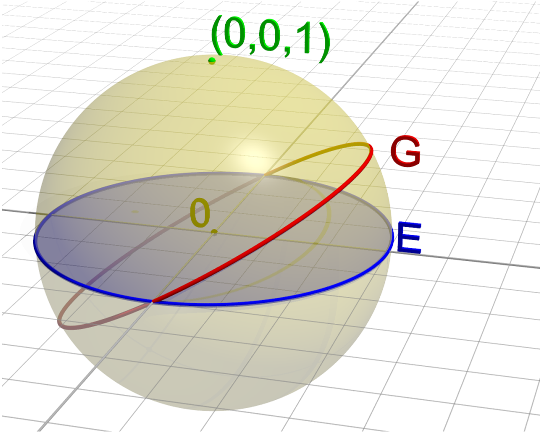
1) Stereographic projection π

- G ... great circle of $\mathbb{S}^2$
- E ... equator of $\mathbb{S}^2$

- G intersects E in two antipodal points (or is identical to E)

- $\pi(E) = E$
- $\pi(G)$ is circle or line in $\mathbb{R}^2$ and intersects $\pi(E)$ in two antipodal points



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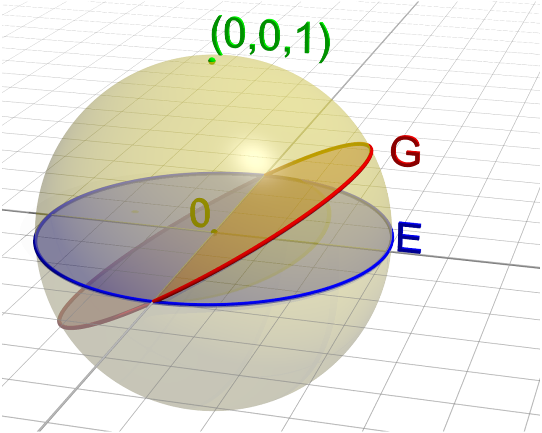

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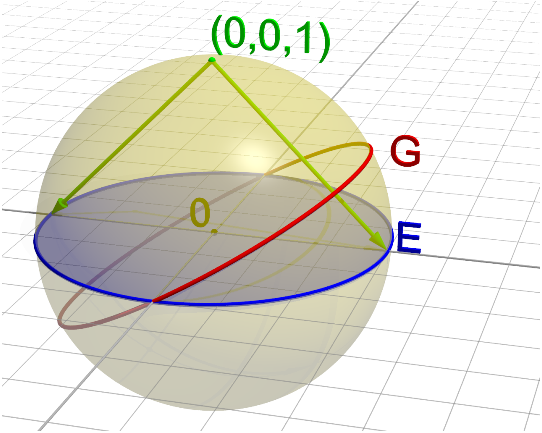

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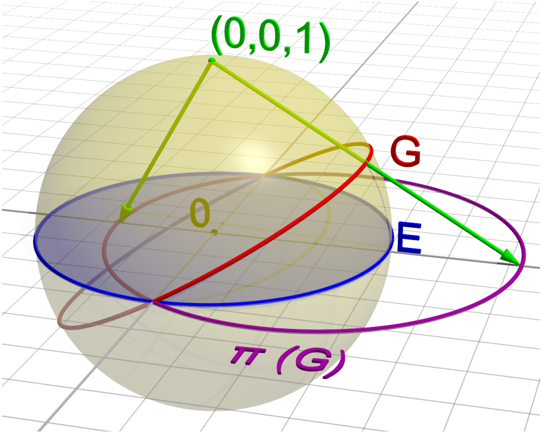
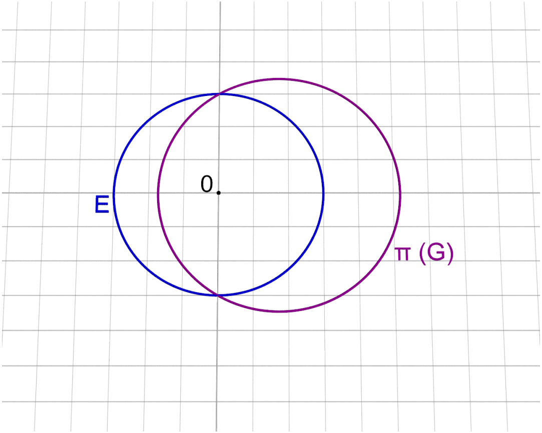


2) Uniform scaling

- Scaling with factor $\sigma = \sqrt{\frac{1+z}{1-z}}$
- Unit circle E becomes circle $\sigma(\pi(E))$ with radius σ
- $\sigma(\pi(G))$ intersects $\sigma(\pi(E))$ in two antipodal points



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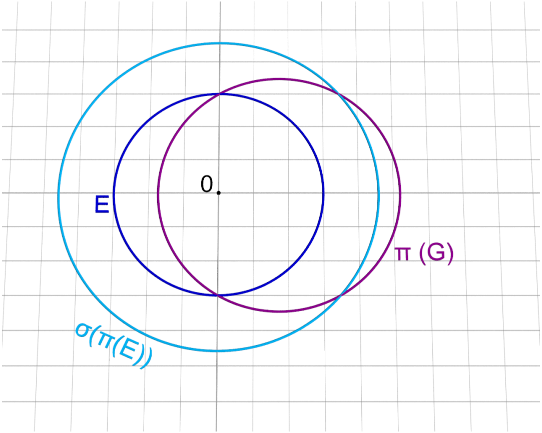

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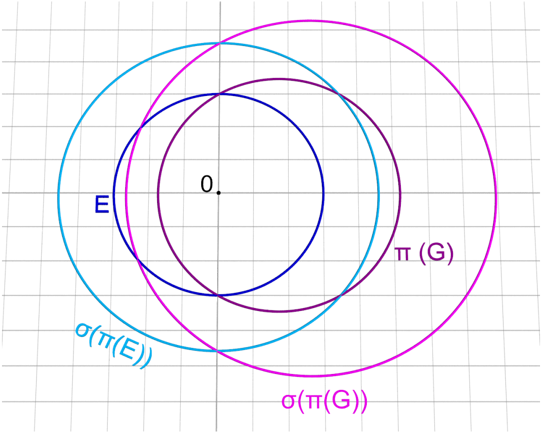

3) Inverse stereographic projection π^{-1}

- Circle with radius s becomes circle of latitude z; $h(E)$
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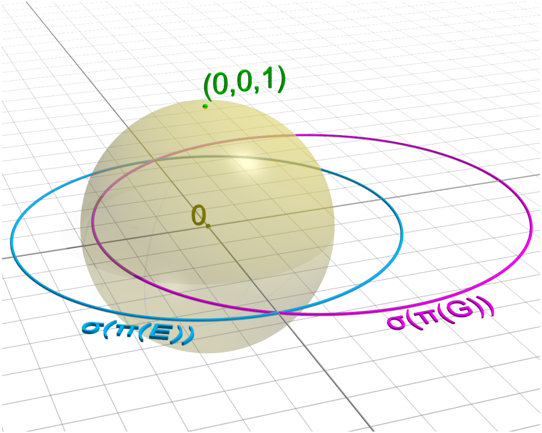

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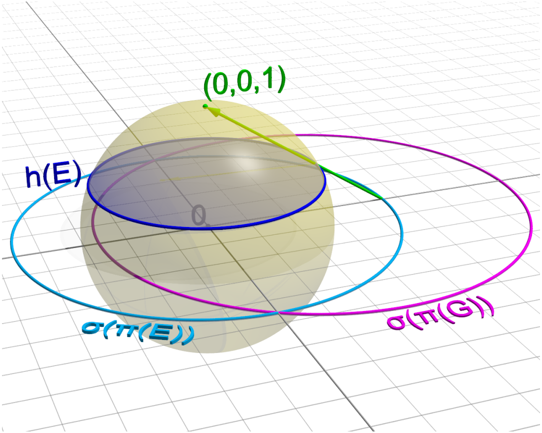

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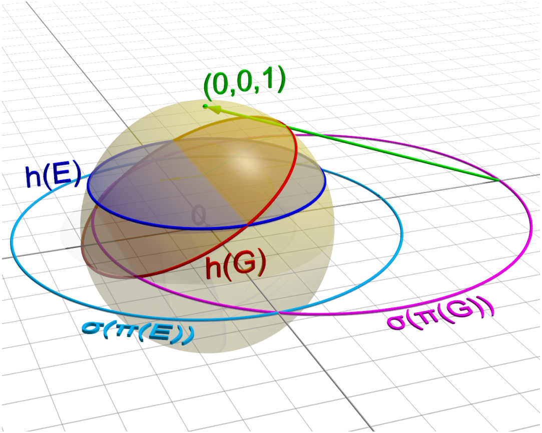

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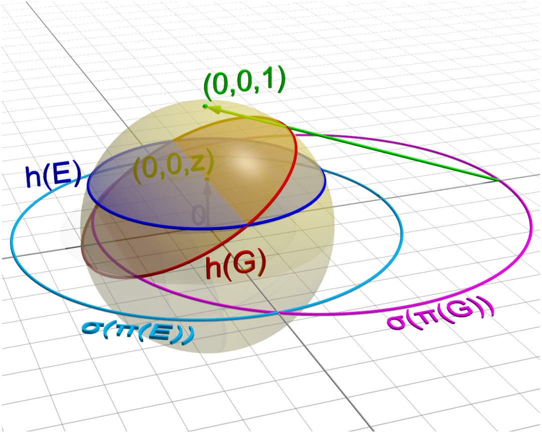
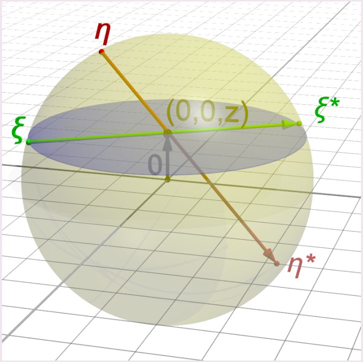


Nullspace of $\mathcal{U}_z$

Theorem [Q., 2017]

For $\xi \in \mathbb{S}^d$ we define $\xi^* \in \mathbb{S}^d$ as the point reflection of the sphere about the point ze^{d+1}.



Let $f \in L^2(\mathbb{S}^d)$. Then

$$\mathcal{U}_z f = 0$$

if and only if for almost every $\xi \in \mathbb{S}^d$

$$f(\xi) = -\left(\frac{1-z^2}{1+z^2-2z\eta_{d+1}}\right)^{d-1} f(\xi^*).$$

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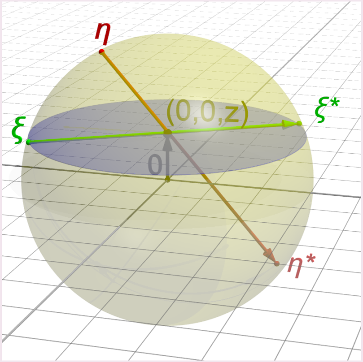

Range of $\mathcal{U}_z$

Theorem [Q., 2017]

The generalized Radon transform

$$\mathcal{U}_z \colon \widetilde{H}^s_{\mathrm{e}}(\mathbb{S}^d) \to H_{\mathrm{e}}^{s+\frac{d-1}{2}}(\mathbb{S}^d)$$

is bijective and continuous.

- $\widetilde{H}^s_{\mathrm{e}}(\mathbb{S}^d) = \left\{ f \in H^s(\mathbb{S}^d) \mid f(\boldsymbol{\xi}) = \left(\frac{1-z^2}{1+z^2-2z\eta_{d+1}} \right)^{d-1} f(\boldsymbol{\xi}) \right\}$
- $H_{\mathrm{e}}^{s+\frac{d-1}{2}}(\mathbb{S}^d)$ ... Sobolev space of even functions

Sketch of proof

$$\mathcal{U}\colon H^s(\mathbb{S}^d) \xrightarrow{\mathcal{M}_z} H^s(\mathbb{S}^d) \xrightarrow{\mathcal{F}} H^{s+\frac{d-1}{2}}(\mathbb{S}^d) \xrightarrow{\mathcal{N}_z} H^{s+\frac{d-1}{2}}(\mathbb{S}^d)$$

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Lemma 1

Let $s \in \mathbb{N}_0$ and $g \in C^s(\mathbb{S}^d)$. Then there exists c_s such that

$$\|fg\|_{H^s(\mathbb{S}^d)} \le c_s \|f\|_{H^s(\mathbb{S}^d)} \|g\|_{C^s(\mathbb{S}^d)} \qquad \forall f \in H^s(\mathbb{S}^d),\ \forall g \in C^s(\mathbb{S}^d).$$

Lemma 2

Let $s \in \mathbb{N}_0$ and $g \in C^s(\mathbb{S}^d \to \mathbb{S}^d)$ be a diffeomorphism. Then there exists $C_{g,s}$ such that

$$\|f \circ g\|_{H^s(\mathbb{S}^d)} \le C_{g,s} \|f\|_{H^s(\mathbb{S}^d)} \qquad \forall f \in H^s(\mathbb{S}^d).$$

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Theorem [Strichartz, 1981]

The Funk–Radon transform

$$\mathcal{F}\colon L^2_{\mathrm{even}}(\mathbb{S}^d) \to H^{\frac{d-1}{2}}_{\mathrm{even}}(\mathbb{S}^d)$$

is continuous and bijective.

Sketch of proof

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□

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