The Funk–Radon transform and its generalizations

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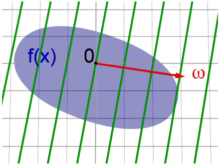
Modeling, analysis, and approximation theory toward applications in tomography and inverse problems
Braunschweig, 6 February 2018

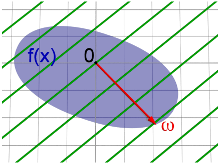
The Radon transform

[Radon, 1917]

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$. Define the **Radon transform**

$$\mathcal{R}f(\boldsymbol{\omega}, s) = \int_{\langle \boldsymbol{x}, \boldsymbol{\omega} \rangle = s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \qquad \boldsymbol{\omega} \in \mathbb{S}^1,\ s \in \mathbb{R}.$$





Questions:

- Injectivity
- Reconstruction formulas
- Stability of reconstruction

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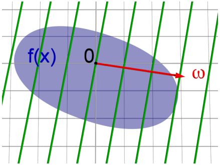

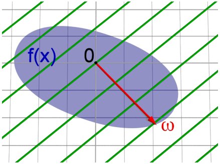

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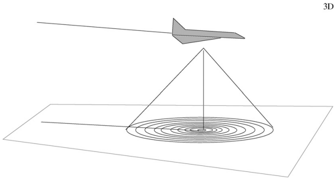
The circular Radon transform

Let $f : \mathbb{R}^2 \to \mathbb{C}$ and $A \subset \mathbb{R}^2$.

Define the **circular Radon transform**

$$\int_{\|\boldsymbol{x}-\boldsymbol{a}\|=t} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}, \qquad t \geq 0, \; \boldsymbol{a} \in A$$



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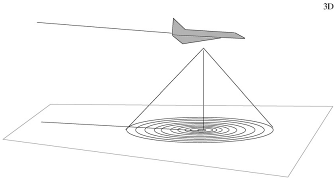

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Explicit reconstruction formulas for

- A is a line [Andersson, 1988]
- A is a circle [Finch, Patch & Rakesh, 2004]
- A is an ellipse [Haltmeier, 2014]

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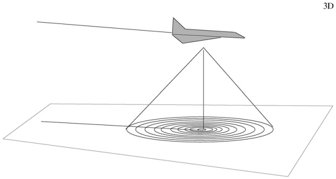

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Solution to the injectivity problem [Agranovsky & Quinto, 1996]

The circular Radon transform of the set A is injective on $C_c(\mathbb{R}^2)$ if and only if A is not contained in any set of the form

$$\omega(\Sigma_N) \cup F,$$

where

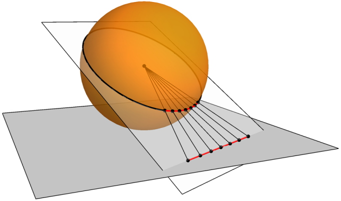
$$\Sigma_N = \{t\mathrm{e}^{\pi \mathrm{i} k/N} : t \in \mathbb{R},\ k = 1, \dots, N\},$$

ω is a rigid motion in $\mathbb{R}^2$ and F is a finite set.

From the plane to the sphere

- Central (gnomonic) projection from the origin to the plane tangential to sphere at the south pole $-e^3$
- Great circles on the sphere are mapped to straight lines in the plane
- Line integrals become great circle integrals

[Gindikin, Reeds, Shepp, 1994]



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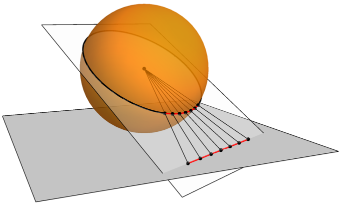

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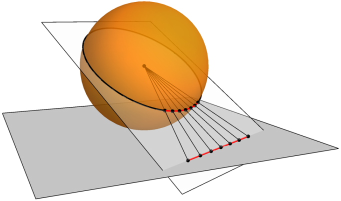

Content

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Funk–Radon transform

[Funk, 1911]

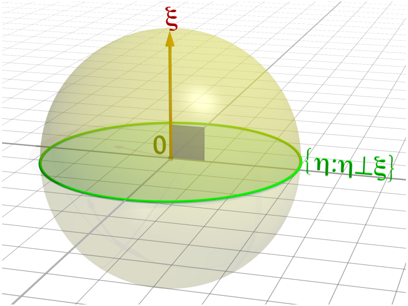
- Sphere $\mathbb{S}^2 = \{\xi \in \mathbb{R}^3 : \|\xi\| = 1\}$
- Function $f\colon \mathbb{S}^2 \to \mathbb{C}$
- Funk–Radon transform

$$\mathcal{F}\colon C(\mathbb{S}^2) \to C(\mathbb{S}^2),$$

$$\mathcal{F}f(\xi) = \frac{1}{2\pi} \int_{\langle \xi,\eta \rangle = 0} f(\eta)\, \mathrm{d}\lambda(\eta)$$

(integrals of f along all great circles)

  - $\mathcal{F}$ is a linear operator
  - Normalization: $f \equiv 1 \implies \mathcal{F}f \equiv 1$
  - $\mathcal{F}f$ is even, i.e. $\mathcal{F}f(\xi) = \mathcal{F}f(-\xi)$
  - If f is odd, then $\mathcal{F}f = 0$



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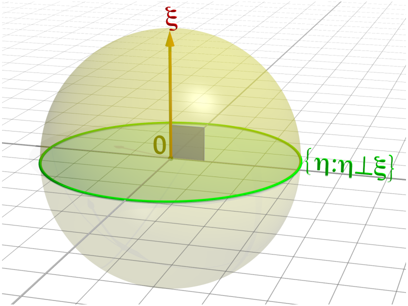

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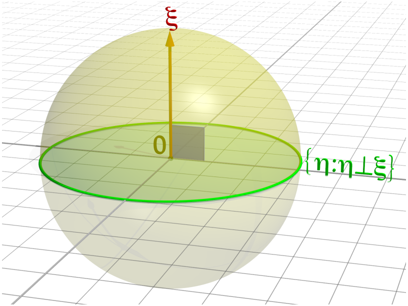

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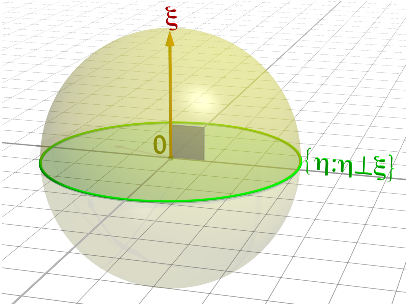

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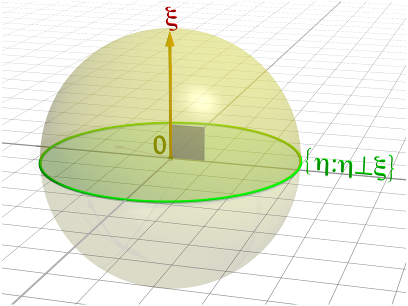

History

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- Integral over a great circle was used to proof that bodies of constant width and bodies of constant circumference are equivalent

Paul Funk, 1911: *Über Flächen mit lauter geschlossenen geodätischen Linien*

Wanted to reconstruct a function $f : \mathbb{S}^2 \to \mathbb{R}$ on the sphere

Given $\mathcal{F}f$ called the **circle-integral function** of f ("Kreis-Integralfunktion")

Johann Radon, 1917: *Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten*

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Given $\mathcal{R}f$, the integrals of f along all lines

Both the great circles on the sphere and the lines in the plane are **geodesics**

The Funk–Radon transform is also known as Minkowski–Funk transform, Funk transform or spherical Radon transform

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- Q–ball imaging in medicine [Tuch, 2004]
- Surface wave models for earthquakes [Amirbekyan, Michel & Simons, 2008]
- Synthetic aperture radar (SAR) [Yarman & Yazici, 2011]
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Spherical harmonics

An orthonormal basis on $\mathbb{S}^2$

Use cylindrical coordinates

$$\boldsymbol{\xi}(\varphi, t) = (\cos\varphi, \sin\varphi, t) \in \mathbb{S}^2$$

Define the **spherical harmonics** of degree n

$$Y_n^k(\varphi, t) = \sqrt{\frac{2n+1}{4\pi}\frac{(n-k)!}{(n+k)!}}\, P_n^k(t)\, \mathrm{e}^{\mathrm{i}k\varphi}$$

Every $f \in L^2(\mathbb{S}^2)$ can be written as series

$$f = \sum_{n=0}^{\infty}\sum_{k=-n}^{n} \hat{f}(n,k) Y_n^k, \qquad \hat{f}(n,k) := \int_{\mathbb{S}^2} f(\boldsymbol{\xi}) \overline{Y_n^k(\boldsymbol{\xi})}\, \mathrm{d}\boldsymbol{\xi}$$

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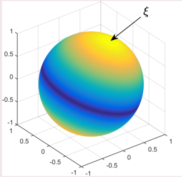
Funk–Hecke formula

Let $g\colon [-1,1] \to \mathbb{C}$. Then

$$\int_{\mathbb{S}^2} Y_n^k(\boldsymbol{\eta})\, g(\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle)\, \mathrm{d}\boldsymbol{\eta} = Y_n^k(\boldsymbol{\xi}) \int_{-1}^{1} g(x)\, P_n(x)\, \mathrm{d}x$$

P_n – Legendre polynomial of degree n



$g(\langle \boldsymbol{\xi}, \cdot \rangle)$

For the Funk–Radon transform: Insert $g(t) = \delta(t)$

Singular value decomposition (SVD) [Minkowski, 1904]

The Funk–Radon transform is given by

$$\mathcal{F} Y_n^k(\xi) = \lambda_n\, Y_n^k(\xi), \qquad \lambda_n = P_n(0) = \begin{cases} \frac{(n-1)\,(n-3)\cdots 1}{n\,(n-2)\cdots 2}, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

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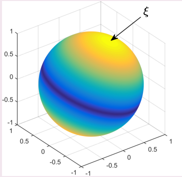

$g(\langle \boldsymbol{\xi}, \cdot \rangle)$

For the Funk–Radon transform: Insert $g(t) = \delta(t)$

Singular value decomposition (SVD) [Minkowski, 1904]

The Funk–Radon transform is given by

$$\mathcal{F} Y_n^k(\xi) = \lambda_n Y_n^k(\xi), \qquad \lambda_n = P_n(0) = \begin{cases} \frac{(n-1)\,(n-3)\cdots 1}{n\,(n-2)\cdots 2}, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

Funk–Hecke formula

Let $g\colon [-1,1] \to \mathbb{C}$. Then

$$\int_{\mathbb{S}^2} Y_n^k(\boldsymbol{\eta})\, g(\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle)\, \mathrm{d}\boldsymbol{\eta} = Y_n^k(\boldsymbol{\xi}) \int_{-1}^{1} g(x)\, P_n(x)\, \mathrm{d}x$$

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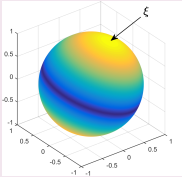

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SVD for the inversion

Want to solve

$$\mathcal{F}f = g$$

We have

$$\mathcal{F}f = \sum_{n=0}^{\infty} \sum_{k=-n}^{n} \lambda_n \hat{f}(n,k) Y_n^k$$

If f is even, we reconstruct

$$f = \sum_{\substack{n=0 \\ 2|n}}^{\infty} \sum_{k=-n}^{n} \frac{1}{\lambda_n} \hat{g}(n,k) Y_n^k$$

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Sobolev spaces

For $s \geq 0$, the **Sobolev space** $H^s(\mathbb{S}^2)$ is the completion of the space of polynomials $f\colon \mathbb{S}^2 \to \mathbb{C}$ with the norm

$$\|f\|_s^2 = \sum_{n=0}^{\infty} \sum_{k=-n}^{n} \left|\hat{f}(n,k)\right|^2 \left(n + \frac{1}{2}\right)^{2s}.$$

Theorem [Strichartz, 1981]

The Funk–Radon transform is bijective

$$\mathcal{F}\colon L^2_{\mathrm{even}}(\mathbb{S}^2) \to H^{\frac{1}{2}}_{\mathrm{even}}(\mathbb{S}^2).$$

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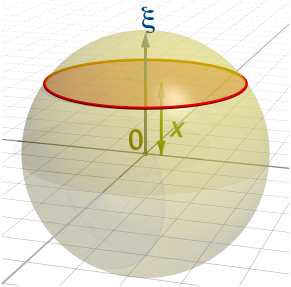
Content

Circles on the sphere

A circle on the sphere is the intersection of the sphere with a plane:

$$\{\boldsymbol{\eta} \in \mathbb{S}^2 : \langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = x\},$$

$\boldsymbol{\xi} \in \mathbb{S}^2,\ x \in [-1, 1]$



Mean operator

$$\mathcal{S} \colon C(\mathbb{S}^2) \to C(\mathbb{S}^2 \times [-1,1]),$$

$$\mathcal{S} f(\boldsymbol{\xi}, x) = \int_{\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = x} f(\boldsymbol{\eta}) \, \mathrm{d}\lambda(\boldsymbol{\eta})$$

$\mathcal{S} f(\boldsymbol{\xi}, 0) = \mathcal{F} f(\boldsymbol{\xi})$ is the Funk–Radon transform
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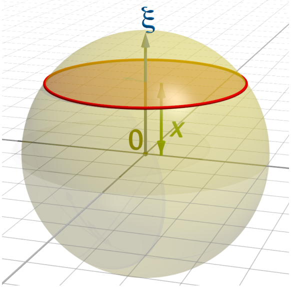

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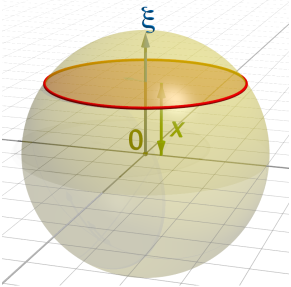

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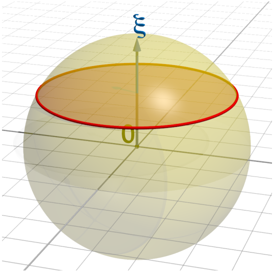
Circles with fixed radius

For fixed $x_0 \in [-1, 1]$, compute

$$\mathcal{S}_{x_0} f(\boldsymbol{\xi}) = \int_{\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = x_0} f(\boldsymbol{\eta}) \, \mathrm{d}\boldsymbol{\eta}$$



Eigenvalue decomposition

$$\mathcal{S}_{x_0} Y_n^k = P_n(x_0) Y_n^k$$

"Freak theorem" [Schneider, 1969]

The set of values x_0 for which $\mathcal{S}_{x_0}$ is **not** injective is countable and dense in $[-1, 1]$.

This is because $\mathcal{S}_{x_0}$ is injective if and only if $P_n(x_0) = 0 \; \forall n \in \mathbb{N}$.

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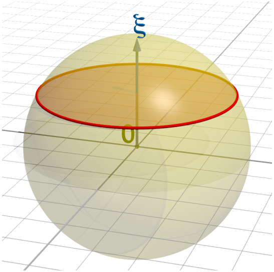

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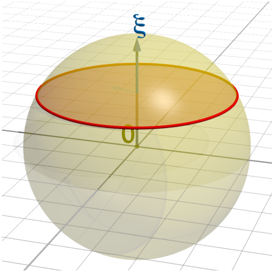

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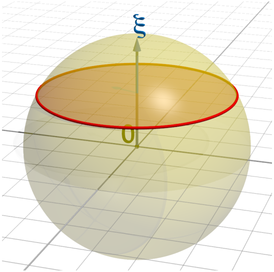

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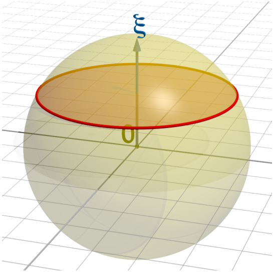

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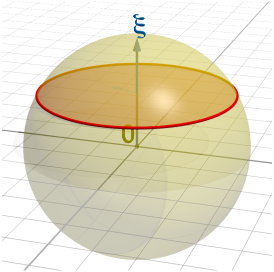

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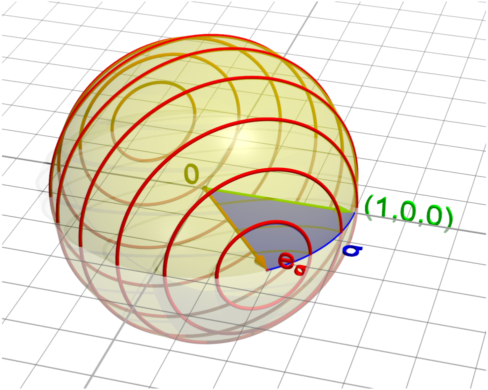
Vertical slices

$$\mathcal{S}(\boldsymbol{\xi}, x) = \int_{\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = x} f(\boldsymbol{\eta})\, \mathrm{d}s(\boldsymbol{\eta}), \qquad \xi_3 = 0$$



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- Injective for symmetric functions $f(\xi_1, \xi_2, \xi_3) = f(\xi_1, \xi_2, -\xi_3)$
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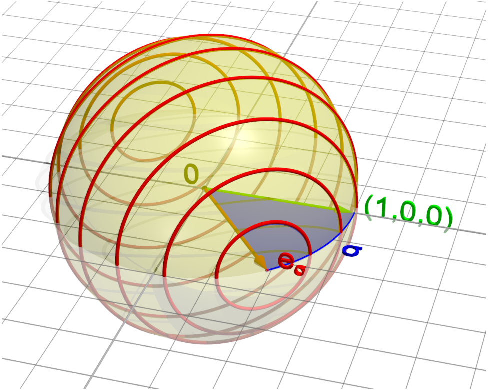

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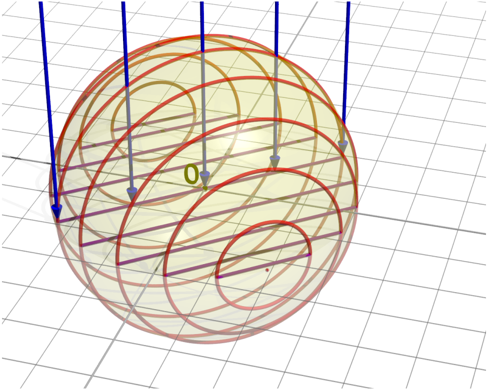

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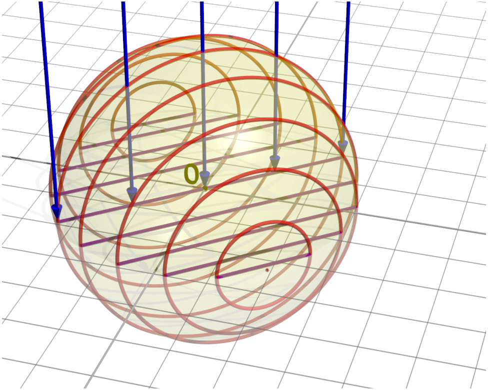

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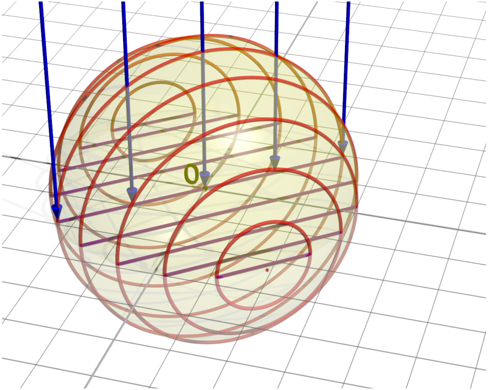

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Circles with all values of x

The vertical slices are a special case of

$$\mathcal{S}f(\boldsymbol{\xi}, x), \qquad \boldsymbol{\xi} \in A \subset \mathbb{S}^2,\ x \in [-1, 1].$$

Centers are on an arbitrary set $A \subset \mathbb{S}^2$

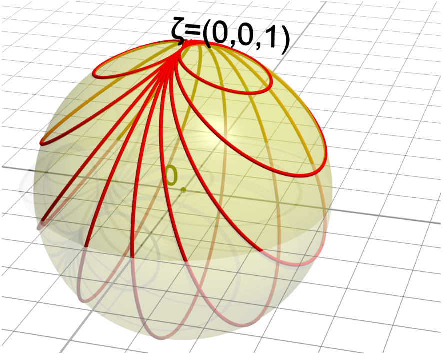
Theorem [Agranovsky & Quinto, 1996] [Agranovsky, Volchkov & Zalcman, 1999]

The spherical mean operator $\mathcal{S}$ restricted a set $A \subset \mathbb{S}^2$ is injective if and only if A is not a subset of the zero set of a nontrivial spherical harmonic $Y_n \in \mathscr{H}_n$ for any $n \in \mathbb{N}$.

Spherical slice transform

[Abouelaz & Daher, 1993]

$$\mathcal{S}f(\boldsymbol{\xi},\xi_3) = \int_{\langle \boldsymbol{\xi},\boldsymbol{\eta}\rangle=\xi_3} f(\boldsymbol{\eta})\,\mathrm{d}s(\boldsymbol{\eta})$$



Circles through the north pole

- Stereographic projection turns circles into lines in the plane
  ↗ Radon transform in $\mathbb{R}^2$
- Injective if f is differentiable and vanishes at $(0,0,1)$
  [Helgason, 1999]
- Injective for all functions $L^2(\mathbb{S}^2)$ vanishing around $(0,0,1)$
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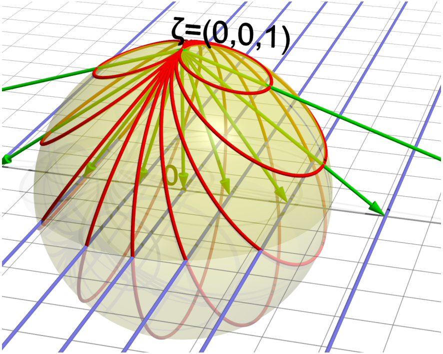

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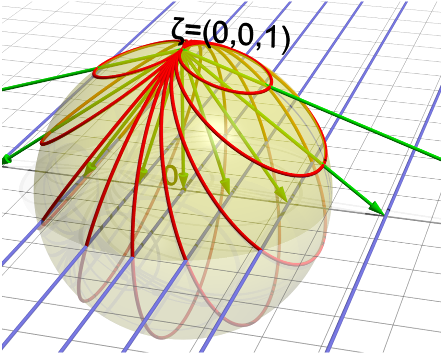

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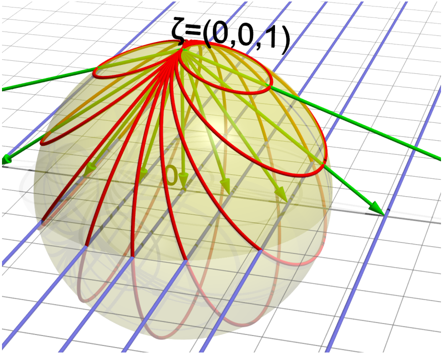

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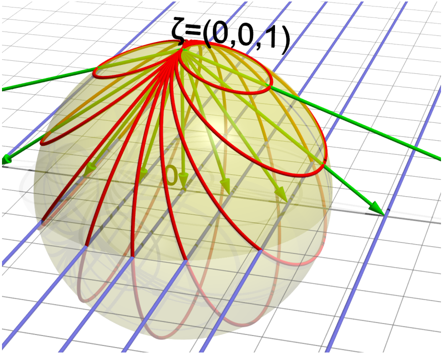

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Planes through a fixed point

[Salman, 2016]

Consider an arbitrary point inside the sphere:

$$(0,0,z), \qquad 0 \le z < 1$$

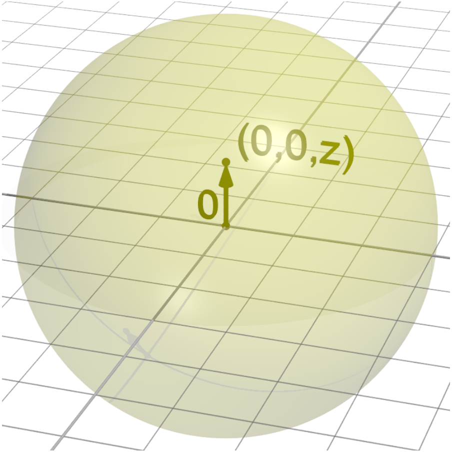
Plane section through $(0,0,z)$ is

$$\{\boldsymbol{\eta} \in \mathbb{S}^2 : \langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = z\xi_3\}.$$

Definition

$$\mathcal{U}_z : C(\mathbb{S}^2) \to C(\mathbb{S}^2),$$
$$\mathcal{U}_z f(\boldsymbol{\xi}) = \int_{\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = z\xi_3} f(\boldsymbol{\eta}) \, \mathrm{d}\lambda(\boldsymbol{\eta})$$

$z = 0$: Funk–Radon transform
$z = 1$: Spherical slice transform



Planes through a fixed point

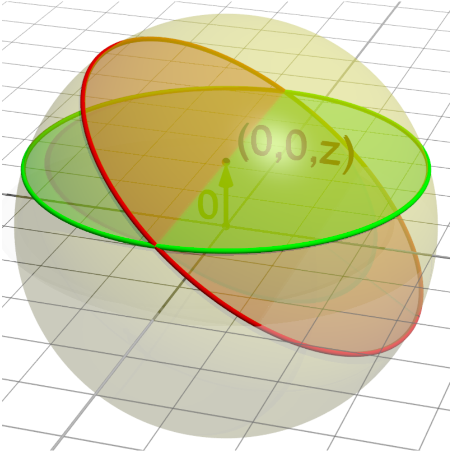
[Salman, 2016]

Consider an arbitrary point inside the sphere:

$$(0,0,z), \qquad 0 \le z < 1$$

Plane section through $(0,0,z)$ is

$$\{\boldsymbol{\eta} \in \mathbb{S}^2 : \langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle = z\xi_3\}.$$



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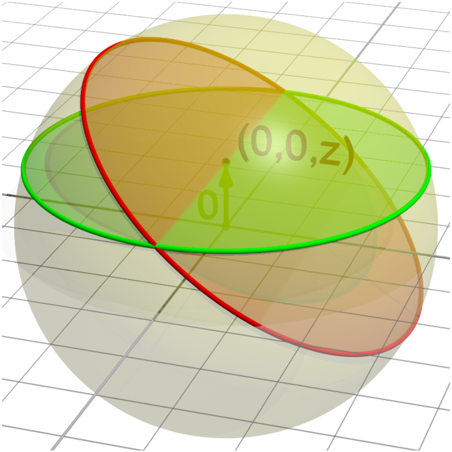

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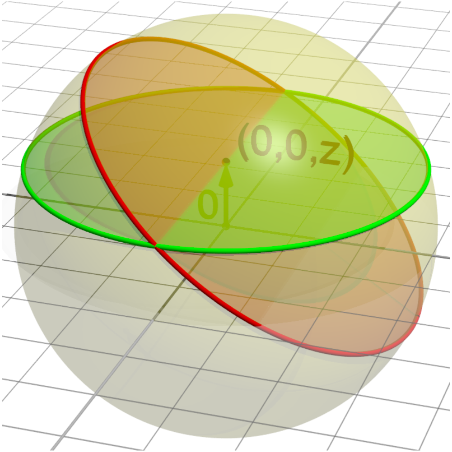

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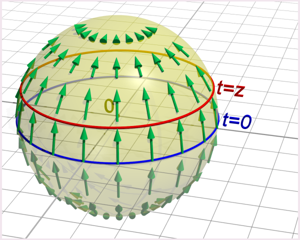
From great circles to small circles

Definition

Define the conformal map $h\colon \mathbb{S}^2 \to \mathbb{S}^2$

$$h(\boldsymbol{\xi}) = \pi^{-1}\left(\sqrt{\frac{1+z}{1-z}}\,\pi(\boldsymbol{\xi})\right)$$



consisting of

1. Stereographic projection $\pi\colon \mathbb{S}^2 \to \mathbb{R}^2$
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We are going to see that

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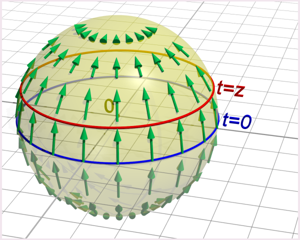

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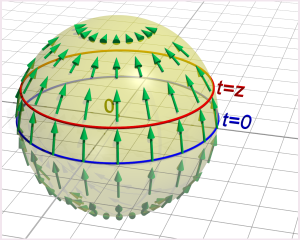

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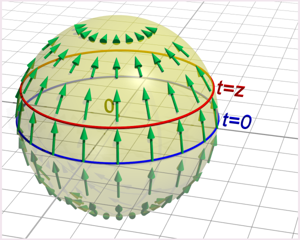

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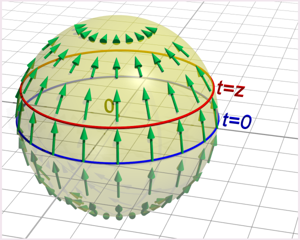

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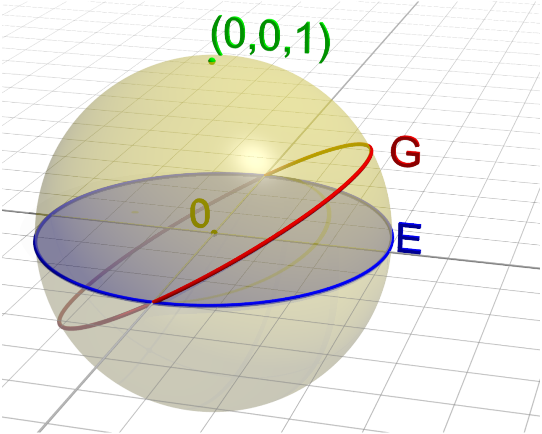
1) Stereographic projection π

- G ... Great circle of $\mathbb{S}^2$
- E ... Equator of $\mathbb{S}^2$

- G intersects E in two antipodal points (or is identical to E)

- $\pi(E) = E$
- $\pi(G)$ is a circle or line in $\mathbb{R}^2$ that intersects $\pi(E)$ in two antipodal points



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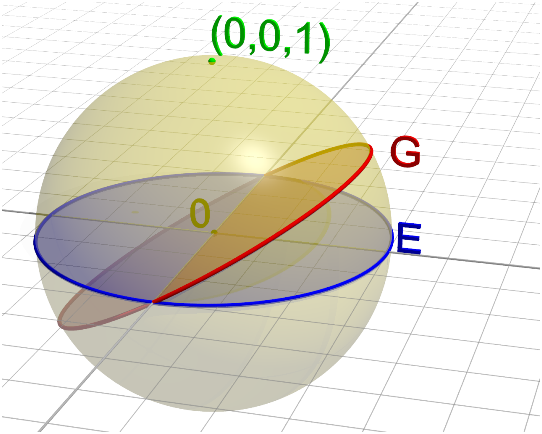

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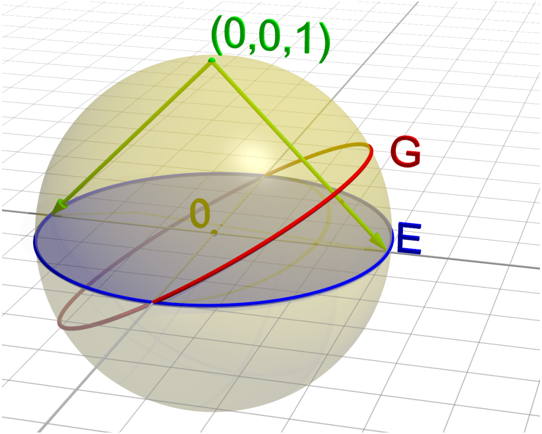

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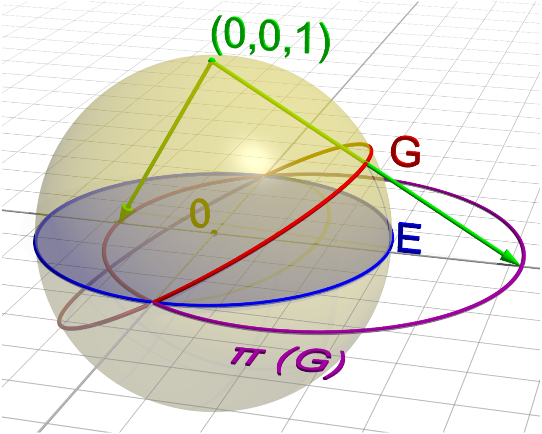
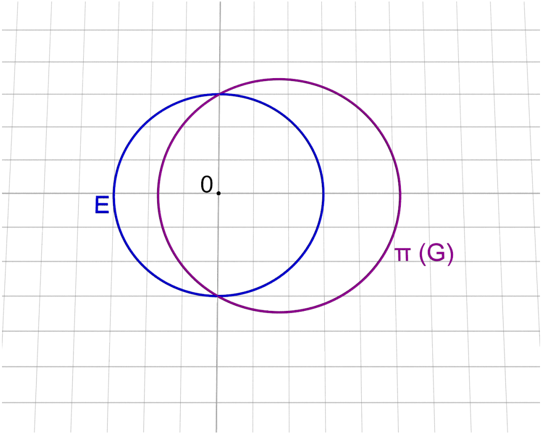


2) Scaling σ in the plane

- Uniform scaling with scale factor $s = \sqrt{\frac{1+z}{1-z}}$
- Unit circle E is mapped to the circle $\sigma(\pi(E))$ with radius s
- $\sigma(\pi(G))$ intersects $\sigma(\pi(E))$ in two antipodal points



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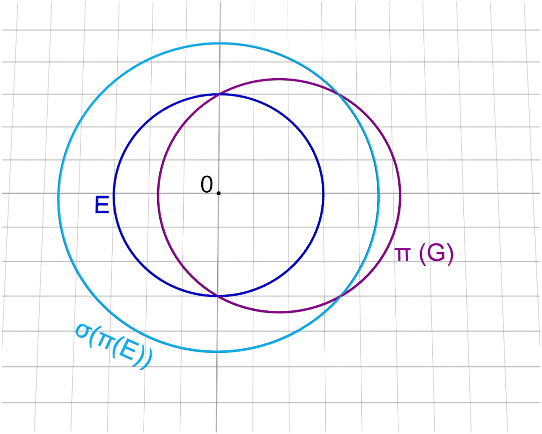

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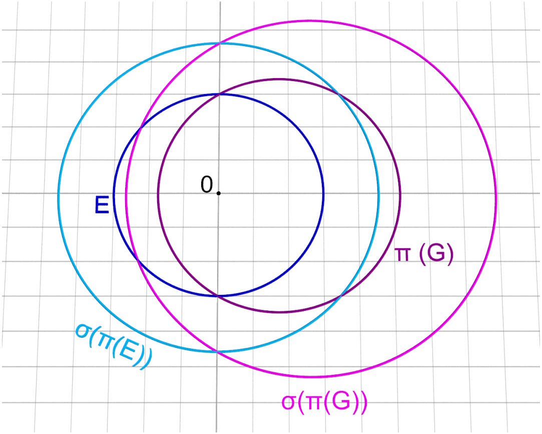
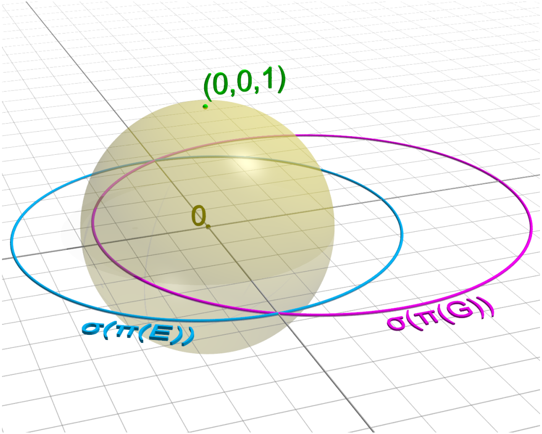


3) Inverse stereographic projection π^{-1}

- The circle with radius s is mapped to the circle of latitude z; $h(E)$
- $h(G) = \pi^{-1}(\sigma(\pi(G)))$ intersects $h(E)$ in two antipodal points
- $h(G)$ is a small circle through $(0,0,z)$



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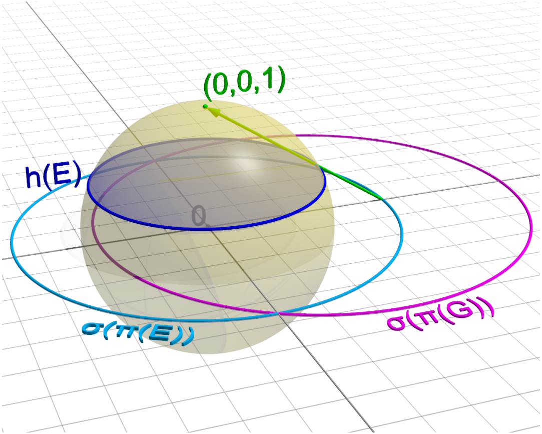

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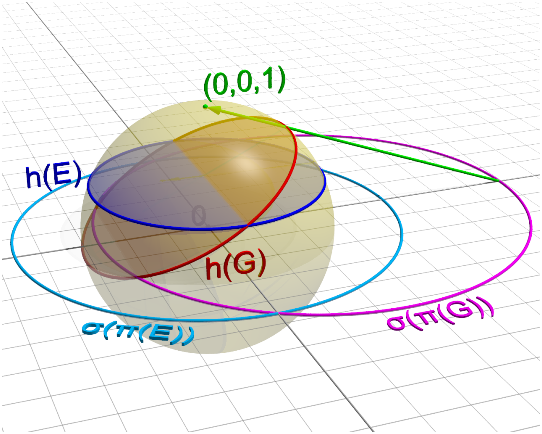

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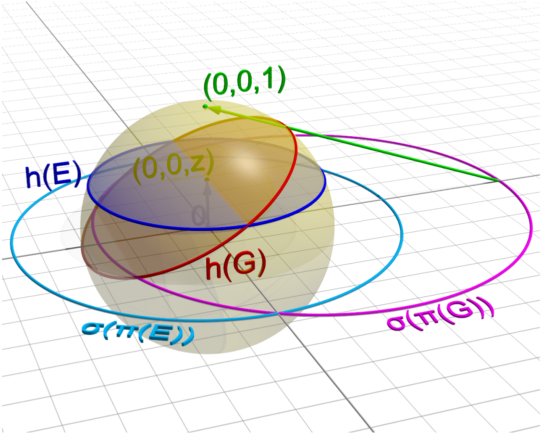
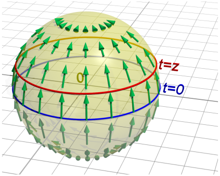


Theorem [Q., 2017]

Let $z \in [0,1)$. The generalized Radon transform $\mathcal{U}_z$ can be represented through

$$\mathcal{U}_z = \mathcal{N}_z \mathcal{F} \mathcal{M}_z.$$

These operators are defined for $f \in C(\mathbb{S}^2)$ by



- $\mathcal{M}_z f(\boldsymbol{\xi}) = \frac{\sqrt{1-z^2}}{1+z\xi_3}[f \circ h](\boldsymbol{\xi})$

  $h(\boldsymbol{\xi}) = \frac{\sqrt{1-z^2}}{1+z\xi_3}(\xi_1 \boldsymbol{e}^1 + \xi_2 \boldsymbol{e}^2) + \frac{z+\xi_3}{1+z\xi_3}\boldsymbol{e}^3$
- $\mathcal{F}$ ... Funk–Radon transform
- $\mathcal{N}_z f(\boldsymbol{\xi}) = f\left(\frac{1}{\sqrt{1-z^2\xi_3^2}}\left(\xi_1, \xi_2, \sqrt{1-z^2}\xi_3\right)\right)$

Nullspace of $\mathcal{U}_z$

Theorem [Q., 2017]

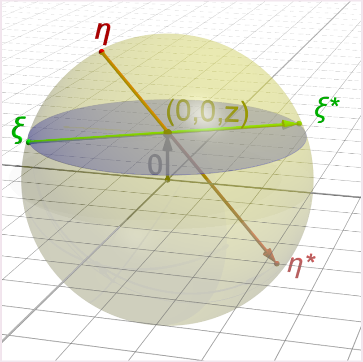
For $\xi \in \mathbb{S}^2$, we define $\xi^* \in \mathbb{S}^2$ as the point reflection of the sphere about the point $(0, 0, z)$.

Let $f \in L^2(\mathbb{S}^2)$. Then

$$\mathcal{U}_z f = 0$$

if and only if for almost every $\xi \in \mathbb{S}^2$

$$f(\xi) = -\frac{1 - z^2}{1 + z^2 - 2z\eta_3} f(\xi^*).$$



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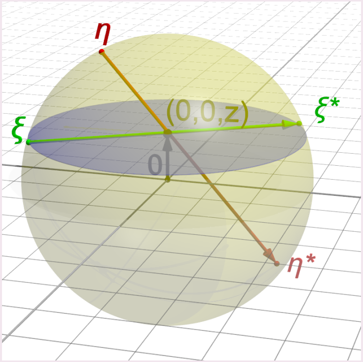

Range of $\mathcal{U}_z$

Theorem [Q., 2017]

The generalized Radon transform

$$\mathcal{U}_z \colon \widetilde{L}^2_{\mathrm{e}}(\mathbb{S}^2) \to H^{1/2}_{\mathrm{e}}(\mathbb{S}^2)$$

is continuous and bijective.

- $\widetilde{L}^2_{\mathrm{e}}(\mathbb{S}^2) = \left\{ f \in L^2(\mathbb{S}^2) \mid f(\boldsymbol{\eta}) = f(\boldsymbol{\eta}^*) \dfrac{1-z^2}{1+z^2-2z\eta_3} \right\}$
- $H^{1/2}_{\mathrm{e}}(\mathbb{S}^2)$ ... Sobolev space of smoothness $1/2$ that contains only even functions

Content

Motivation: Spherical surface wave tomography

- Seismic waves propagate along the surface of the earth
- Speed of propagation depends on the position on $\mathbb{S}^2$

Method

- Measure the traveltimes of surface waves between many pairs of epicenter and detector
- Reconstruct the local speed of propagation

Assumption

A wave propagates along the arc of a great circle.

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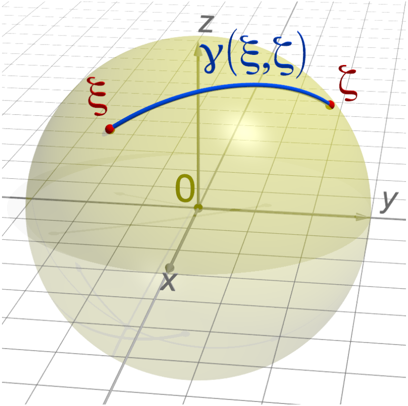
The arc transform

- Function $f\colon \mathbb{S}^2 \to \mathbb{R}$
  - Surface waves: $f = \frac{1}{c}$ (c ... speed of sound)
- $\xi, \zeta \in \mathbb{S}^2$ not antipodal
- $\gamma(\xi, \zeta)$ great circle arc



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$$\mathcal{B}f(\xi,\zeta) = \int_{\gamma(\xi,\zeta)} f \,\mathrm{d}\gamma$$

$\mathcal{B}\colon C(\mathbb{S}^2) \to C(\mathbb{S}^2 \times \mathbb{S}^2)$ is not continuous

We choose a different parameterization

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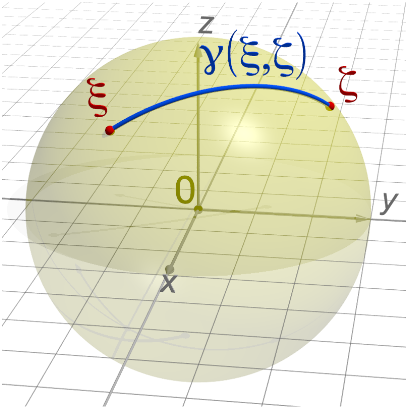

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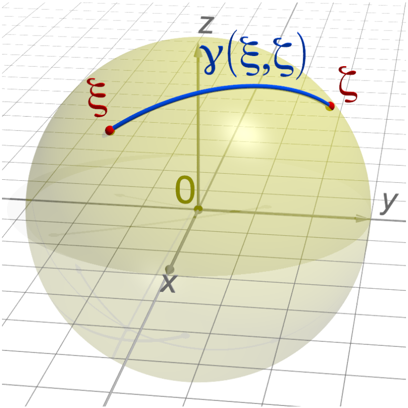

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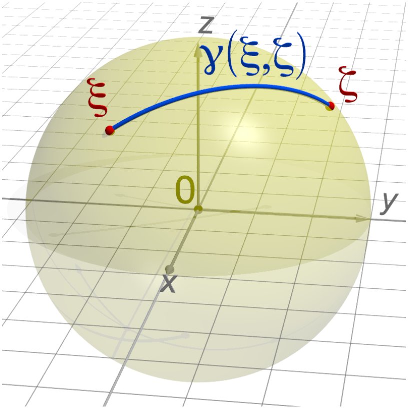

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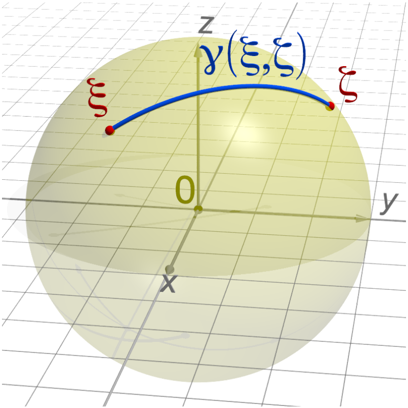

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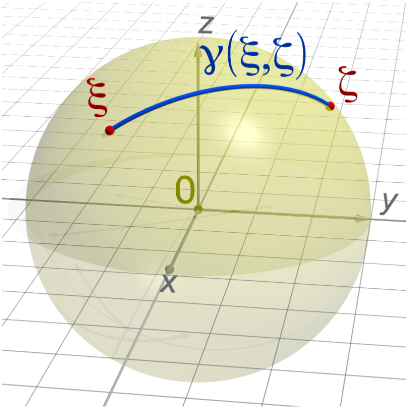

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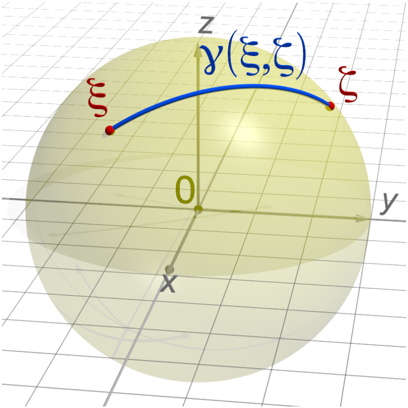

Definition

$$\mathcal{B}f(\xi,\zeta) = \int_{\gamma(\xi,\zeta)} f \,\mathrm{d}\gamma$$

$\mathcal{B}\colon C(\mathbb{S}^2) \to C(\mathbb{S}^2 \times \mathbb{S}^2)$ is not continuous

We choose a different parameterization

The arc transform: alternative parameterization

- $\psi = \arccos(\boldsymbol{\xi}^\top \boldsymbol{\zeta})$ ... length of γ
- $Q \in \mathrm{SO}(3)$ such that
  - $Q\boldsymbol{\xi} = \boldsymbol{e}_{-\psi/2}$ and
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  where $\boldsymbol{e}_\psi = (\sin\psi, \cos\psi, 0)$

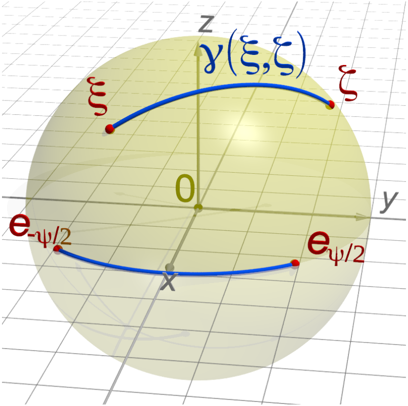
Definition

$$\mathcal{A}\colon C(\mathbb{S}^2) \to C(\mathrm{SO}(3) \times [0, 2\pi]),$$

$$\mathcal{A}f(Q,\psi) = \int_{-\psi/2}^{\psi/2} f(Q^{-1}\boldsymbol{e}_\varphi)\,\mathrm{d}\varphi$$

Theorem

$\mathcal{A}\colon C(\mathbb{S}^2) \to C(\mathrm{SO}(3) \times [0, 2\pi])$ is continuous.



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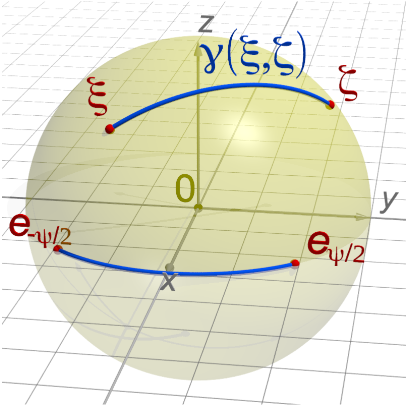

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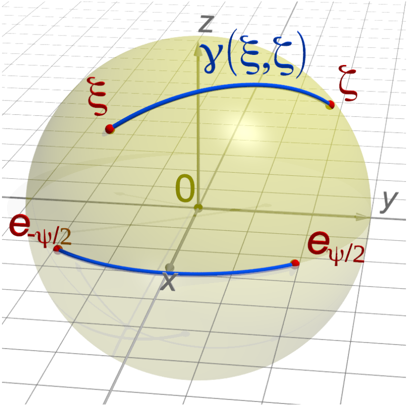

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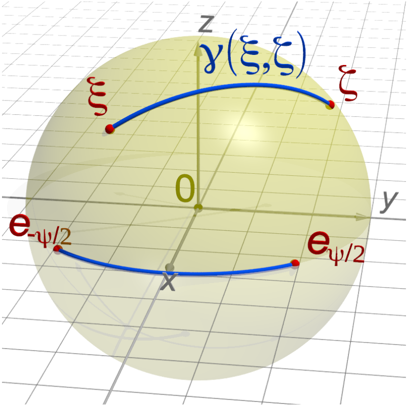

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Notation: On the rotation group $\mathrm{SO}(3)$

- Rotation group

$$\mathrm{SO}(3) = \{Q \in \mathbb{R}^{3\times 3} : Q^{-1} = Q^\top,\ \det(Q) = 1\}$$

- Orthogonal basis on $L^2(\mathrm{SO}(3))$: **rotational harmonics** (Wigner D-functions)

$$D_n^{j,k}(Q) = \int_{\mathbb{S}^2} Y_n^k(Q^{-1}\xi)\,\overline{Y_n^j(\xi)}\,\mathrm{d}\xi$$

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Theorem [Dahlen & Tromp 1998]

Let $n \in \mathbb{N}$ and $k \in \{-n, \ldots, n\}$. Then

$$\mathcal{A}Y_n^k(Q, \psi) = \sum_{j=-n}^{n} \widetilde{P}_n^j(0)\, D_n^{j,k}(Q)\, s_j(\psi),$$

where

$$s_j(\psi) = \begin{cases} \psi, & j = 0 \\ \frac{2\sin(j\psi/2)}{j}, & j \neq 0 \end{cases}$$

and

$$\widetilde{P}_n^j(0) = \begin{cases} (-1)^{\frac{n+j}{2}} \sqrt{\frac{2n+1}{4\pi} \frac{(n-j-1)!!(n+j-1)!!}{(n-j)!!(n+j)!!}}, & n+j \text{ even} \\ 0, & n+j \text{ odd.} \end{cases}$$

Singular value decomposition [Hielscher, Potts, Q. 2017]

The operator $\mathcal{A}\colon L^2(\mathbb{S}^2) \to L^2(\mathrm{SO}(3) \times [0, 2\pi])$ is compact with the singular value decomposition

$$\mathcal{A} Y_n^k = \sigma_n \, E_n^k, \quad n \in \mathbb{N},\ k \in \{-n, \ldots, n\},$$

with singular values

$$\sigma_n = \sqrt{\frac{32\pi^3}{2n+1}} \sqrt{\frac{\pi^2}{3} \left| \widetilde{P}_n^0(0) \right|^2 + \sum_{j=1}^{n} \frac{1}{j^2} \left| \widetilde{P}_n^j(0) \right|^2} \in \mathcal{O}\left(\frac{1}{\sqrt{n}} \right)$$

and the orthonormal functions in $L^2(\mathrm{SO}(3) \times [0, 2\pi])$

$$E_k^n = \sigma_n^{-1} \sum_{j=-n}^{n} \widetilde{P}_n^j(0) \, D_n^{j,k}(Q) \, s_j(\psi).$$

Arcs from the north pole

- Fix one endpoint of the arcs as the north pole e^3:

$$\mathcal{B}f(\boldsymbol{\xi}(\varphi,\vartheta)) = \int_{\gamma(\boldsymbol{e}^3,\,\boldsymbol{\xi}(\varphi,\vartheta))} f \,\mathrm{d}\gamma$$

- Then f can be recovered from $\mathcal{B}f$ by

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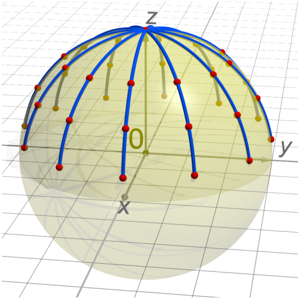

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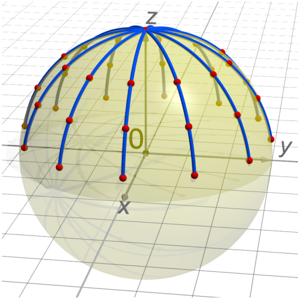


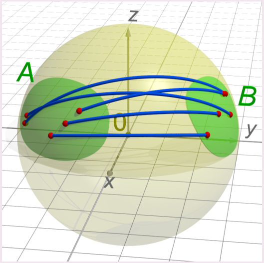
Arcs between two sets

[Amirbekyan 2007]

Let $A, B \subset \mathbb{S}^2$ nonempty. If $f \in C(\mathbb{S}^2)$ and

$$\int_{\gamma(\boldsymbol{\xi},\boldsymbol{\zeta})} f \, \mathrm{d}\gamma = 0 \qquad \forall \boldsymbol{\xi} \in A, \boldsymbol{\zeta} \in B,$$

then $f \equiv 0$ on $\overline{A \cup B}$.



Arcs from the boundary of a set

[Hielscher, Potts, Q. 2017]

Let $\Omega \subset \mathbb{S}^2$ be convex and strictly contained in a hemisphere. If $f \in C(\mathbb{S}^2)$ and

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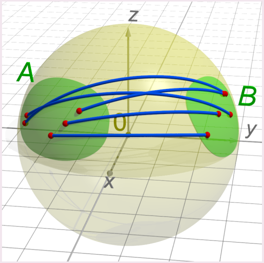

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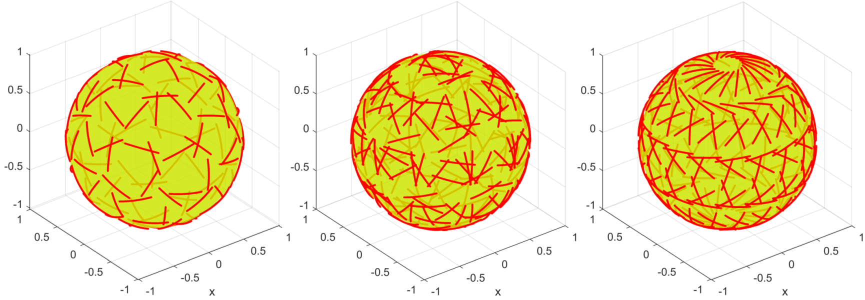
Arcs with fixed length

We fix the arclength $\psi \in [0, 2\pi]$ and define

$$\mathcal{A}_\psi = \mathcal{A}(\cdot, \psi) \colon L^2(\mathbb{S}^2) \to L^2(\mathrm{SO}(3)).$$



Singular Value Decomposition

[Hielscher, Potts, Q. 2017]

Let $\psi \in (0, 2\pi)$ be fixed. The operator $\mathcal{A}_\psi \colon L^2(\mathbb{S}^2) \to L^2(\mathrm{SO}(3))$ has the SVD

$$\mathcal{A}_\psi Y_n^k = \mu_n(\psi)\, Z_{n,\psi}^k, \quad n \in \mathbb{N},\, k \in \{-n, \ldots, n\},$$

with singular values

$$\mu_n(\psi) = \sqrt{\sum_{j=-n}^{n} \frac{8\pi^2}{2n+1} \left| \widetilde{P}_n^j(0) \right|^2 s_j(\psi)^2}$$

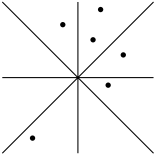
and singular functions

$$Z_{n,\psi}^k = \frac{1}{\mu_n(\psi)} \sum_{j=-n}^{n} \widetilde{P}_n^j(0)\, s_j(\psi)\, D_n^{j,k} \in L^2(\mathrm{SO}(3)).$$

Hence $\mathcal{A}_\psi$ is injective.

Singular values $\mu_n(\psi)$: dependency on n

$\psi = 0.02\,\pi$



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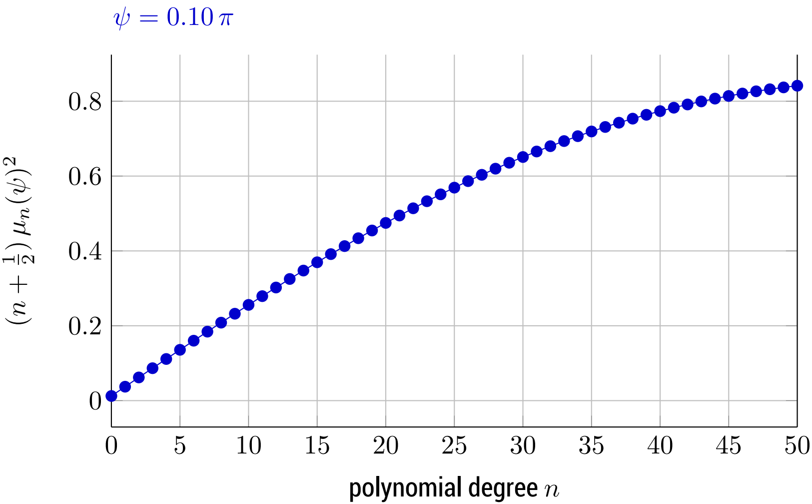

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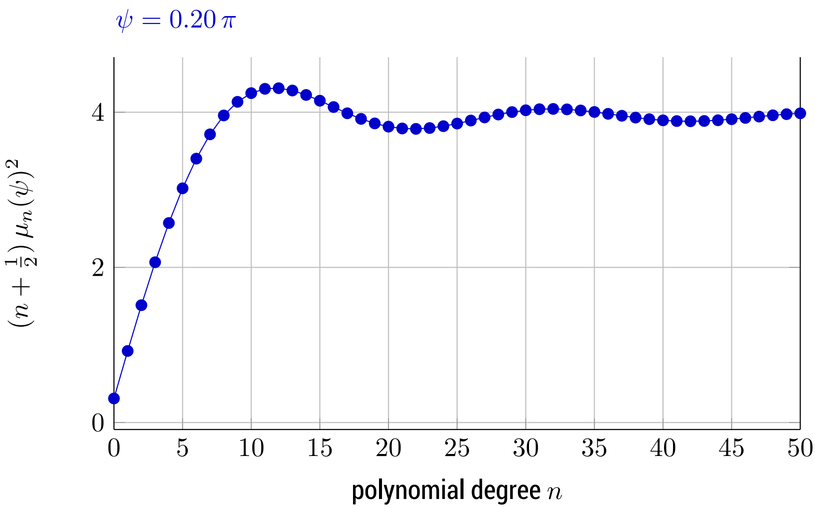

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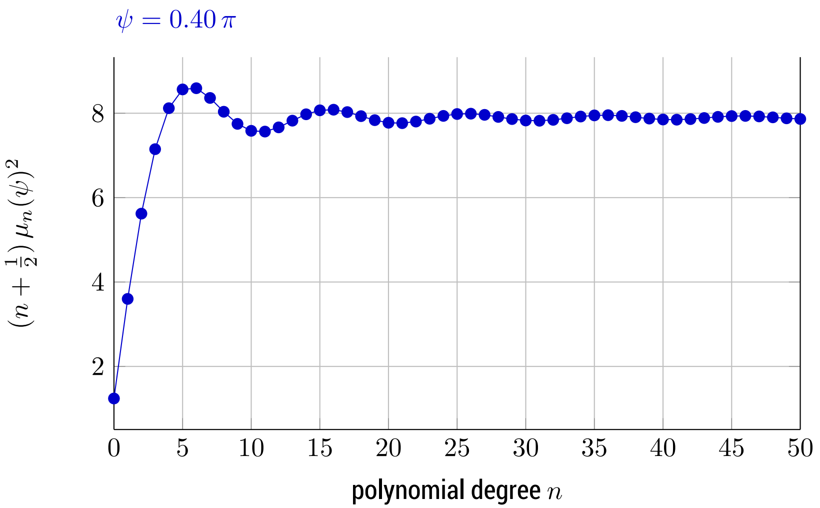
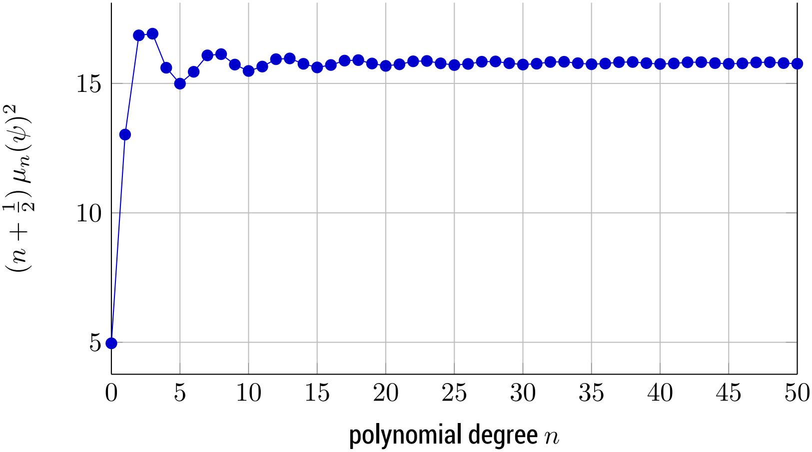


Singular values $\mu_n(\psi)$: dependency on n

$\psi = 1.00\,\pi$ **(half circle)**



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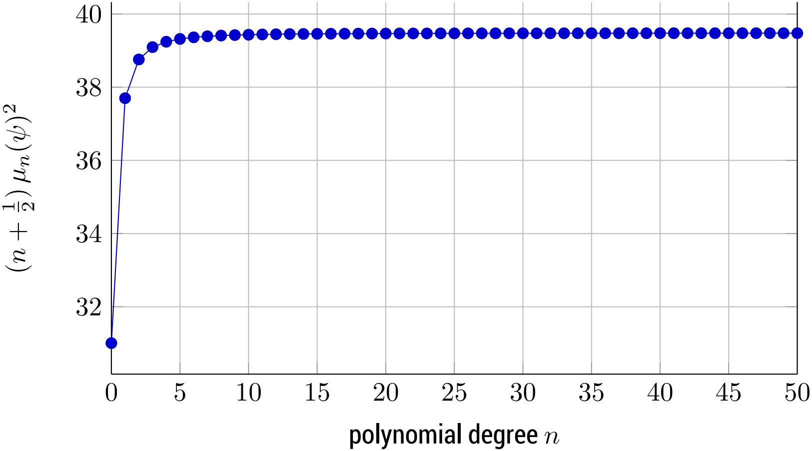
$\psi = 1.04\,\pi$



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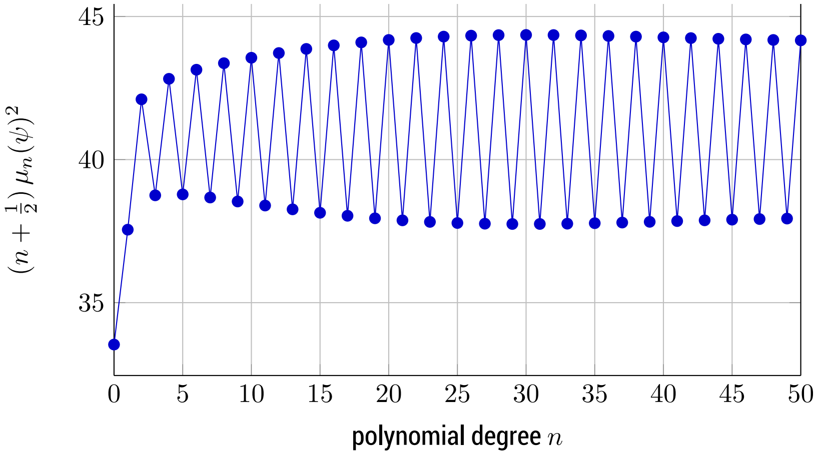
$\psi = 1.90\,\pi$



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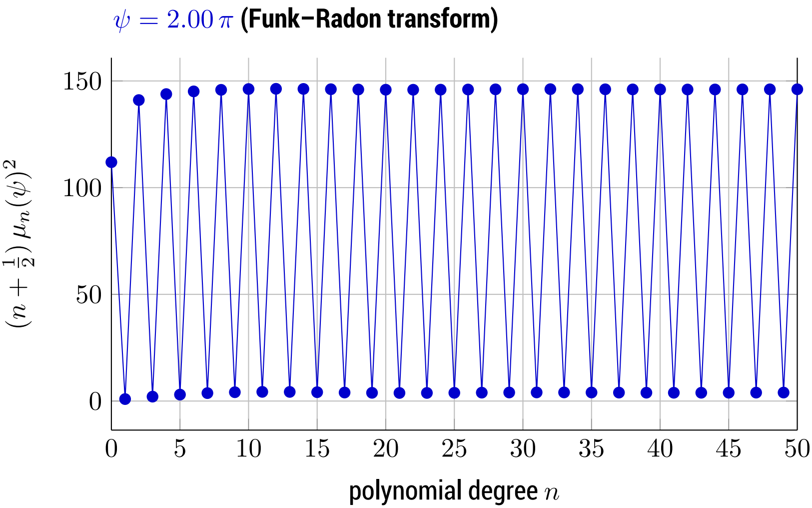
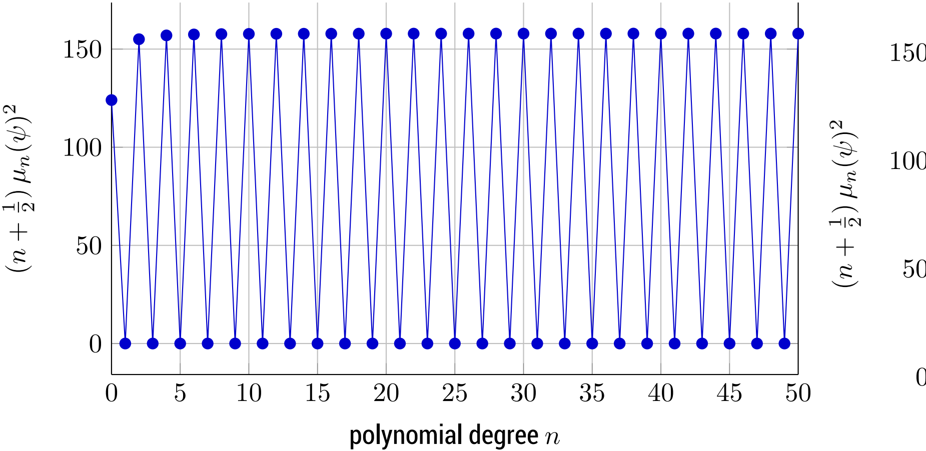


Singular values $\mu_n(\psi)$: dependency on arc-length ψ



Singular values: asymptotic behavior

Theorem

[Hielscher, Potts, Q. 2017]

The singular values $\mu_n(\psi)$ of $\mathcal{A}_\psi$ satisfy

$$\lim_{\substack{n\to\infty \\ n \text{ even}}} n\,\mu_n(\psi)^2 = \begin{cases} 2\pi\psi, & \psi \in [0,\pi] \\ 12\pi\psi - 2\pi^2, & \psi \in [\pi, 2\pi], \end{cases}$$

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Special cases

- $\psi = 2\pi$: **Funk–Radon transform**: Injective only for even functions
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  half circles in one hemisphere [Rubin 2017]

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Content

3D transforms

Cone-beam transform (divergent beam X-ray transform), for a scanning set $\Gamma \subset \mathbb{R}^d$

$$\mathcal{D}_{\boldsymbol{a}} f(\boldsymbol{\omega}) = \int_0^\infty f(\boldsymbol{a} + t\boldsymbol{\omega})\,\mathrm{d}t,$$
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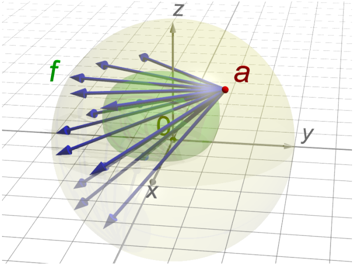

Radon transform

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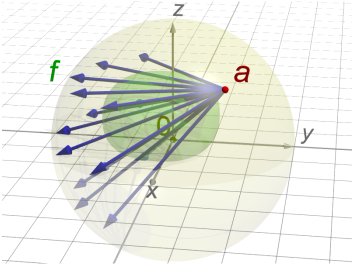
3D transforms



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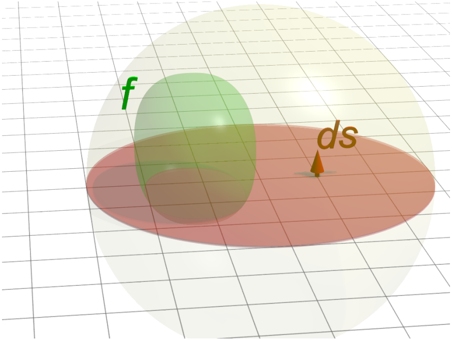
Cone-beam and Radon transform

Grangeat's formula

[Grangeat, 1991]

$$\frac{\partial}{\partial s}\mathcal{R}f(\boldsymbol{\omega}, \boldsymbol{a}^\top\boldsymbol{\omega}) = \int_{\boldsymbol{\xi}\in\mathbb{S}^2, \boldsymbol{\xi}^\top\boldsymbol{\omega}=0} \frac{\partial}{\partial\boldsymbol{\omega}}\mathcal{D}_{\boldsymbol{a}}f(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}$$



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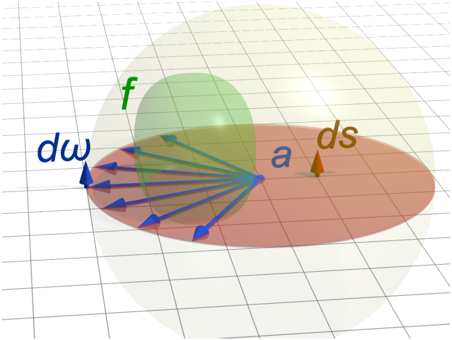

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$$\mathcal{S}^{(j)}f(\boldsymbol{\xi}) = \int_{\mathbb{S}^2} \delta^{(j)}(\boldsymbol{\xi}^\top\boldsymbol{\eta})\, f(\boldsymbol{\eta})\,\mathrm{d}\boldsymbol{\eta}, \qquad \boldsymbol{\xi}\in\mathbb{S}^2,$$

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$$\frac{\partial}{\partial s}\mathcal{R}f(\boldsymbol{\omega}, s)|_{s=\boldsymbol{a}^\top\boldsymbol{\omega}} = -\mathcal{S}^{(1)}\mathcal{D}_{\boldsymbol{a}}f(\boldsymbol{\omega})$$

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Grangeat's formula becomes

$$\frac{\partial}{\partial s}\mathcal{R}f(\boldsymbol{\omega}, s)|_{s=\boldsymbol{a}^\top\boldsymbol{\omega}} = -\mathcal{S}^{(1)}\mathcal{D}_{\boldsymbol{a}} f(\boldsymbol{\omega})$$

$$-\left(\mathcal{S}^{(1)}\right)^{-1} \frac{\partial}{\partial s}\mathcal{R}f(\boldsymbol{\omega}, s)|_{s=\boldsymbol{a}^\top\boldsymbol{\omega}} = \mathcal{D}_{\boldsymbol{a}} f(\boldsymbol{\omega})$$

Generalized Funk–Radon transform

Theorem

Let $j \in \mathbb{N}$. The generalized Funk–Radon transform $\mathcal{S}^{(j)} \colon C(\mathbb{S}^2) \to C(\mathbb{S}^2)$ satisfies the eigenvalue decomposition

$$\mathcal{S}^{(j)} Y_n^k = P_n^{(j)}(0)\, Y_n^k, \qquad n \in \mathbb{N},\ k = -n, \dots, n$$

with eigenvalues

$$P_n^{(j)}(0) = \begin{cases} 2\pi\,(-1)^{\frac{n-j}{2}} \dfrac{(n+j-1)!!}{(n-j)!!}, & n+j \text{ even and } (n \geq j) \\ 0, & \text{otherwise} \end{cases}$$

for $j = 1$: [Makai, Martini, Odor, 2000]
general: [Q., Hielscher, Louis, 2018]

Cone-beam transform

Singular value decomposition

[Maaß, 1987] [Kazantsev, 2015] [Q., Hielscher, Louis, 2018]

The cone-beam transform $\mathcal{D}$ with sources $\boldsymbol{a}$ on the sphere $\mathbb{S}^2$

$$\mathcal{D}\widetilde{V}_{m,l,k}(\boldsymbol{a},\boldsymbol{\omega}) =$$

$$\frac{4\pi}{\sqrt{2m+3}} \sum_{j=-m-1}^{m+1} \overline{Y_{m+1}^{j}(\boldsymbol{a})} \sideset{}{'}\sum_{n=|m+1-l|}^{l+m+1} \frac{(-1)^{\frac{n+1}{2}}\,(n-1)!!}{n!!} G_{m+1,j,l,k}^{n,j+k} Y_n^{j+k}(\boldsymbol{\omega})$$

with the ball polynomials

$$\widetilde{V}_{m,l,k}(s\boldsymbol{\omega}) = \sqrt{2m+3}\, s^l P_{\frac{m-l}{2}}^{(0,l+\frac{1}{2})}(2s^2-1)\, Y_l^k(\boldsymbol{\omega}), \qquad s \in [0,1],\ \boldsymbol{\omega} \in \mathbb{S}^2,$$

and the Gaunt coefficients $G_{n_1,k_1,n_2,k_2}^{n,k} = \int_{\mathbb{S}^2} Y_{n_1}^{k_1}(\boldsymbol{\xi})\, Y_{n_2}^{k_2}(\boldsymbol{\xi})\, \overline{Y_n^k(\boldsymbol{\xi})}\, \mathrm{d}\boldsymbol{\xi}$.

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