



The cone-beam transform and spherical convolution operators

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(joint work with Ralf Hielscher and Alfred K. Louis)

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Content

1. The generalized Funk–Radon transform

Definition

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2. Cone-beam transform

Cone-beam and Radon transform in 3D

Connection with the Radon transform

Singular value decomposition

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Funk–Radon transform

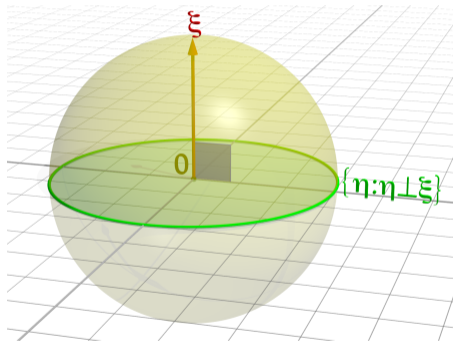
[Funk, 1911]

- ▶ Sphere $\mathbb{S}^{d-1} = \{\xi \in \mathbb{R}^d : \|\xi\| = 1\}$
- ▶ Function $f: \mathbb{S}^{d-1} \rightarrow \mathbb{C}$
- ▶ Funk–Radon transform

$$\begin{aligned} \mathcal{S}^{(0)} f(\xi) &= \int_{\mathbb{S}^{d-1}} \delta(\xi^\top \eta) f(\eta) d\eta \\ &= \int_{\xi^\top \eta=0} f(\eta) d\lambda(\eta) \end{aligned}$$

(integrals of f along all great circles)

- ▶ Take derivatives of the delta function



Funk–Radon transform

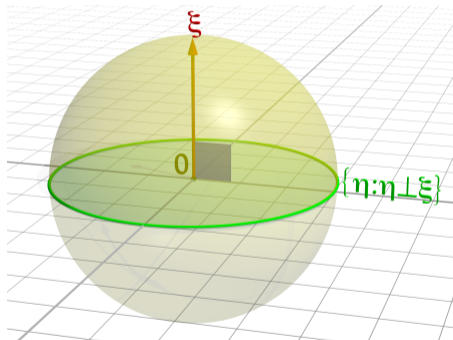
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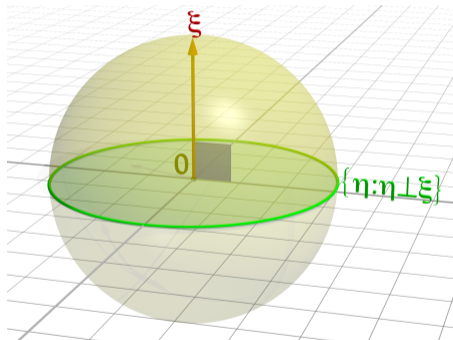
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Generalized Funk–Radon transform

[Louis, 2016]

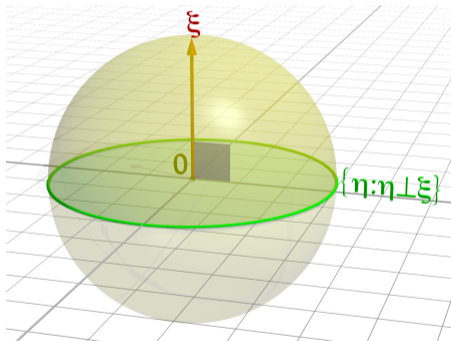
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$$\begin{aligned} \mathcal{S}^{(j)} f(\boldsymbol{\xi}) &= \int_{\mathbb{S}^{d-1}} \delta^{(j)}(\boldsymbol{\xi}^\top \boldsymbol{\eta}) f(\boldsymbol{\eta}) \, d\boldsymbol{\eta} \\ &= (-1)^j \int_{\boldsymbol{\xi}^\top \boldsymbol{\eta}=0} \left(\frac{\partial}{\partial \boldsymbol{\xi}} \right)^j f(\boldsymbol{\eta}) \, d\lambda(\boldsymbol{\eta}) \end{aligned}$$

► $\frac{\partial}{\partial \boldsymbol{\xi}}$... directional derivative

► Similar definition for $j = 1$:

[Makai, Martini, Odor, 2000]



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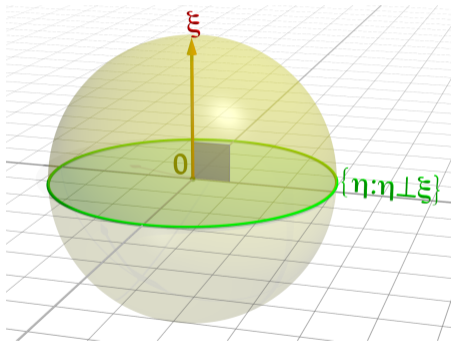
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Spherical harmonics: An orthonormal basis on \mathbb{S}^{d-1}

The spherical harmonics

$$Y_n^k : \mathbb{S}^{d-1} \rightarrow \mathbb{C}, \quad n \in \mathbb{N}_0, \quad k = 1, \dots, N_{n,d}$$

form an orthonormal basis of $L^2(\mathbb{S}^{d-1})$.

Any $f \in L^2(\mathbb{S}^{d-1})$ can be written as series

$$f = \sum_{n=0}^{\infty} \sum_{k=1}^{N_{n,d}} \hat{f}(n, k) Y_n^k, \quad \hat{f}(n, k) := \int_{\mathbb{S}^{d-1}} f(\xi) \overline{Y_n^k(\xi)} \, d\xi$$

Fast algorithms for spherical Fourier transforms on \mathbb{S}^2

[Driscoll & Healy, 1994] [Potts, Steidl & Tasche, 1998] [Kunis & Potts, 2003] [Schaeffer, 2013]

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Spherical convolution

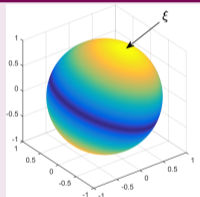
Funk–Hecke formula

[Funk, 1915] [Hecke, 1917]

Let $g: [-1, 1] \rightarrow \mathbb{C}$. Then

$$\int_{\mathbb{S}^{d-1}} Y_n^k(\boldsymbol{\eta}) g(\boldsymbol{\xi}^\top \boldsymbol{\eta}) d\boldsymbol{\eta} = Y_n^k(\boldsymbol{\xi}) \int_{-1}^1 g(t) P_{n,d}(t) (1-t^2)^{\frac{d-3}{2}} dt$$

$P_{n,d}$ – Legendre (ultraspherical) polynomial of degree n in dimension d
 orthogonal polynomial on $[-1, 1]$ w.r.t. the weight $(1-t^2)^{\frac{d-3}{2}}$


 $g(\boldsymbol{\xi}^\top \cdot)$

For the generalized Funk–Radon transform: Insert $g(t) = \delta^{(j)}(t)$

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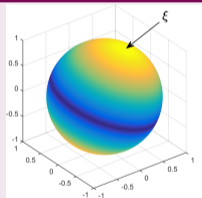
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Eigenvalue decomposition of the generalized Funk–Radon transform

Theorem

[Q., Hielscher, Louis, 2018]

Let $j \in \mathbb{N}_0$. The generalized Funk–Radon transform $\mathcal{S}^{(j)} : C(\mathbb{S}^{d-1}) \rightarrow C(\mathbb{S}^{d-1})$ satisfies the eigenvalue decomposition

$$\mathcal{S}^{(j)} Y_n^k = P_{n,d}^{(j)}(0) Y_n^k, \quad n \in \mathbb{N}_0, k = -n, \dots, n$$

with eigenvalues

$$P_{n,d}^{(j)}(0) = \begin{cases} |\mathbb{S}^{d-2}| (-1)^{\frac{n+j}{2}} \frac{(n+j-1)!! (d-3)!!}{(n-j+d-3)!!}, & n+j \text{ even and } (n \geq j) \\ 0, & \text{otherwise} \end{cases}$$

Decay of the eigenvalues of $\mathcal{S}^{(j)}$

Lemma

Let $j \in \mathbb{N}_0$, $n + j$ even and $n \geq j$. The eigenvalues of $\mathcal{S}^{(j)}$ satisfy

$$\left| P_{n,d}^{(j)}(0) \right| \simeq n^{j - \frac{d-2}{2}} \pi^{\frac{d-1}{2}} 2^{\frac{d}{2}} \quad \text{for } n \rightarrow \infty.$$

Theorem

[Q., Hielscher, Louis, 2018]

Let $s \in \mathbb{R}$ and $j \in \mathbb{N}_0$. The generalized Funk–Radon transform $\mathcal{S}_d^{(j)}$ extends to a continuous operator

$$\mathcal{S}^{(j)} : H^s(\mathbb{S}^{d-1}) \rightarrow H^{s-j+\frac{d-2}{2}}(\mathbb{S}^{d-1}).$$

The nullspace of $\mathcal{S}^{(j)}$ is the closed linear span

$$\overline{\text{span}} \left\{ Y_n^k : n + j \text{ odd or } (n \leq j - d + 1 \text{ and } d \text{ odd}), k = 1, \dots, N_{n,d} \right\}.$$

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Special cases of j

$j = -1$: **modified hemispherical transform**

[Ungar 1954] [Rubin, 1999]

$$\mathcal{S}^{(-1)} f(\xi) = \frac{1}{2} \int_{\mathbb{S}^{d-1}} \operatorname{sgn}(\xi^\top \eta) f(\eta) d\eta.$$

$j = -2$: **spherical cosine transform**

[Petty, 1961] [Schneider, 1967] [Groemer, 1996]

$$\mathcal{S}^{(-2)} f(\xi) = \frac{1}{2} \int_{\mathbb{S}^{d-1}} |\xi^\top \eta| f(\eta) d\eta.$$

If $d = 2j + 2$, the absolute value of the eigenvalues

$$P_{n,d}^{(j)}(0) = |\mathbb{S}^{2j}| (-1)^{\frac{n+j}{2}} (2j-1)!!, \quad n+j \text{ even}$$

is constant. Hence $\mathcal{S}_{2j+2}^{(j)} : L^2(\mathbb{S}^{2j+1}) \rightarrow L^2(\mathbb{S}^{2j+1})$ is a partial isometry.

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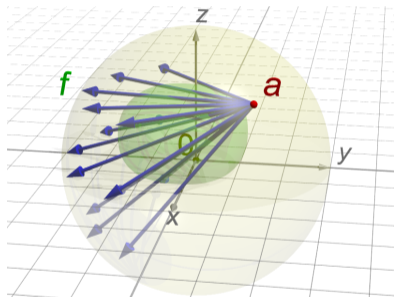
Connection with the Radon transform

Singular value decomposition

Cone-beam transform

- ▶ $f: \mathbb{R}^d \rightarrow \mathbb{R}$
- ▶ $\mathbf{a} \in \mathbb{R}^d$... source of the ray
- ▶ $\boldsymbol{\omega} \in \mathbb{S}^{d-1}$... direction of the ray
- ▶ **Cone-beam transform**
(a.k.a. divergent beam X-ray transform)

$$\mathcal{D}f(\mathbf{a}, \boldsymbol{\omega}) = \int_0^\infty f(\mathbf{a} + t\boldsymbol{\omega}) dt$$



[Hamaker et al. 1980] [Tuy, 1983] [Finch, 1985] [Feldkamp, Davis, Kress, 1984]

Radon transform

▶ $f: \mathbb{R}^d \rightarrow \mathbb{R}$

▶ **Radon transform**

$$\mathcal{R}f(\boldsymbol{\omega}, s) = \int_{\mathbf{x}^\top \boldsymbol{\omega} = s} f(\mathbf{x}) \, d\mathbf{x}$$

▶ Integral along (hyper-)plane with normal $\boldsymbol{\omega} \in \mathbb{S}^{d-1}$ and $s \in \mathbb{R}$

Cone-beam and Radon transform (in 3D)

[Grangeat, 1991]

Consider a “fan” of ray integrals orthogonal to ω

$$\int_{\xi \in \mathbb{S}^2, \xi^\top \omega = 0} \mathcal{D}f(a, \xi) d\xi$$

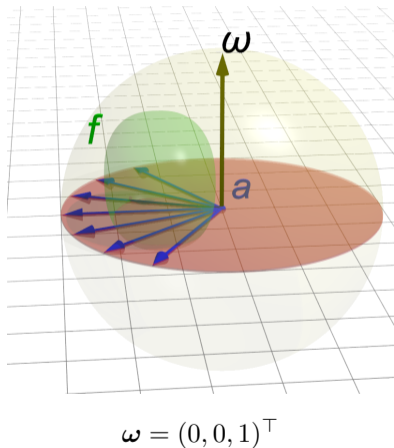
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$$\mathcal{R}f(\omega, a^\top \omega) = \int_{x^\top \omega = a^\top \omega} f(x) dx$$

Grangeat's formula

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$\frac{\partial}{\partial \omega}$... directional derivative w.r.t. ξ



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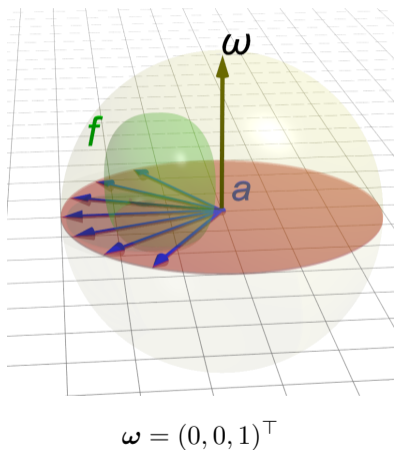
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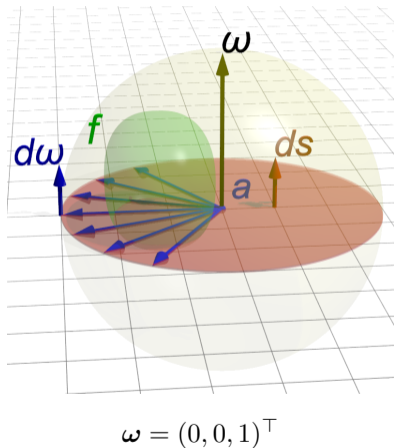
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Recall the generalized Funk–Radon transform

$$\mathcal{S}^{(j)} f(\omega) = \int_{\mathbb{S}^2} \delta^{(j)}(\xi^\top \omega) f(\xi) \, d\xi, \quad \omega \in \mathbb{S}^2.$$

Write Grangeat's formula as

[Louis, 2016]

$$\left. \frac{\partial}{\partial s} \mathcal{R}f(\omega, s) \right|_{s=\mathbf{a}^\top \omega} = -\mathcal{S}_\omega^{(1)} \mathcal{D}f(\mathbf{a}, \omega)$$

In general dimension d

$$\left. (-1)^d \left(\frac{\partial}{\partial s} \right)^{d-2} \mathcal{R}f(\omega, s) \right|_{s=\mathbf{a}^\top \omega} = \mathcal{S}_\omega^{(d-2)} \mathcal{D}f(\mathbf{a}, \omega)$$

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Singular Value decomposition of the Radon transform \mathcal{R}

Theorem

[Louis, 1984]

For $m \in \mathbb{N}_0$, $l = 0, \dots, m$ with $m + l$ even and $k = 1, \dots, N_{l,d}$

$$\mathcal{R}V_{m,l,k}(\boldsymbol{\omega}, s) = \frac{\sqrt{2m+d} \Gamma(\frac{d}{2}) m!}{2^{1-d} \pi^{1-\frac{d}{2}} (m+d-1)!} (1-s^2)^{\frac{d-1}{2}} C_m^{(\frac{d}{2})}(s) Y_l^k(\boldsymbol{\omega}),$$

where

$$V_{m,l,k}(s\boldsymbol{\omega}) = \sqrt{2m+d} s^l P_{\frac{m-l}{2}}^{(0, l + \frac{d-2}{2})}(2s^2 - 1) Y_l^k(\boldsymbol{\omega}), \quad s \in [0, 1], \boldsymbol{\omega} \in \mathbb{S}^{d-1}$$

and $P_n^{(\alpha, \beta)}$ is the Jacobi polynomial of degree n and orders $\alpha, \beta > -1$.

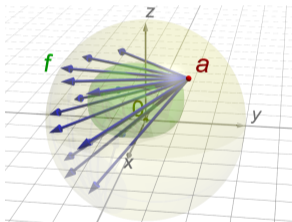
Cone-beam transform

- ▶ Let $f: \mathbb{B}^d = \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| \leq 1\} \rightarrow \mathbb{R}$
- ▶ Consider the cone-beam transform

$$\mathcal{D}: L^2(\mathbb{B}^d) \rightarrow L^2(\mathbb{S}^{d-1} \times \mathbb{S}^{d-1})$$

$$\mathcal{D}f(\mathbf{a}, \boldsymbol{\omega}) = \int_0^\infty f(\mathbf{a} + t\boldsymbol{\omega}) dt, \quad \boldsymbol{\omega} \in \mathbb{S}^{d-1}$$

with sources $\mathbf{a} \in \mathbb{S}^{d-1}$ on the sphere



Cone-beam transform

Singular value decomposition

[Q., Hielscher, Louis, 2018]

The cone-beam transform \mathcal{D} with sources \mathbf{a} on the sphere \mathbb{S}^{d-1} and d odd has the SVD

$$\mathcal{D}V_{m,l,k}(\mathbf{a}, \boldsymbol{\omega}) = \mu_{m,d} \sum_{j=1}^{N_{m+1,d}} Y_{m+1,d}^j(\mathbf{a}) \sum'_{n=m+1-l}^{l+m+1} \nu_{n,d} \sum_{i=1}^{N_{n,d}} G_{m+1,j,l,k}^{n,i,d} Y_{n,d}^i(\boldsymbol{\omega}),$$

where \sum' denotes the summation over odd indices and

$$\mu_{m,d} = \sqrt{\frac{2^{d+1} \pi^{d-1}}{2m+d}}, \quad \nu_{n,d} = \frac{(-1)^{\frac{n+1}{2}} (n-1)!!}{(n+d-3)!!}$$

and the Gaunt coefficients $G_{n_1,k_1,n_2,k_2}^{n,k} = \int_{\mathbb{S}^{d-1}} Y_{n_1}^{k_1}(\boldsymbol{\xi}) Y_{n_2}^{k_2}(\boldsymbol{\xi}) \overline{Y_n^k(\boldsymbol{\xi})} d\boldsymbol{\xi}$.

Remarks

▶ **dimension $d = 3$**

[Kazantsev, 2015]

▶ d odd

[Q., Hielscher, Louis, 2018]

▶ Similar result for full lines (X-ray transform)

[Maaß, 1987]

Decay of the singular values $\lambda_{m,l,d}$ of the cone-beam transform \mathcal{D}

▶ Lower bound: $\lambda_{m,l,d} \geq c_d m^{-1/2}$ for $m \rightarrow \infty$

▶ Upper bound: $\lambda_{m,l,d} \leq C_d$

▶ for dimension $d = 3$: $\lambda_{m,l,3} \leq C m^{-1/4}$ [Kazantsev, 2015]

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Conclusion

- ▶ Investigated the **generalized Funk–Radon transform**
- ▶ Characterization of nullspace and range
- ▶ Grangeat's formula connects the cone-beam transform, Radon transform and generalized Funk–Radon transform
- ▶ SVD of the **cone-beam transform** for sources on the sphere
- ▶ **Future:** Limited data, algorithm for the cone-beam transform, ...

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