The cone-beam transform and spherical convolution operators

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(joint work with Ralf Hielscher and Alfred K. Louis)

Chemnitz University of Technology
Faculty of Mathematics

Workshop on Mathematical Signal and Image Analysis
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Content

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Funk–Radon transform

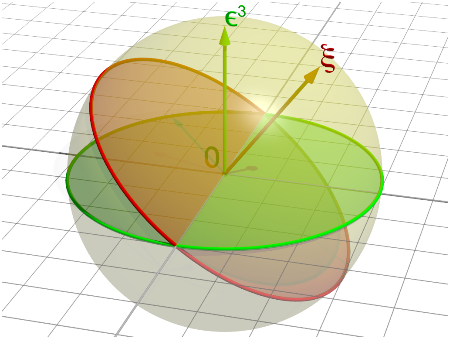
[Funk, 1911]

- Sphere $\mathbb{S}^{d-1} = \{\boldsymbol{\xi} \in \mathbb{R}^d : \|\boldsymbol{\xi}\| = 1\}$
- Function $f\colon \mathbb{S}^{d-1} \to \mathbb{C}$
- Funk–Radon transform

$$\mathcal{S}^{(0)} f(\boldsymbol{\xi}) = \int_{\mathbb{S}^{d-1}} \delta(\boldsymbol{\xi}^\top \boldsymbol{\eta})\, f(\boldsymbol{\eta})\, \mathrm{d}\boldsymbol{\eta} = \int_{\boldsymbol{\xi}^\top \boldsymbol{\eta}=0} f(\boldsymbol{\eta})\, \mathrm{d}\lambda(\boldsymbol{\eta})$$

(integrals of f along all great circles)

- Take derivatives of the delta function



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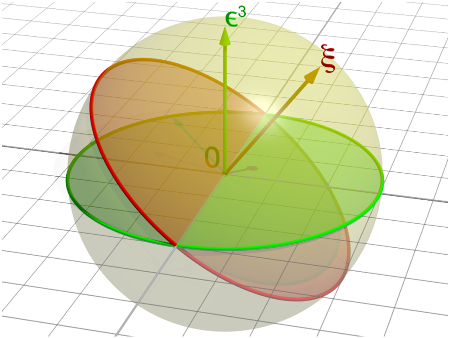

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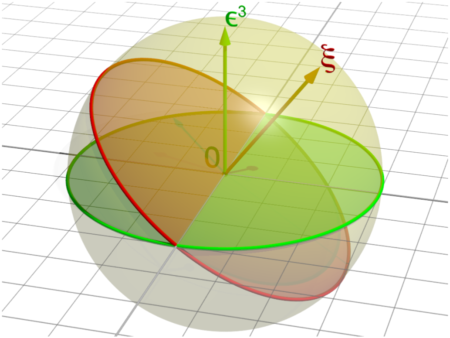

Generalized Funk–Radon transform

[Louis, 2016]

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- Similar definition for $j = 1$:
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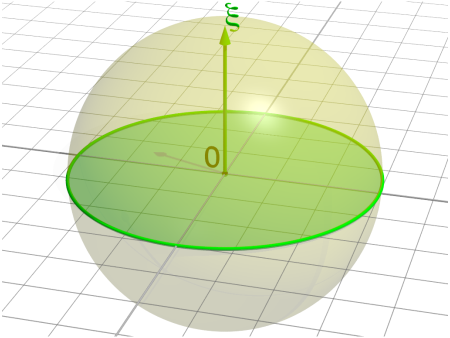

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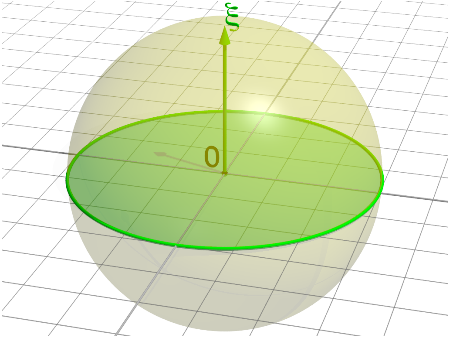

Spherical harmonics: An orthonormal basis on $\mathbb{S}^{d-1}$

The **spherical harmonics**

$$Y_n^k \colon \mathbb{S}^{d-1} \to \mathbb{C}, \qquad n \in \mathbb{N}_0,\ k = 1, \dots, N_{n,d}$$

form an orthonormal basis of $L^2(\mathbb{S}^{d-1})$.

Any $f \in L^2(\mathbb{S}^{d-1})$ can be written as series

$$f = \sum_{n=0}^{\infty} \sum_{k=1}^{N_{n,d}} \hat{f}(n,k)\, Y_n^k, \qquad \hat{f}(n,k) := \int_{\mathbb{S}^{d-1}} f(\xi)\, \overline{Y_n^k(\xi)}\, \mathrm{d}\xi$$

Fast algorithms for spherical Fourier transforms on $\mathbb{S}^2$

[Driscoll & Healy, 1994] [Potts, Steidl & Tasche, 1998] [Kunis & Potts, 2003] [Schaeffer, 2013]

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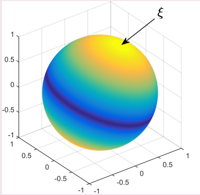
Spherical convolution

Funk–Hecke formula [Funk, 1915] [Hecke, 1917]

Let

- $g\colon [-1,1] \to \mathbb{C}$
- Y_n^k spherical harmonic of degree n
- $P_{n,d}$ Legendre (ultraspherical) polynomial of degree n in dimension d, orthogonal polynomial on $[-1,1]$ w.r.t. the weight $(1-t^2)^{\frac{d-3}{2}}$



$\boldsymbol{\eta} \mapsto g(\boldsymbol{\xi}^\top \boldsymbol{\eta})$

Then

$$\int_{\mathbb{S}^{d-1}} Y_n^k(\boldsymbol{\eta})\, g(\boldsymbol{\xi}^\top \boldsymbol{\eta})\, \mathrm{d}\boldsymbol{\eta} = Y_n^k(\boldsymbol{\xi}) \int_{-1}^{1} g(t)\, P_{n,d}(t)\, (1-t^2)^{\frac{d-3}{2}}\, \mathrm{d}t$$

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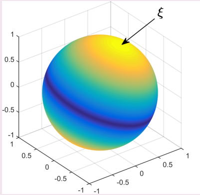

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Eigenvalue decomposition of $\mathcal{S}^{(j)}$

Theorem [Q., Hielscher, Louis, 2018]

Let $j \in \mathbb{N}_0$. The generalized Funk–Radon transform $\mathcal{S}^{(j)} \colon C(\mathbb{S}^{d-1}) \to C(\mathbb{S}^{d-1})$ has the eigenvalue decomposition

$$\mathcal{S}^{(j)} Y_n^k = \hat{\mathcal{S}}^{(j)}(n)\, Y_n^k, \qquad n \in \mathbb{N}_0,\ k = -n, \dots, n$$

with eigenvalues

$$\hat{\mathcal{S}}^{(j)}(n) = \begin{cases} \left|\mathbb{S}^{d-2}\right| \, (-1)^{\frac{n+j}{2}} \dfrac{(n+j-1)!!\,(d-3)!!}{(n-j+d-3)!!}, & n+j \text{ even and } (n \ge j) \\ 0, & \text{otherwise} \end{cases}$$

Sobolev estimates of $\mathcal{S}^{(j)}$

Define the Sobolev space $H^s(\mathbb{S}^{d-1})$ with order $s \ge 0$

$$H^s(\mathbb{S}^{d-1}) = \left\{ f \in L^2(\mathbb{S}^{d-1}) \,\middle|\, \sum_{n=0}^{\infty} \sum_{k=1}^{N_{n,d}} \left|\hat{f}(n,k)\right|^2 \left(n + \tfrac{d-2}{2}\right)^{2s} < \infty \right\}$$

Theorem [Q., Hielscher, Louis, 2018]

Let $s \in \mathbb{R}$ and $j \in \mathbb{N}_0$. The generalized Funk–Radon transform $\mathcal{S}^{(j)}$ extends to a continuous operator

$$\mathcal{S}^{(j)} \colon H^s(\mathbb{S}^{d-1}) \to H^{s-j+\frac{d-2}{2}}(\mathbb{S}^{d-1}).$$

The nullspace of $\mathcal{S}^{(j)}$ is the closed linear span

$$\overline{\operatorname{span}} \left\{ Y_n^k : n+j \text{ odd or } (n \le j-d+1 \text{ and } d \text{ odd}),\ k = 1, \dots, N_{n,d} \right\}.$$

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Special cases of j

$j=-1$: **modified hemispherical transform** [Ungar 1954] [Rubin, 1999]

$$\mathcal{S}^{(-1)} f(\boldsymbol{\xi}) = \frac{1}{2} \int_{\mathbb{S}^{d-1}} \operatorname{sgn}(\boldsymbol{\xi}^\top \boldsymbol{\eta})\ f(\boldsymbol{\eta}) \,\mathrm{d}\boldsymbol{\eta}.$$

$j=-2$: **spherical cosine transform** [Petty, 1961] [Schneider, 1967] [Groemer, 1996]

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$j=\frac{d-2}{2}$: the absolute value of the eigenvalues

$$\hat{\mathcal{S}}^{(j)}(n) = 2(2\pi)^{\frac{d-2}{2}} (-1)^{\frac{2n+d-2}{4}}, \qquad n+j \text{ even}$$

is constant. Hence $\mathcal{S}^{(j)} : L^2(\mathbb{S}^{2j+1}) \to L^2(\mathbb{S}^{d-1})$ is a partial isometry.

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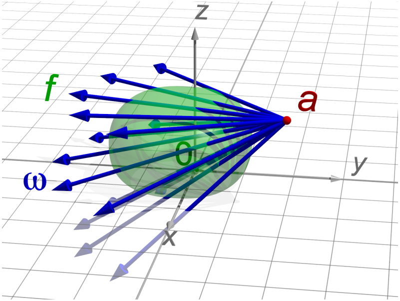
Cone-beam transform

- $f \colon \mathbb{R}^d \to \mathbb{R}$
- $\boldsymbol{a} \in \mathbb{R}^d$ ... source of the ray
- $\boldsymbol{\omega} \in \mathbb{S}^{d-1}$ ... direction of the ray

- **Cone-beam transform**
  (or divergent beam X-ray transform)

$$\mathcal{D}f(\boldsymbol{a},\boldsymbol{\omega}) = \int_0^\infty f(\boldsymbol{a}+t\boldsymbol{\omega})\,\mathrm{d}t$$



[Hamaker et al. 1980] [Tuy, 1983] [Finch, 1985] [Feldkamp, Davis, Kress, 1984]

Radon transform

- $f\colon \mathbb{R}^d \to \mathbb{R}$
- **Radon transform**

$$\mathcal{R}f(\boldsymbol{\omega}, s) = \int_{\boldsymbol{x}^\top \boldsymbol{\omega} = s} f(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}$$

- Integral along (hyper-)plane with normal $\boldsymbol{\omega} \in \mathbb{S}^{d-1}$ and $s \in \mathbb{R}$
- In **2D**: Line integrals
(both Radon transform on $\mathbb{R}^2$ and the cone-beam transform $\mathcal{D}$)
- In **3D**: Plane integrals

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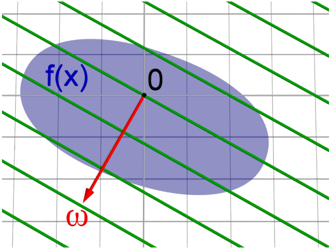

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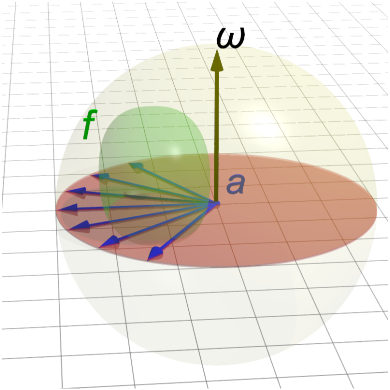
Cone-beam and Radon transform (in 3D)

[Grangeat, 1991]

Consider a “fan” of ray integrals orthogonal to ω

$$\int_{\xi \in \mathbb{S}^2,\, \xi^\top \omega = 0} \mathcal{D}f(a, \xi)\, \mathrm{d}\xi$$



$$\omega = (0, 0, 1)^\top$$

Want the plane integral

$$\mathcal{R}f(\omega, a^\top \omega) = \int_{x^\top \omega = a^\top \omega} f(x)\, \mathrm{d}x$$

Grangeat's formula

$$\frac{\partial}{\partial s} \mathcal{R}f(\omega, a^\top \omega) = \int_{\xi \in \mathbb{S}^2,\, \xi^\top \omega = 0} \frac{\partial}{\partial \omega} \mathcal{D}f(a, \xi)\, \mathrm{d}\xi$$

$\frac{\partial}{\partial \omega}$ ... directional derivative w.r.t. ξ

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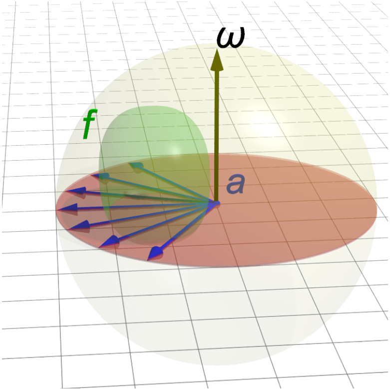

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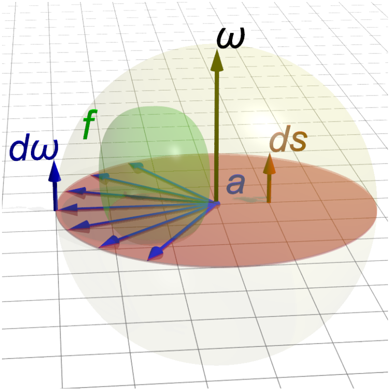

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In general dimension d

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Singular Value decomposition of the Radon transform $\mathcal{R}$

- Let $m \in \mathbb{N}_0$, $l = 0, \dots, m$ with $m + l$ even and $k = 1, \dots, N_{l,d}$.
  Define the orthogonal polynomials on the ball $\mathbb{B}^d$ by

$$V_{m,l,k}(s\boldsymbol{\omega}) = \sqrt{2m+d}\, s^l P_{\frac{m-l}{2}}^{(0,l+\frac{d-2}{2})}(2s^2-1)\, Y_l^k(\boldsymbol{\omega}), \qquad s \in [0,1],\ \boldsymbol{\omega} \in \mathbb{S}^{d-1}$$

  where $P_n^{(\alpha,\beta)}$ is the Jacobi polynomial of degree n and orders $\alpha, \beta > -1$.

- The Radon transform $\mathcal{R}$ admits the **singular value decomposition** [Louis, 1984]

$$\mathcal{R}V_{m,l,k}(\boldsymbol{\omega}, s) = \frac{\sqrt{2m+d}\,\Gamma(\frac{d}{2})\, m!}{2^{1-d}\,\pi^{1-\frac{d}{2}}\,(m+d-1)!}\,(1-s^2)^{\frac{d-1}{2}}\, C_m^{(\frac{d}{2})}(s)\, Y_l^k(\boldsymbol{\omega}).$$

Singular Value decomposition of the Radon transform $\mathcal{R}$

- Let $m \in \mathbb{N}_0$, $l = 0, \dots, m$ with $m + l$ even and $k = 1, \dots, N_{l,d}$.
  Define the orthogonal polynomials on the ball $\mathbb{B}^d$ by

$$V_{m,l,k}(s\boldsymbol{\omega}) = \sqrt{2m+d}\, s^l P_{\frac{m-l}{2}}^{(0,l+\frac{d-2}{2})}(2s^2-1)\, Y_l^k(\boldsymbol{\omega}), \qquad s \in [0,1],\ \boldsymbol{\omega} \in \mathbb{S}^{d-1}$$

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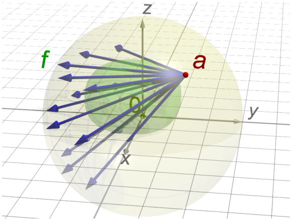
Cone-beam transform

- Let $f\colon \mathbb{B}^d = \{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\| \le 1\} \to \mathbb{R}$
- Consider the cone-beam transform with sources $\boldsymbol{a} \in \mathbb{S}^{d-1}$ on the sphere

$$\mathcal{D}\colon L^2(\mathbb{B}^d) \to L^2(\mathbb{S}^{d-1} \times \mathbb{S}^{d-1})$$

$$\mathcal{D}f(\boldsymbol{a},\boldsymbol{\omega}) = \int_0^\infty f(\boldsymbol{a}+t\boldsymbol{\omega})\,\mathrm{d}t, \qquad \boldsymbol{\omega} \in \mathbb{S}^{d-1}$$



Cone-beam transform

Singular value decomposition

[Q., Hielscher, Louis, 2018]

The cone-beam transform $\mathcal{D}$ with sources $\boldsymbol{a}$ on the sphere $\mathbb{S}^{d-1}$ and d odd has the SVD

$$\mathcal{D}V_{m,l,k}(\boldsymbol{a},\boldsymbol{\omega}) = \mu_{m,d} \sum_{j=1}^{N_{m+1,d}} \overline{Y_{m+1,d}^{j}(\boldsymbol{a})} \sum_{n=m+1-l}^{l+m+1}{}' \nu_{n,d} \sum_{i=1}^{N_{n,d}} G_{m+1,j,l,k}^{n,i,d} Y_{n,d}^{i}(\boldsymbol{\omega}),$$

where $\sum'$ denotes the summation over odd indices and

$$\mu_{m,d} = \sqrt{\frac{2^{d+1}\,\pi^{d-1}}{2m+d}}, \qquad \nu_{n,d} = \frac{(-1)^{\frac{n+1}{2}}\,(n-1)!!}{(n+d-3)!!}$$

and the Gaunt coefficients $G_{n_1,k_1,n_2,k_2}^{n,k} = \int_{\mathbb{S}^{d-1}} Y_{n_1}^{k_1}(\boldsymbol{\xi})\, Y_{n_2}^{k_2}(\boldsymbol{\xi})\, \overline{Y_n^k(\boldsymbol{\xi})}\, \mathrm{d}\boldsymbol{\xi}$.

Remarks

- dimension $d = 3$ [Kazantsev, 2015]
- d odd [Q., Hielscher, Louis, 2018]
- Similar result for full lines (X-ray transform) [Maaß, 1987]

Decay of the singular values $\lambda_{m,l,d}$ of the cone-beam transform $\mathcal{D}$

- Lower bound: $\lambda_{m,l,d} \geq c_d\, m^{-1/2}$ for $m \to \infty$
- Upper bound: $\lambda_{m,l,d} \leq C_d$
- for dimension $d = 3$: $\lambda_{m,l,3} \leq C\, m^{-1/4}$ [Kazantsev, 2015]
- Conjecture: $\lambda_{m,l,d} \leq C_d\, m^{-1/2}$

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Conclusion

- Investigated the **generalized Funk–Radon transform**
- Characterization of its nullspace and range
- Grangeat's formula connects the cone-beam transform, Radon transform and generalized Funk–Radon transform
- SVD of the **cone-beam transform** for sources on the sphere

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