Quicksort and Quickselect

Example run of Quicksort on the array bifgajedhc

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bifgajedhc

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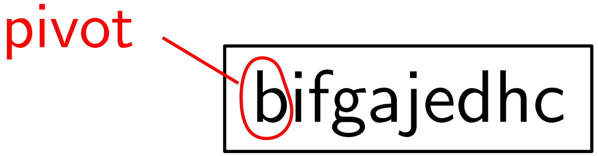

pivot

bifgajedhc

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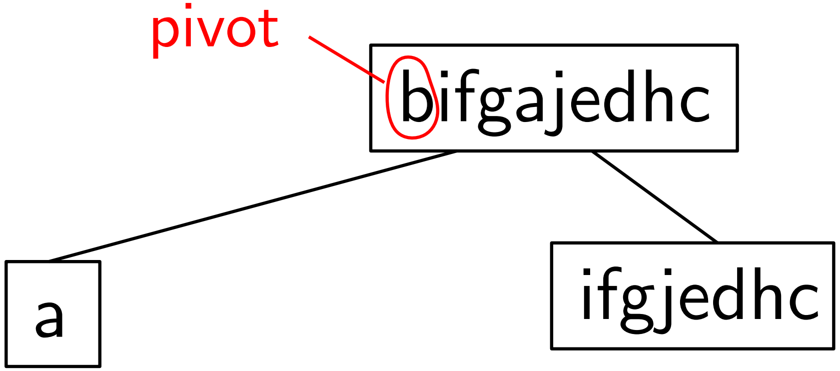

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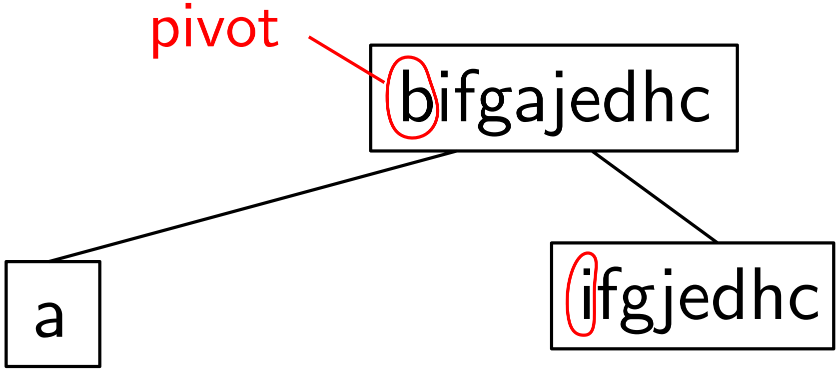

pivot

bifgajedhc

a

ifgjedhc

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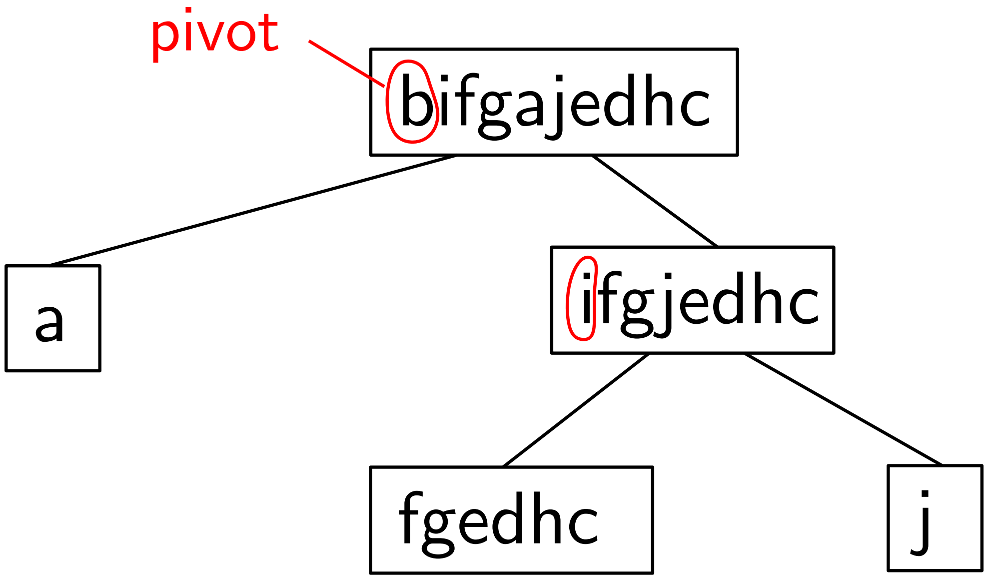

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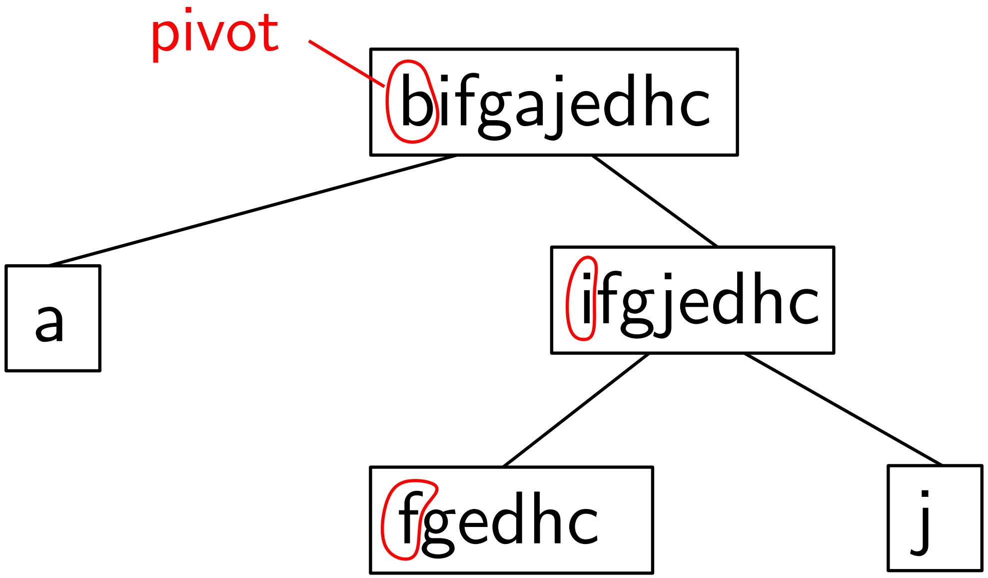

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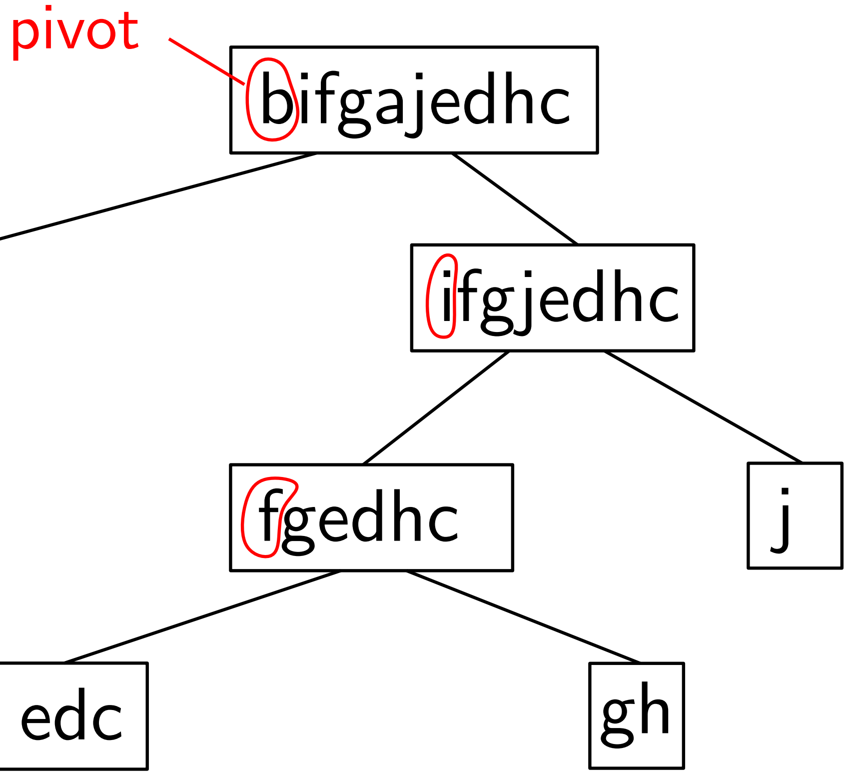

pivot

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ifgjedhc

fgedhc

j

edc

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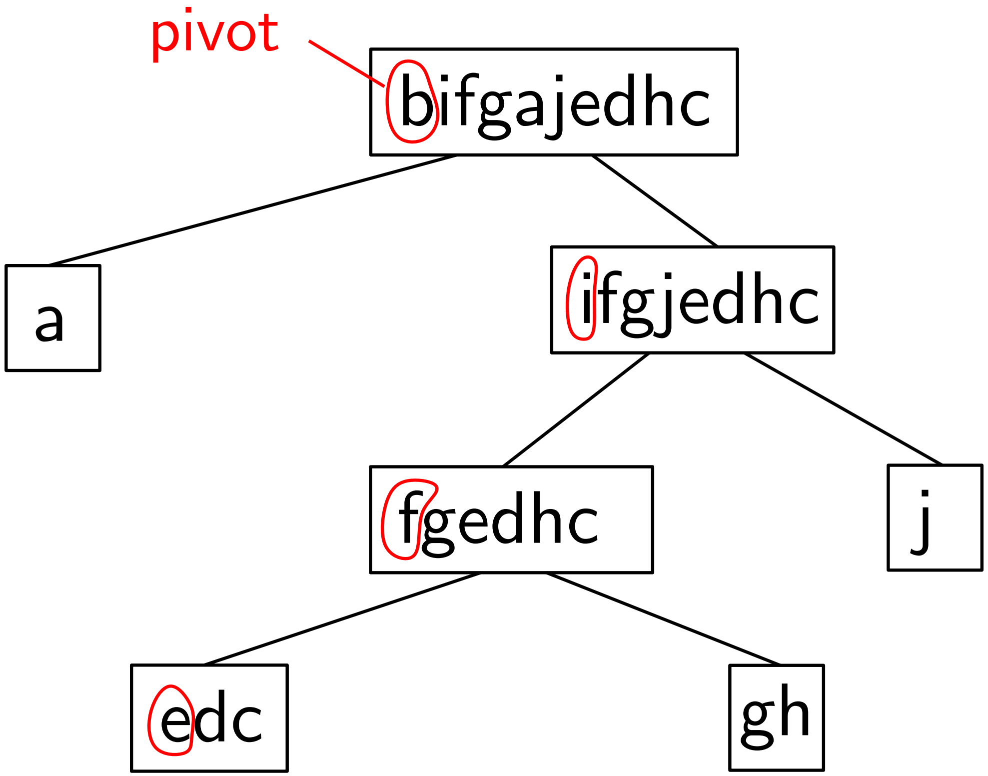

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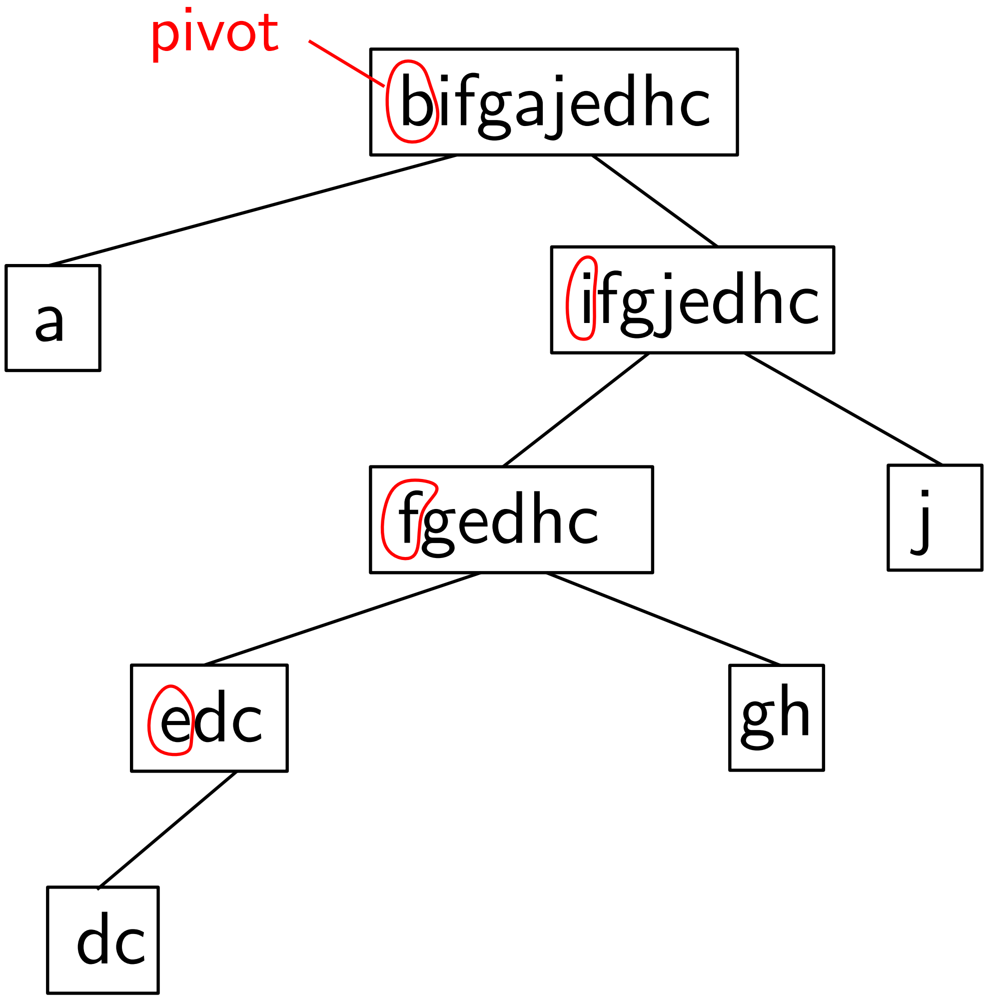

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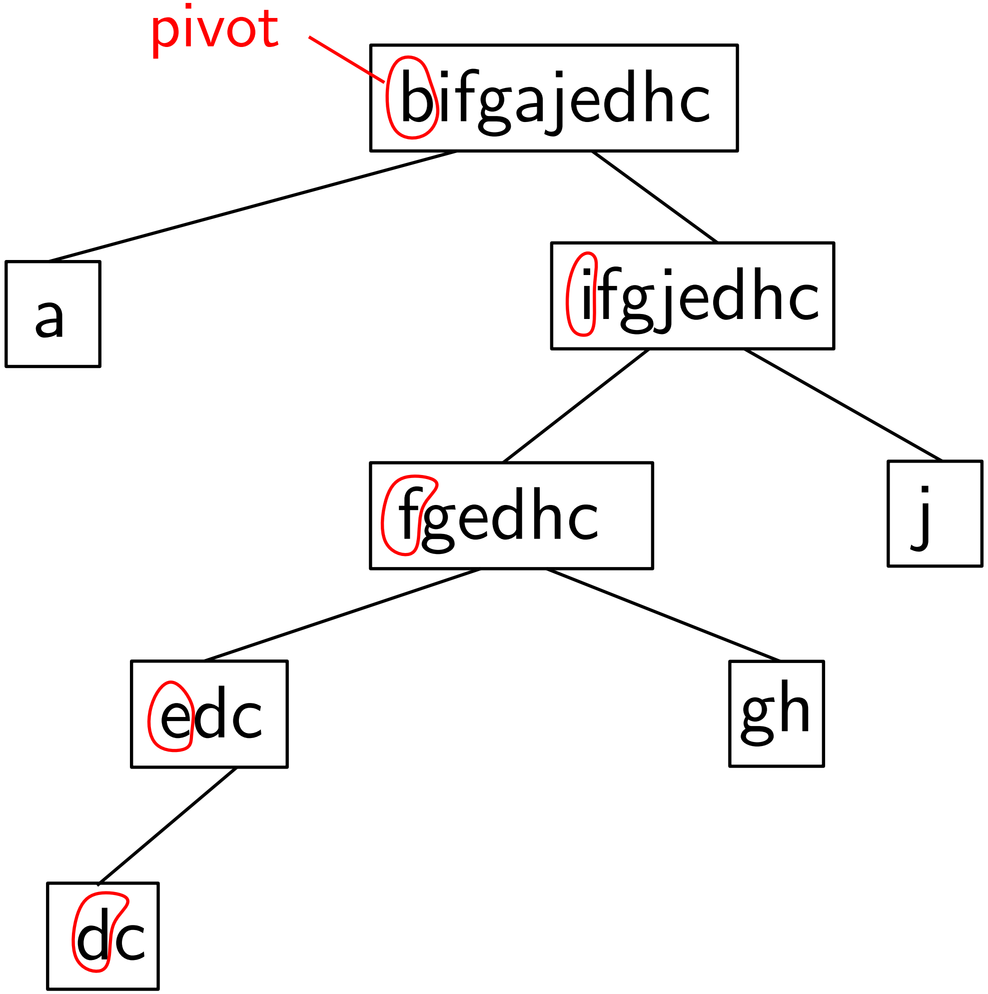

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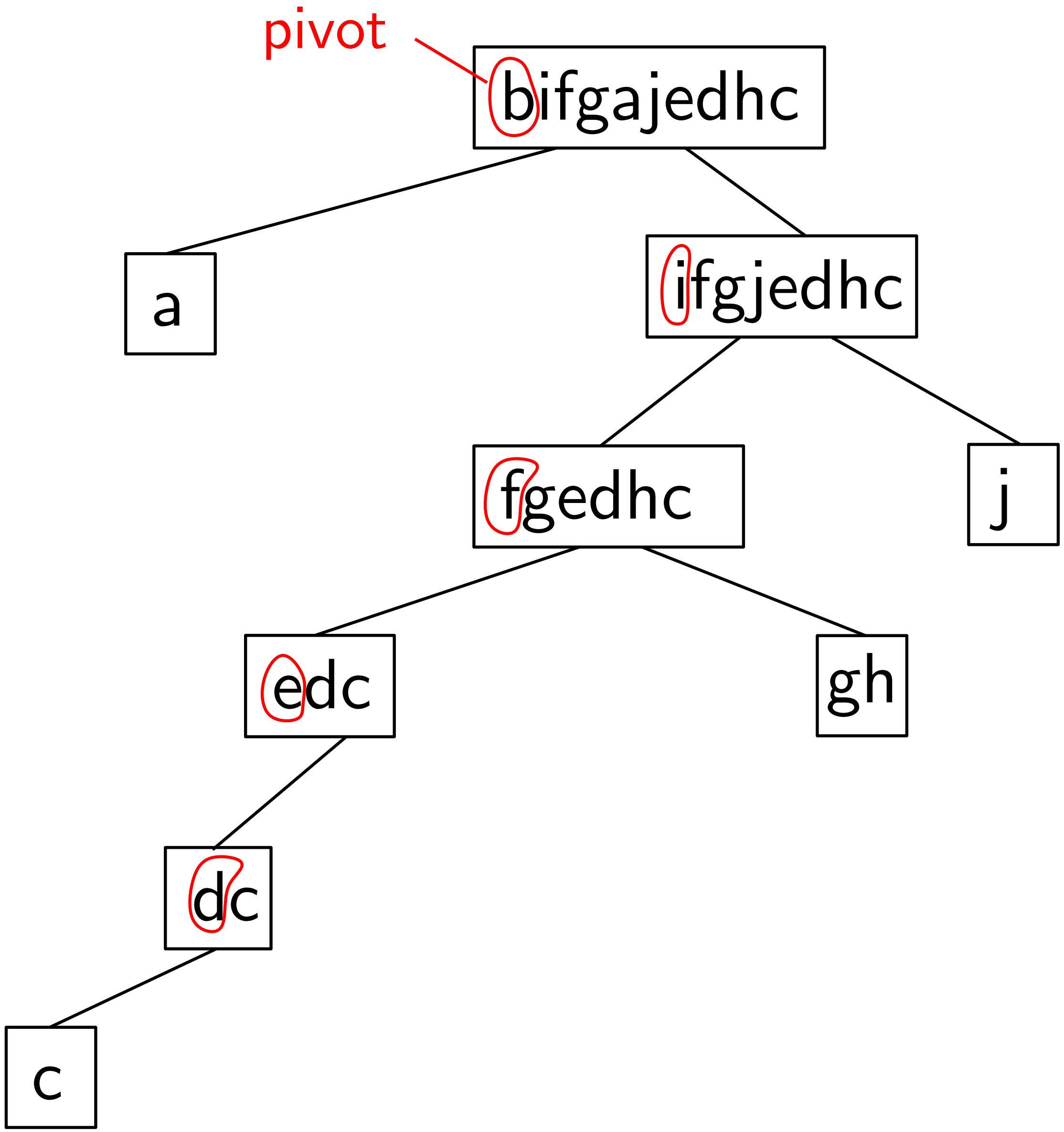

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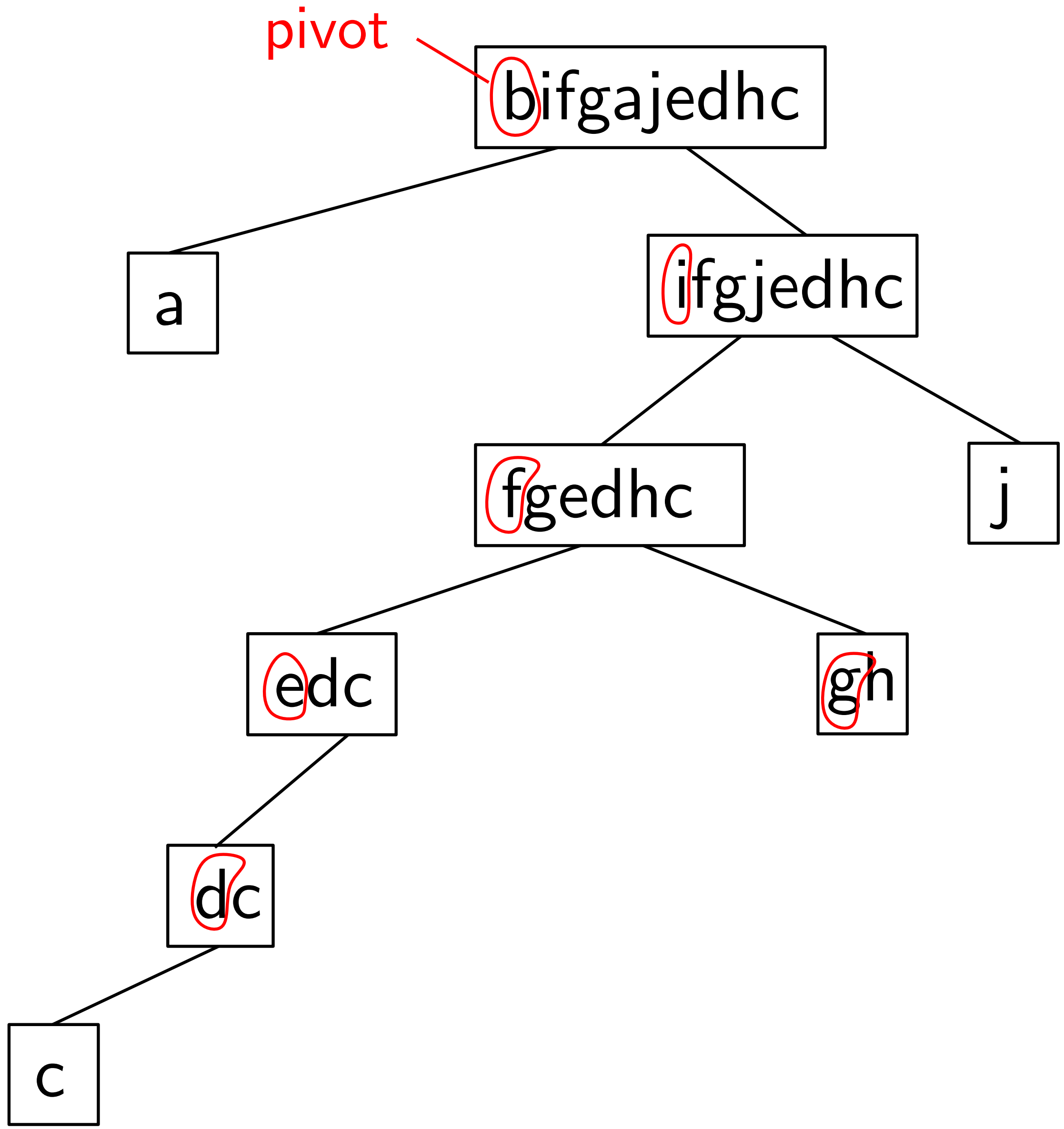

pivot

bifgajedhc

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ifgjedhc

fgedhc

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dc

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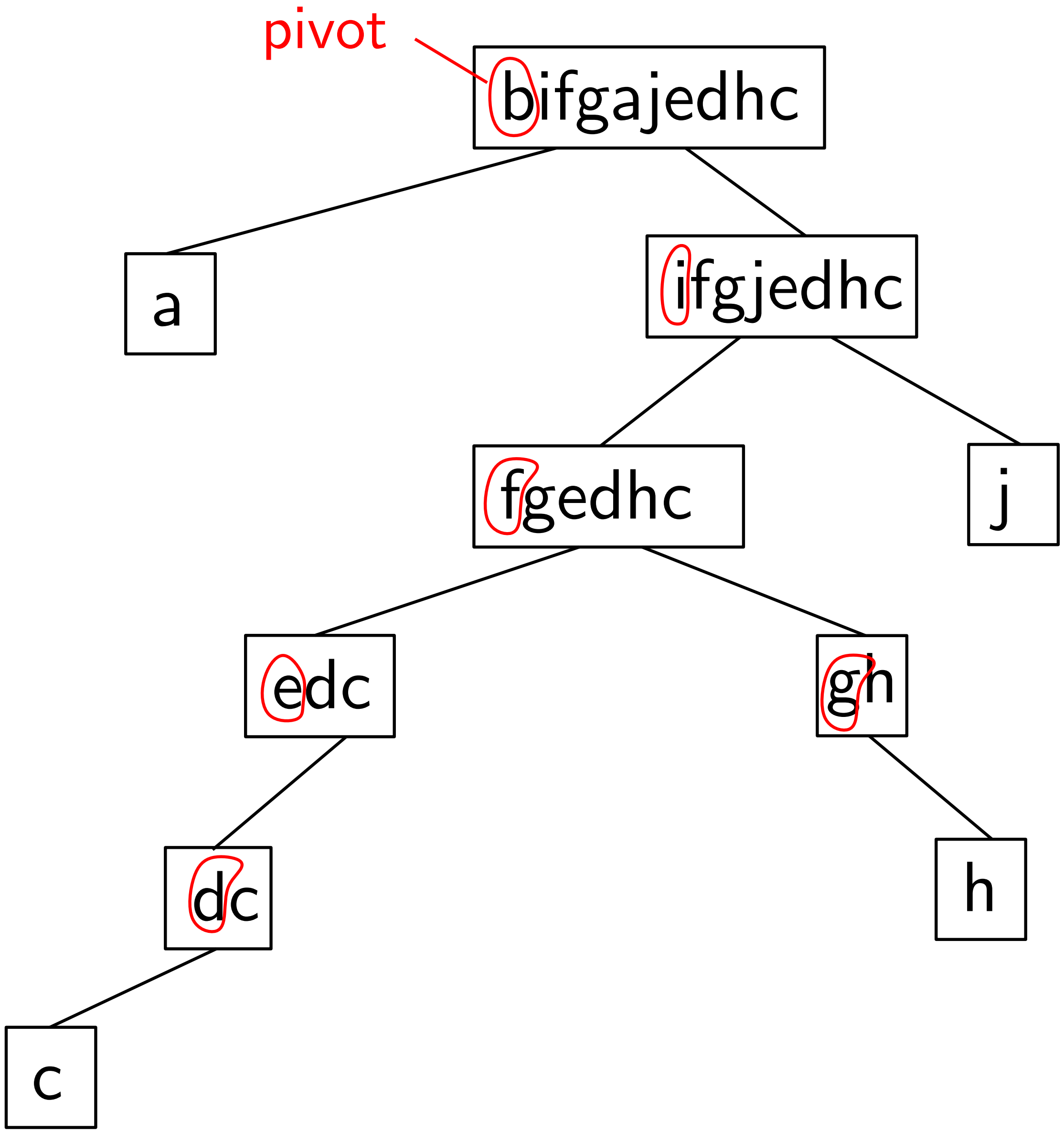

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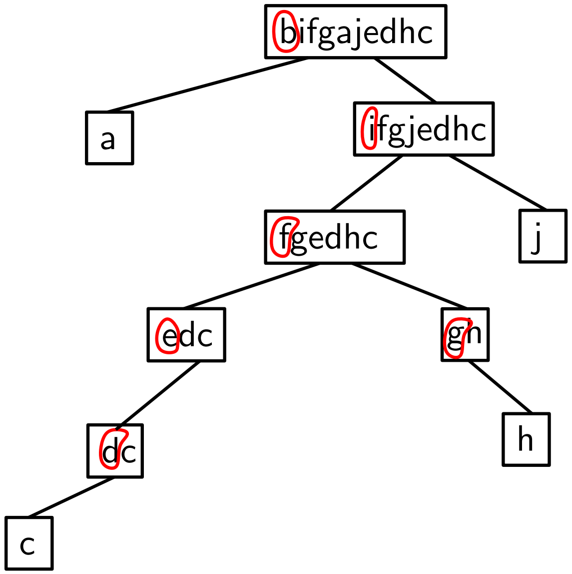

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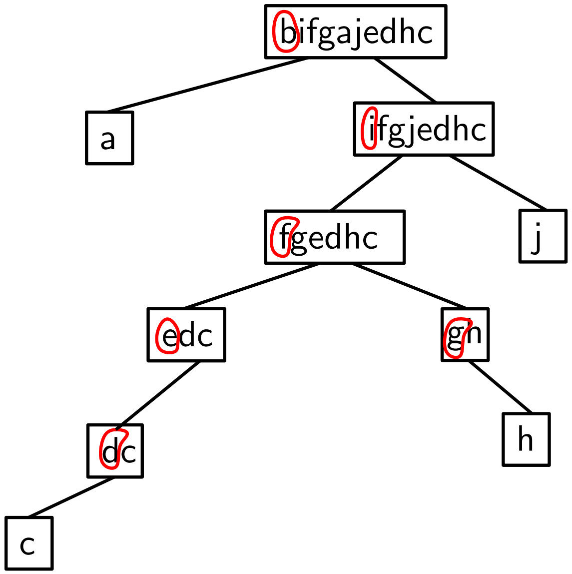

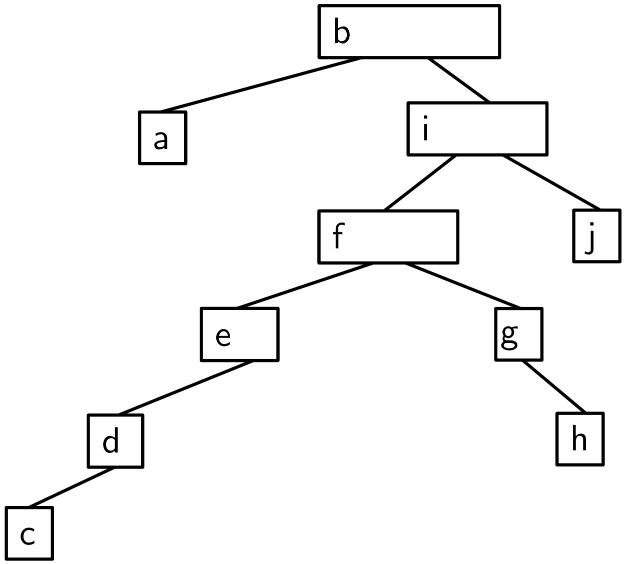

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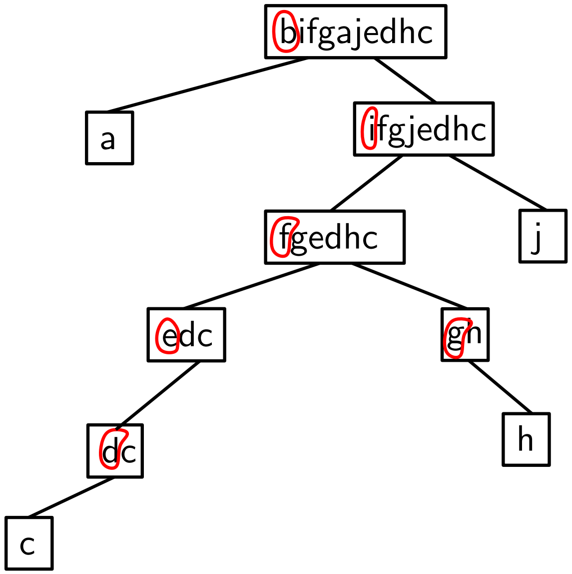

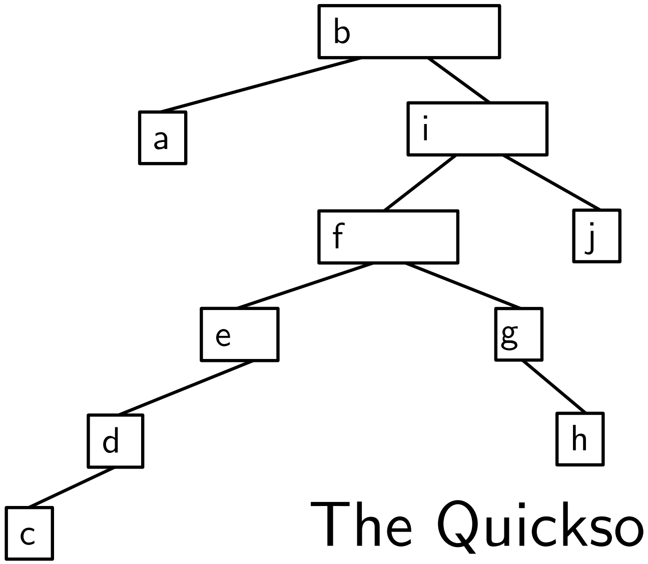

The Quicksort Tree

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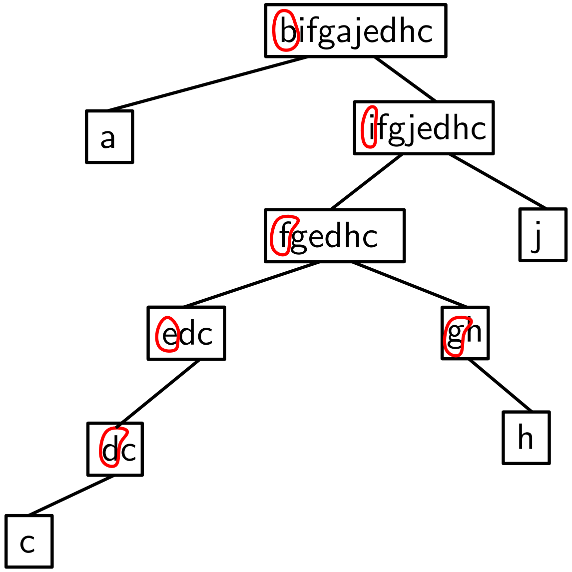

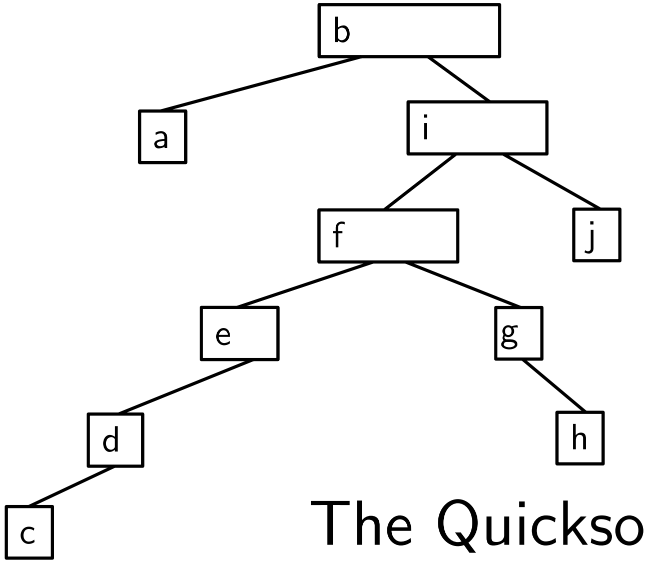

The Quicksort Tree

Definition. $A_{x,y}$ is 1 if x is an ancestor of y in the Quicksort tree; otherwise, $A_{x,y}$ is 0.

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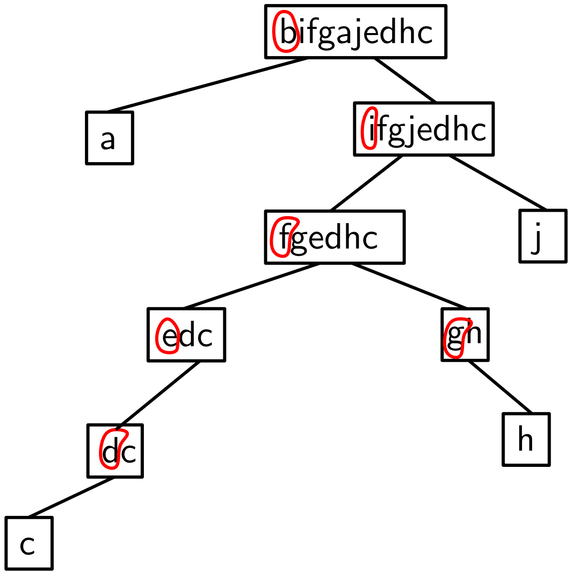

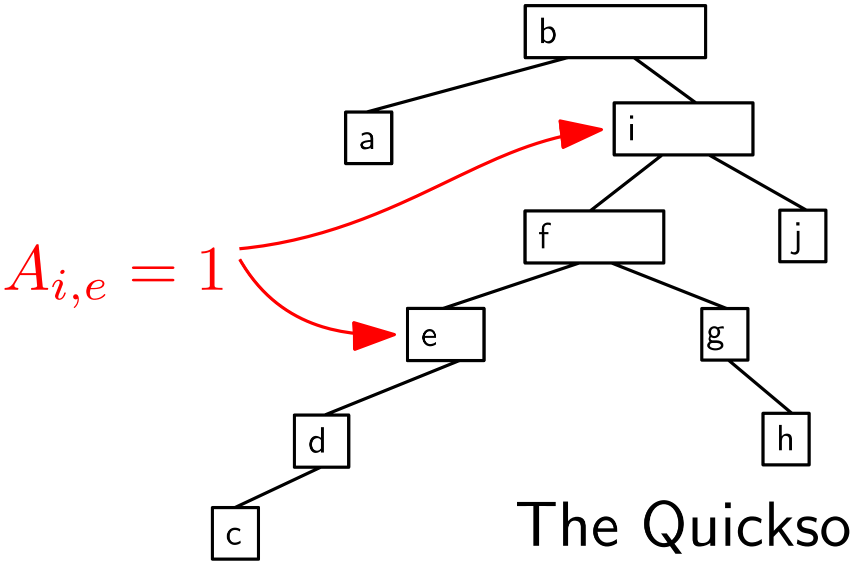

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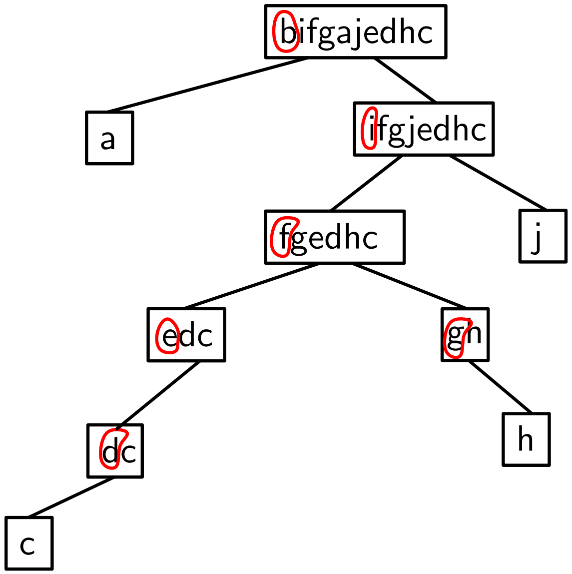

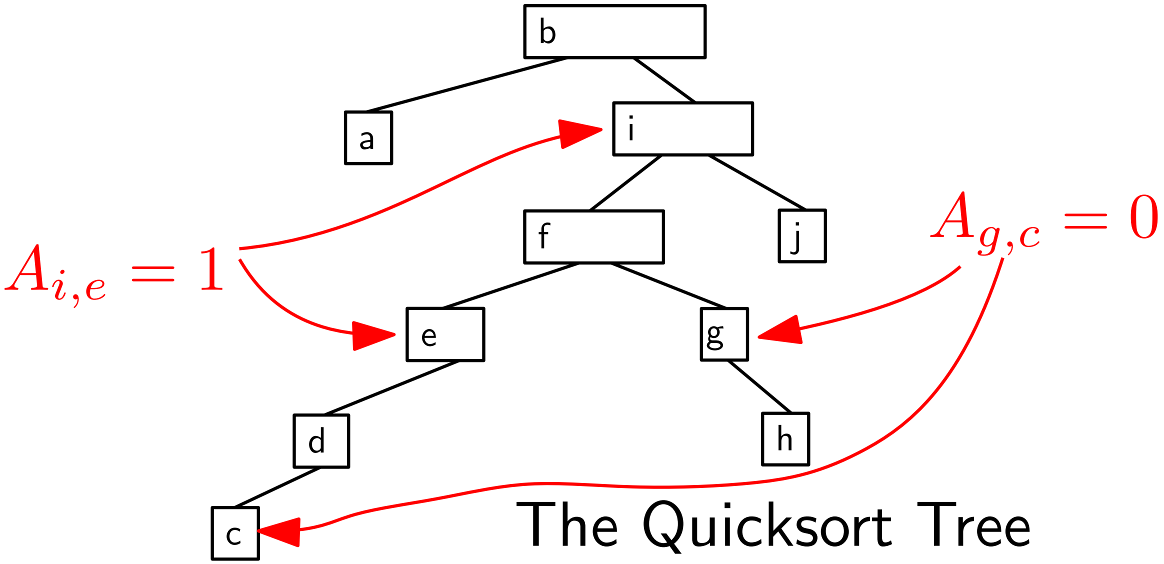

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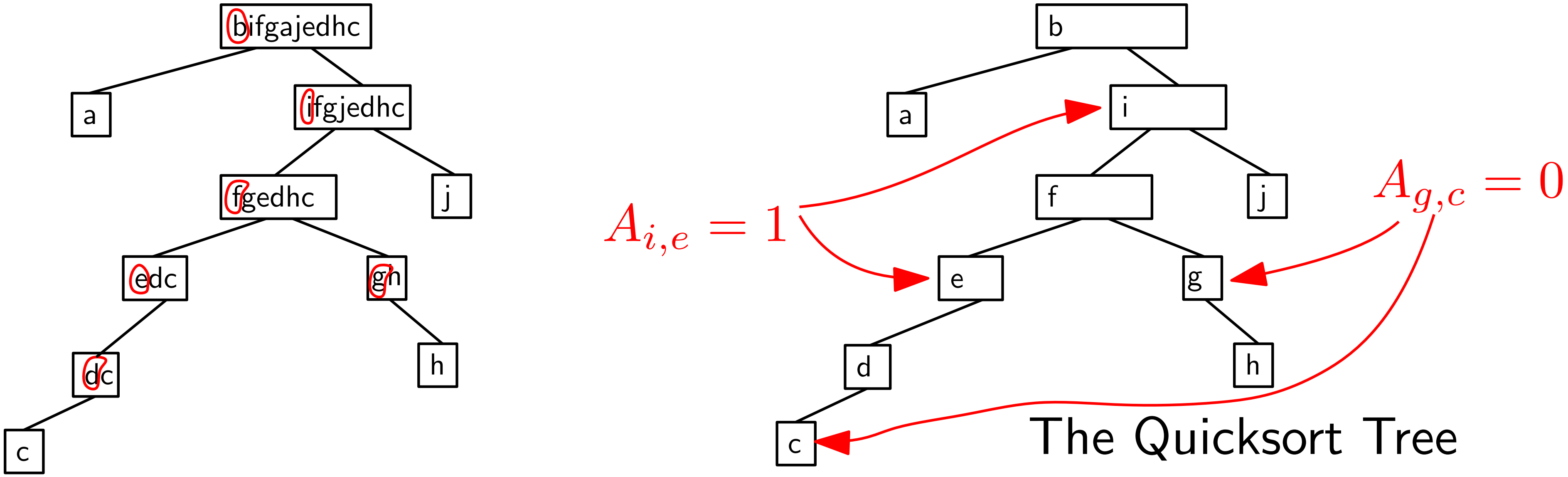

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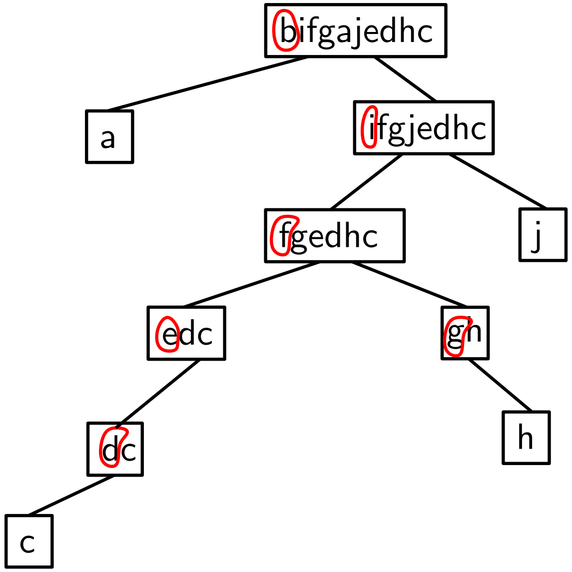

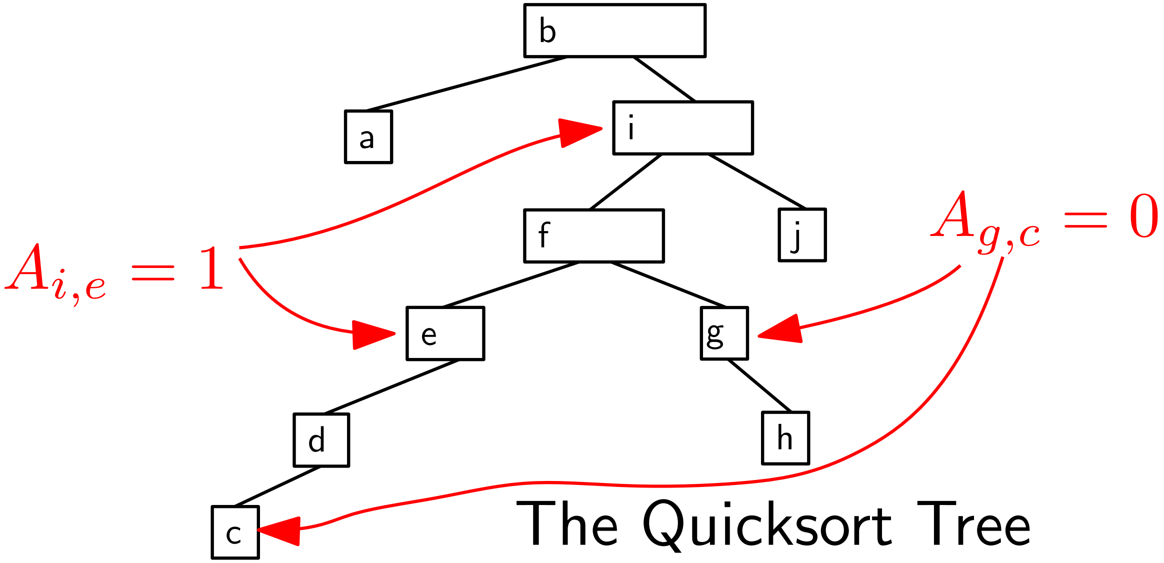

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Observation. Quicksort makes $\sum_x \sum_{y \neq x} A_{x,y}$ comparisons.

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Shuffle the input array randomly!

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$$\begin{aligned}\mathbf{E}[\text{number of comparisons}] &= \mathbf{E}\left[\textstyle\sum_x \sum_{y \neq x} A_{x,y}\right] \\ &= \textstyle\sum_x \sum_{y \neq x} \mathbf{E}[A_{x,y}]\end{aligned}$$

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Shuffle the input array randomly!

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$$= \sum_x \sum_{y \neq x} \mathbf{E}[A_{x,y}]$$

Probability that x is an ancestor of y in the Quicksort tree

Array to sort: bifgajedhc

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Notation: $[x : y]$ is the set of elements between x and y, including x and y.

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Lemma. $A_{x,y} = 1$ if and only if among the elements of $[x : y]$, element x comes first in the input array.

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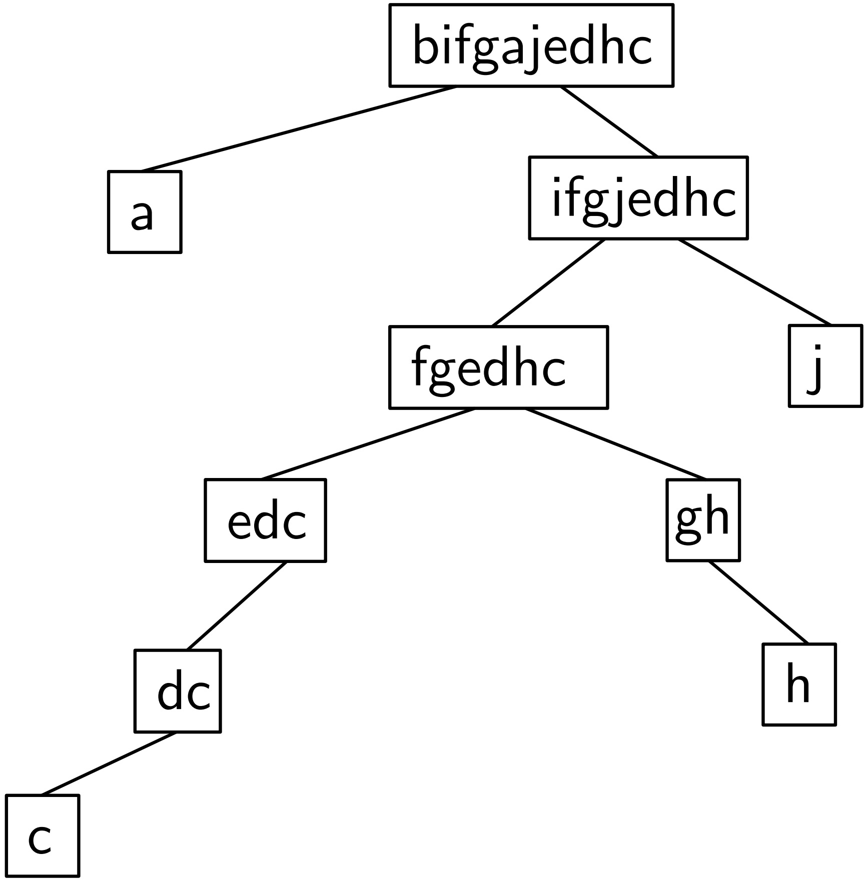

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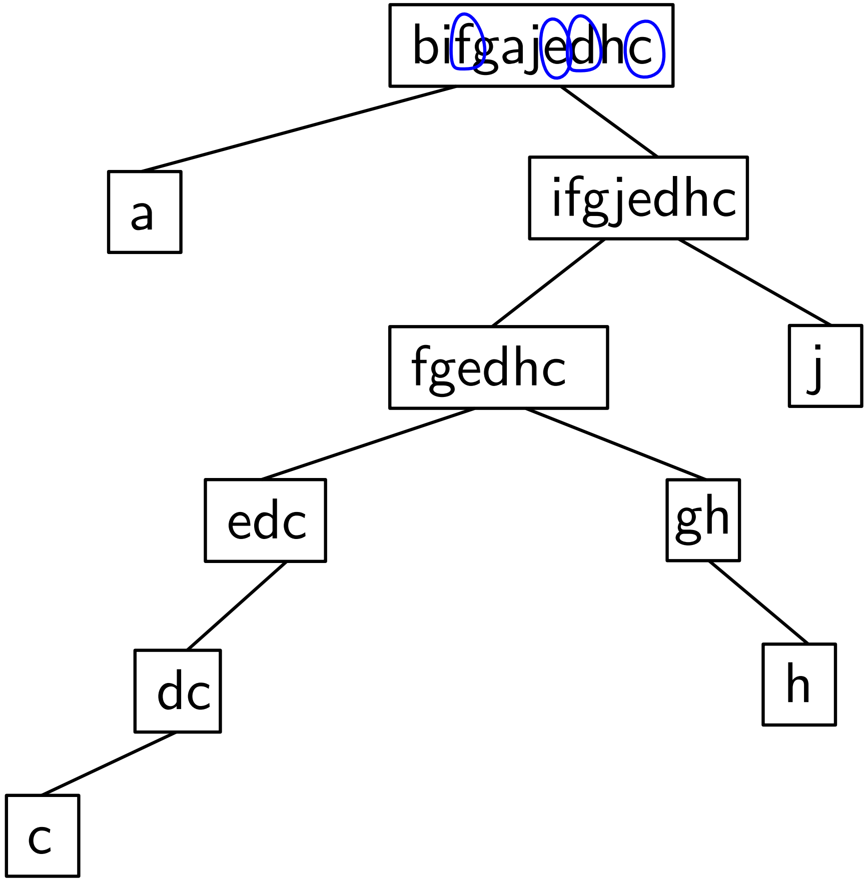

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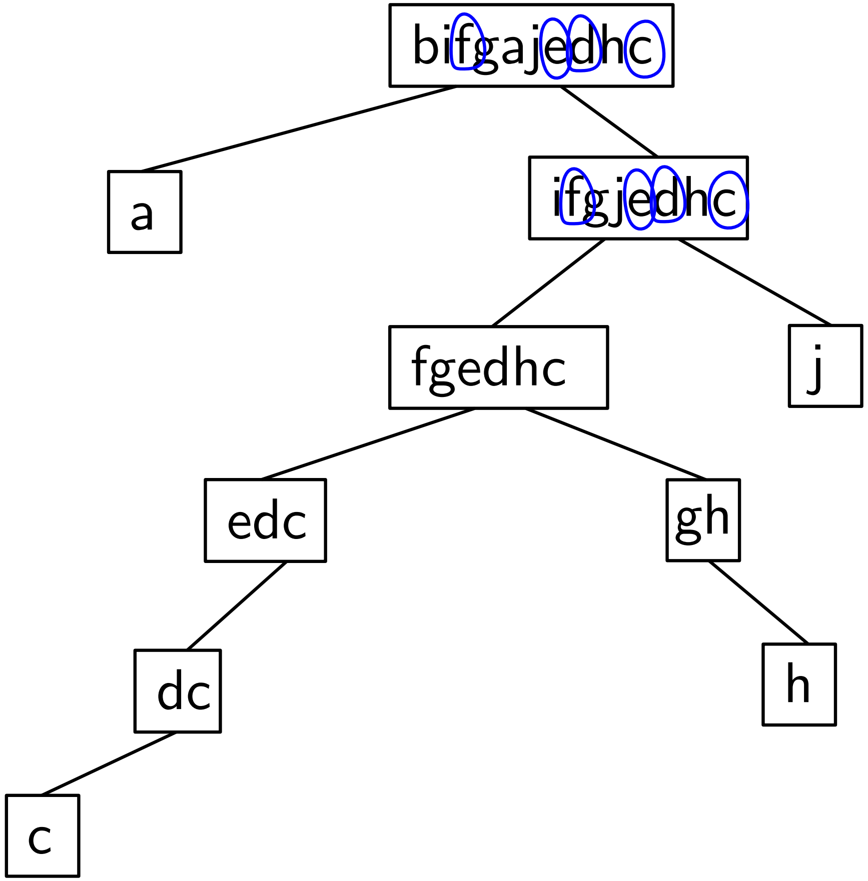

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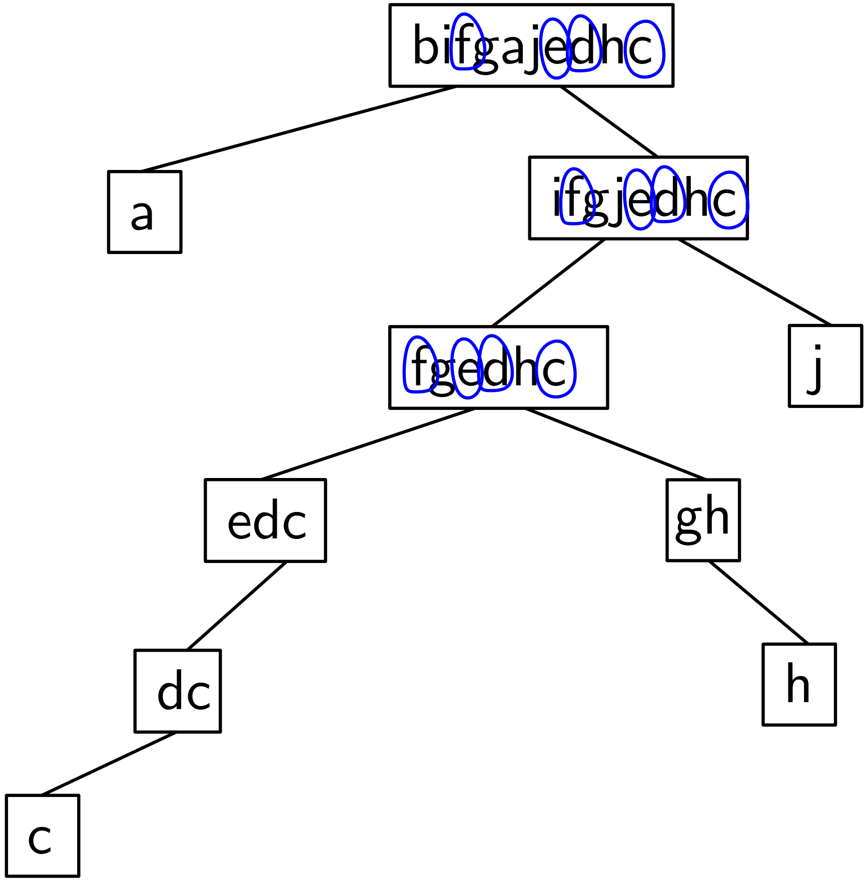

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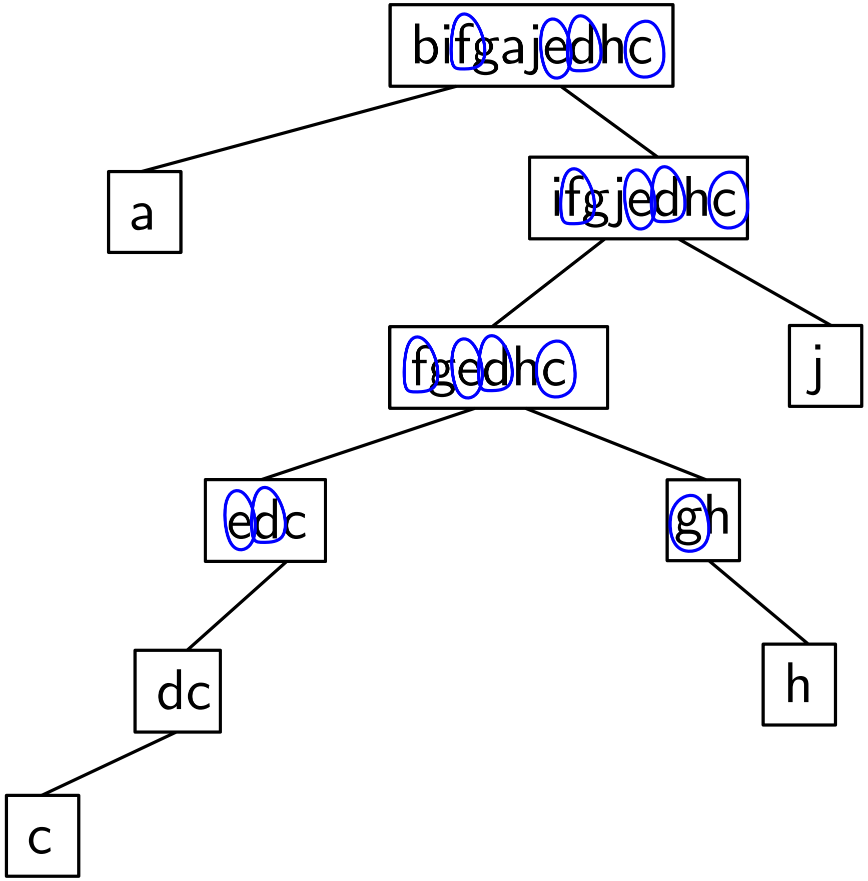

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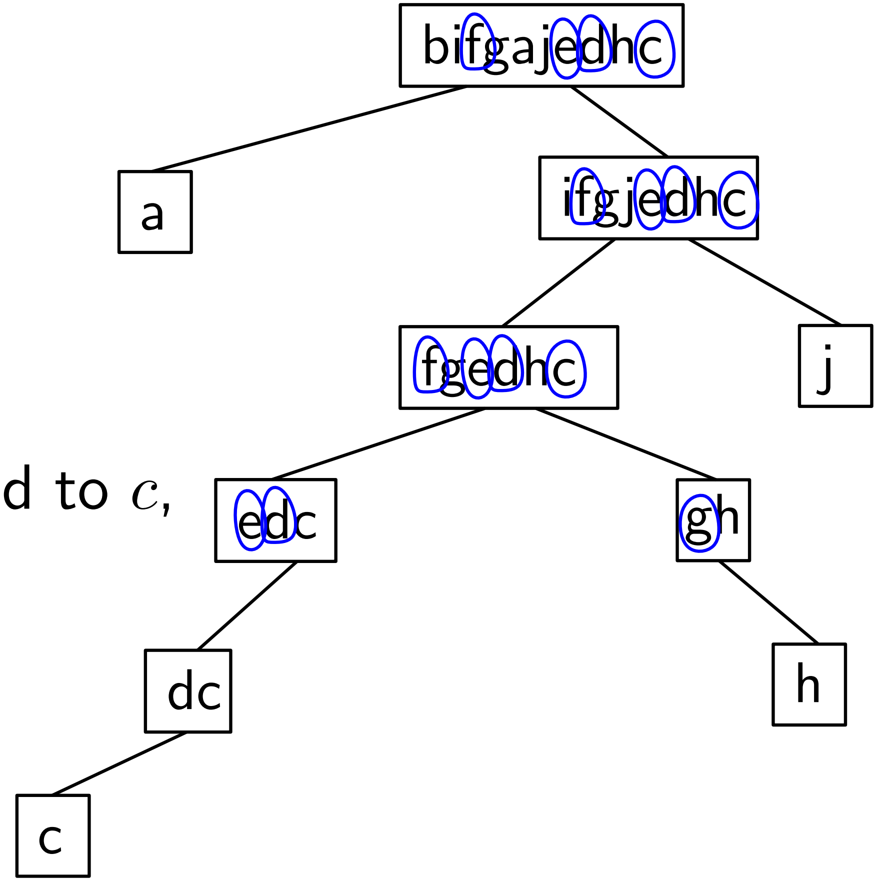

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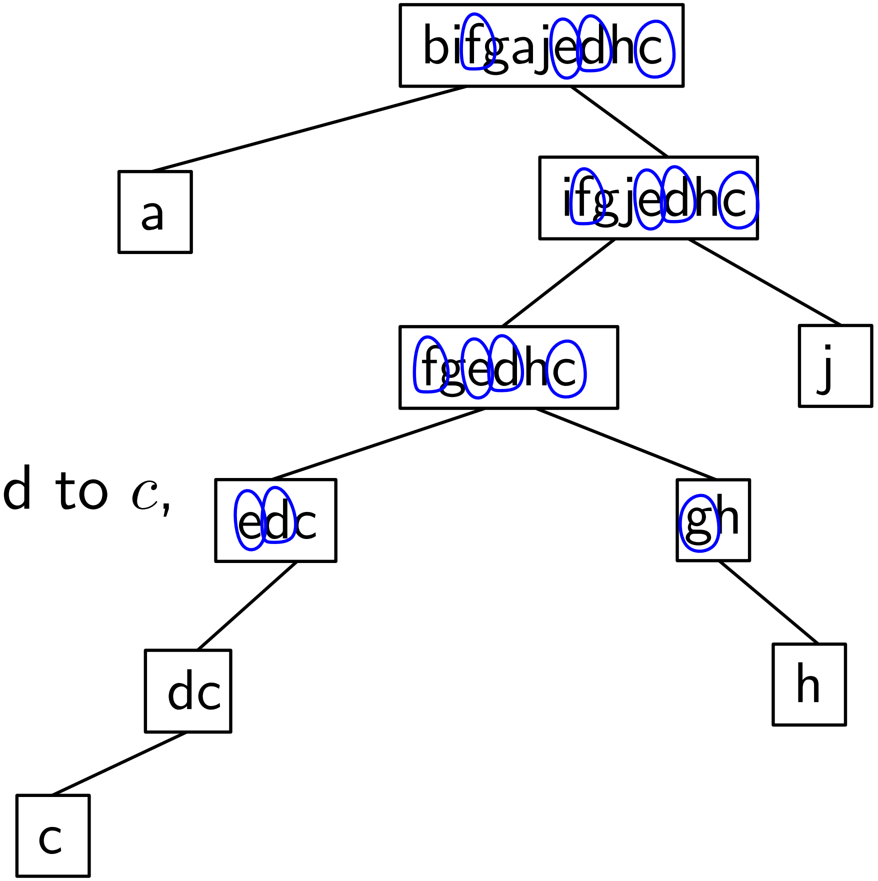

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If the pivot is f, it will be compared to c, and $A_{f,c} = 1$.

Otherwise, f and c will now be separated and $A_{f,c} = 0$.

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$= \frac{1}{4}$.

In general:

$\mathbf{E}[A_{x,y}] = \frac{1}{|[x:y]|}$

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Evaluate this sum (Homework)

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I'm doing it here now.

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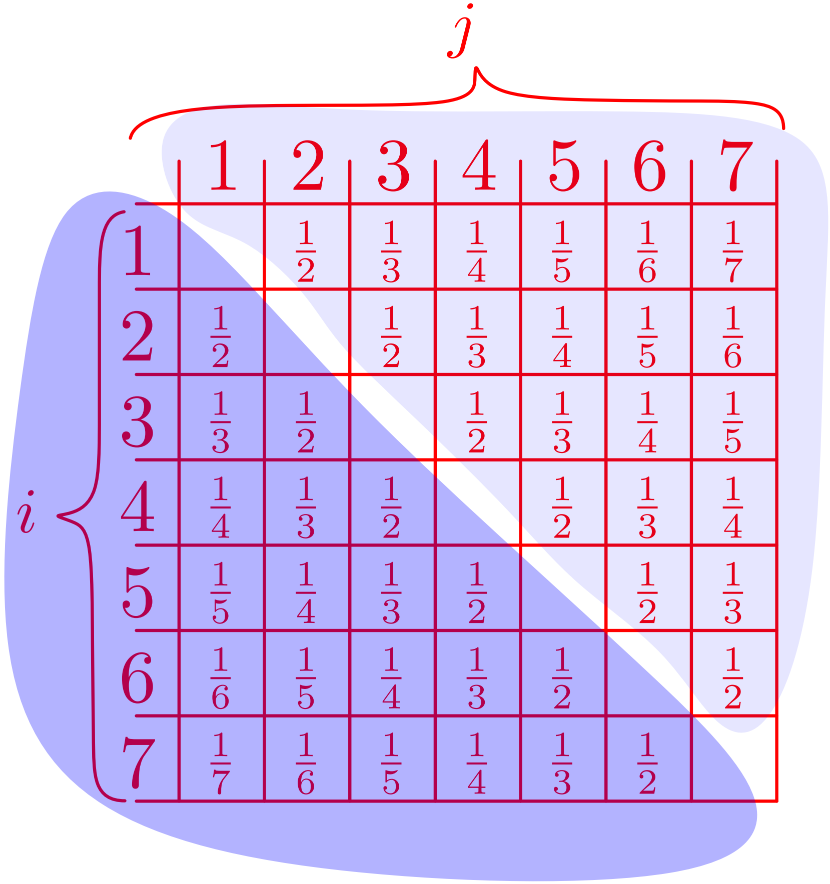
j

i	1	2	3	4	5	6	7
1		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$
2	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$
3	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$
4	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
5	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$
6	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$
7	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	

$$\mathbf{E}[\text{number of comparisons}] = \sum_i \sum_{j\neq i} \mathbf{E}[A_{i,j}] = \sum_i \sum_{j\neq i} \frac{1}{|i-j|+1}$$

$$= 2 \cdot \sum_{i=1}^{n} \sum_{j=1}^{i-1} \frac{1}{i-j+1}$$

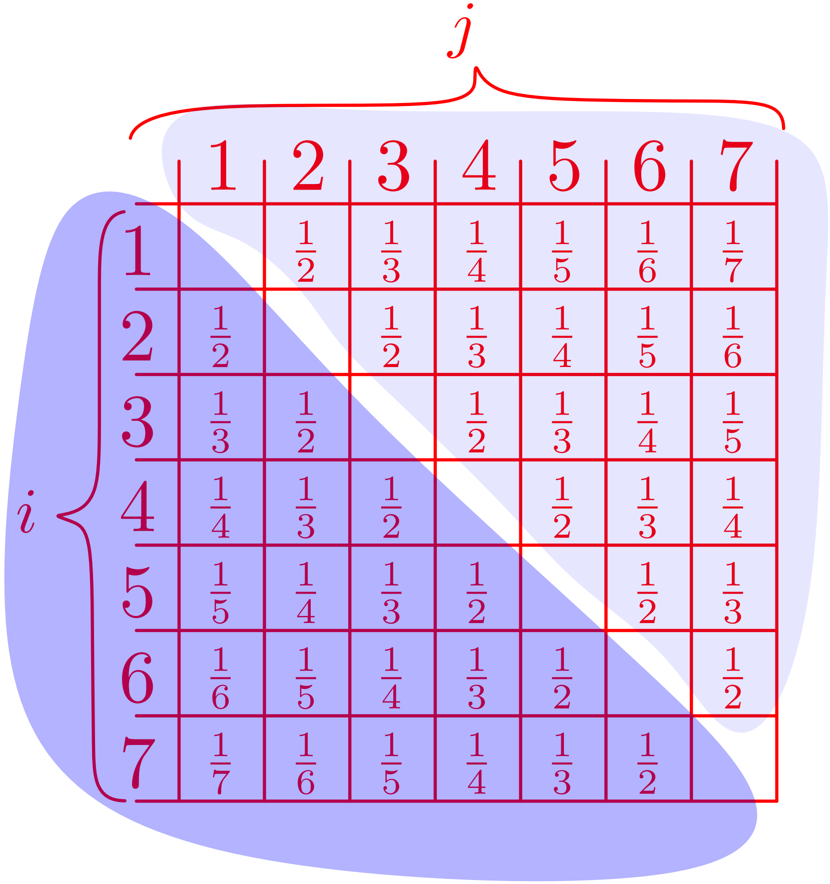


	1	2	3	4	5	6	7
1		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$
2	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$
3	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$
4	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
5	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$
6	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$
7	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	

$$\mathbf{E}[\text{number of comparisons}] = \sum_i \sum_{j\neq i} \mathbf{E}[A_{i,j}] = \sum_i \sum_{j\neq i} \frac{1}{|i-j|+1}$$

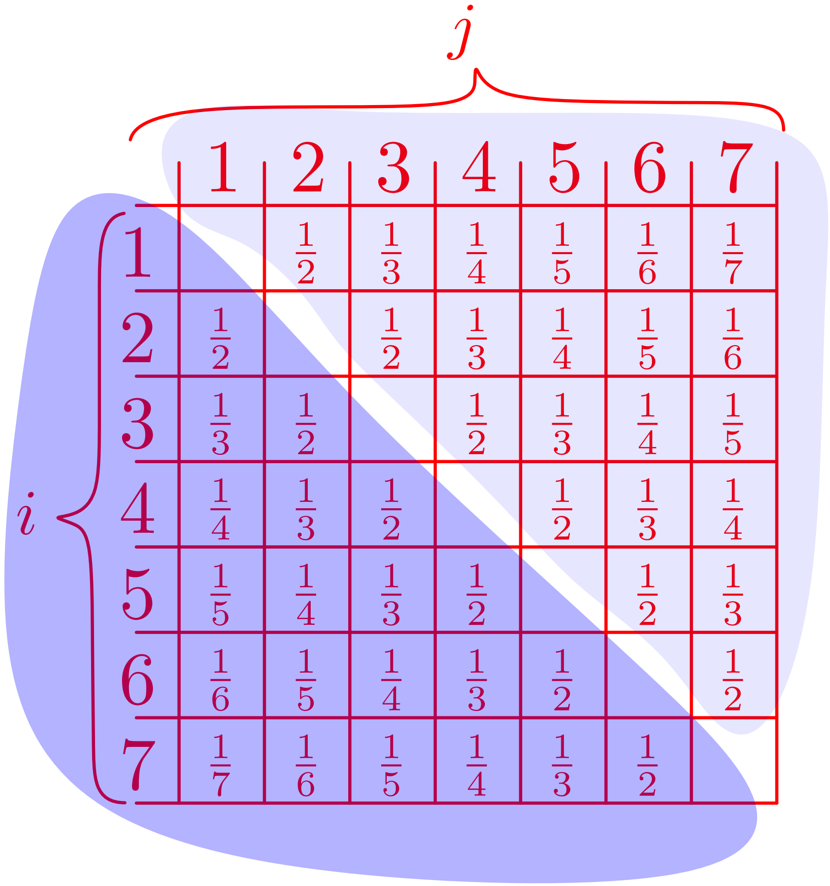
$$= 2 \cdot \sum_{i=1}^{n} \sum_{j=1}^{i-1} \frac{1}{i-j+1}$$

Change of indices: let's write $k := i - j + 1$. When j runs from 1 up to $i-1$, k runs from i down to 2.



	1	2	3	4	5	6	7
1		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$
2	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$
3	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$
4	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
5	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$
6	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$
7	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	

$\mathbf{E}$[number of comparisons] $= \sum_i \sum_{j\neq i} \mathbf{E}[A_{i,j}] = \sum_i \sum_{j\neq i} \frac{1}{|i-j|+1}$



	1	2	3	4	5	6	7
1		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$
2	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$
3	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$
4	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
5	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$
6	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$
7	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	

$$= 2 \cdot \sum_{i=1}^{n} \sum_{j=1}^{i-1} \frac{1}{i-j+1}$$

Change of indices: let's write $k := i - j + 1$. When j runs from 1 up to $i - 1$, k runs from i down to 2.

$$= 2 \cdot \sum_{i=1}^{n} \sum_{k=2}^{i} \frac{1}{k}$$

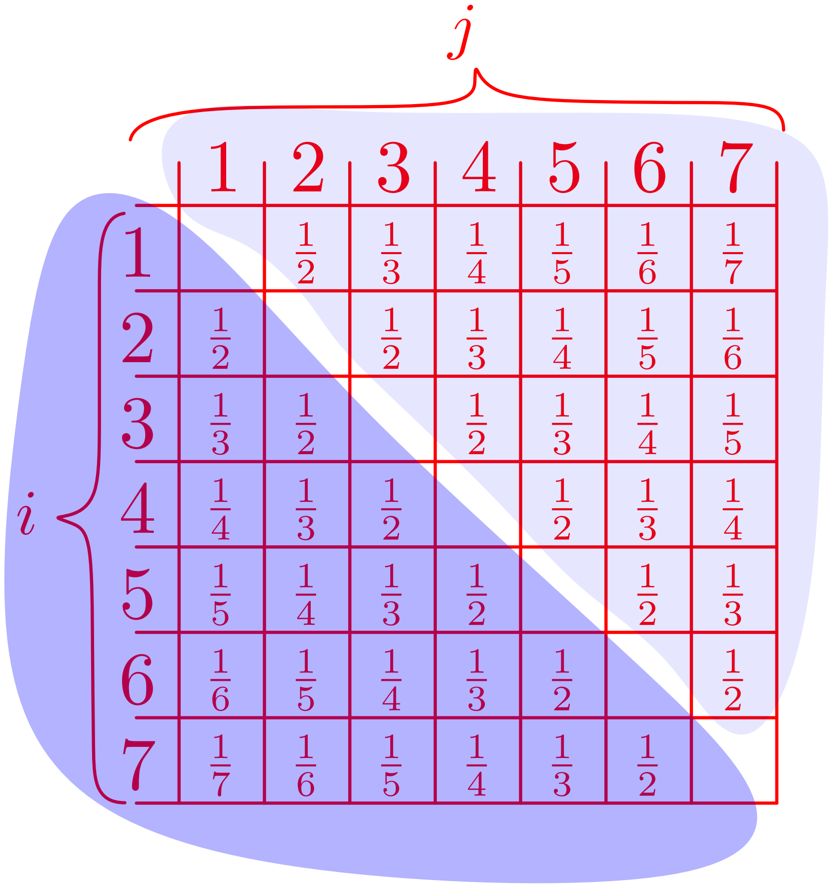
$$\mathbf{E}[\text{number of comparisons}] = \sum_i \sum_{j\neq i} \mathbf{E}[A_{i,j}] = \sum_i \sum_{j\neq i} \frac{1}{|i-j|+1}$$

$$= 2 \cdot \sum_{i=1}^{n} \sum_{j=1}^{i-1} \frac{1}{i-j+1}$$

Change of indices: let's write $k := i - j + 1$. When j runs from 1 up to $i-1$, k runs from i down to 2.

$$= 2 \cdot \sum_{i=1}^{n} \sum_{k=2}^{i} \frac{1}{k}$$

$$\leq 2 \cdot \sum_{i=1}^{n} \sum_{k=2}^{n} \frac{1}{k}$$



	1	2	3	4	5	6	7
1		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$
2	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$
3	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$
4	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
5	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$
6	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$
7	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	

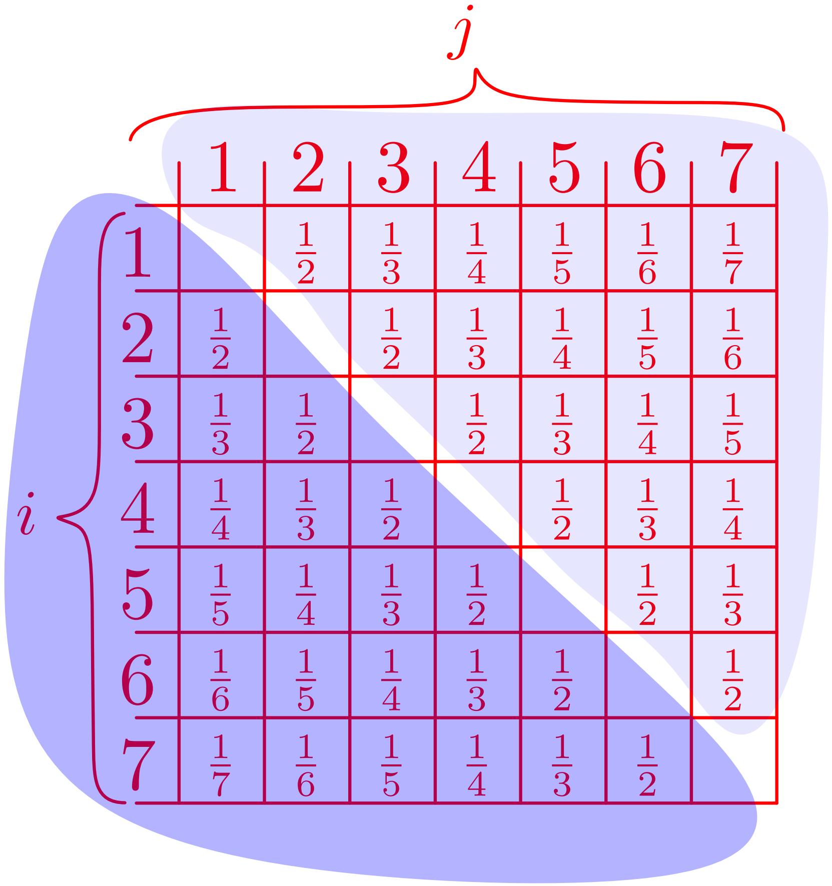
$$\mathbf{E}[\text{number of comparisons}] = \sum_i \sum_{j\neq i} \mathbf{E}[A_{i,j}] = \sum_i \sum_{j\neq i} \frac{1}{|i-j|+1}$$

$$= 2\cdot \sum_{i=1}^{n} \sum_{j=1}^{i-1} \frac{1}{i-j+1}$$

Change of indices: let's write $k := i - j + 1$. When j runs from 1 up to $i-1$, k runs from i down to 2.

$$= 2\cdot \sum_{i=1}^{n} \sum_{k=2}^{i} \frac{1}{k}$$

$$\leq 2\cdot \sum_{i=1}^{n} \sum_{k=2}^{n} \frac{1}{k}$$



	1	2	3	4	5	6	7
1		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$
2	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$
3	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$
4	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
5	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$
6	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$
7	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	

Hier verlieren wir ein bisschen Präzision. Aber die Richtung stimmt: wir wollen eine *Obergrenze* für die erwartete Anzahl von Vergleichen angeben. In diesem Schritt machen wir diese Zahl größer, um uns die Rechnerei leichter zu machen.

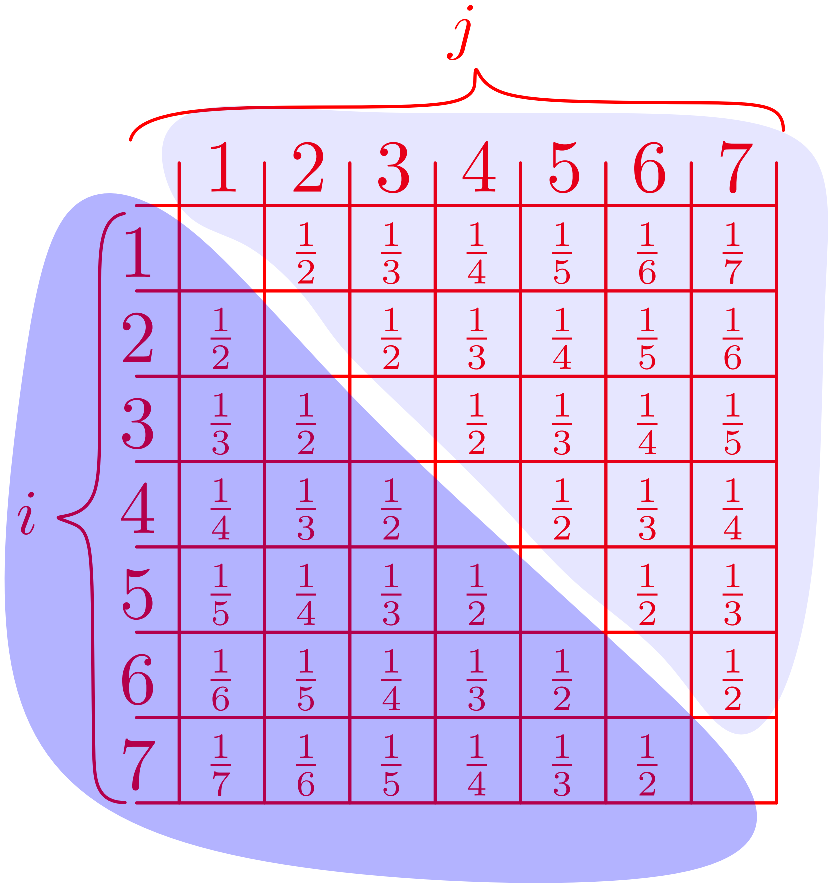
$$\mathbf{E}[\text{number of comparisons}] = \sum_i \sum_{j\neq i} \mathbf{E}[A_{i,j}] = \sum_i \sum_{j\neq i} \frac{1}{|i-j|+1}$$

$$= 2 \cdot \sum_{i=1}^{n} \sum_{j=1}^{i-1} \frac{1}{i-j+1}$$

Change of indices: let's write $k := i - j + 1$. When j runs from 1 up to $i-1$, k runs from i down to 2.

$$= 2 \cdot \sum_{i=1}^{n} \sum_{k=2}^{i} \frac{1}{k}$$

$$\leq 2 \cdot \sum_{i=1}^{n} \sum_{k=2}^{n} \frac{1}{k}$$



$$i \left\{ \begin{array}{c|c|c|c|c|c|c|c} & \multicolumn{7}{c}{\overbrace{\hphantom{1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7}}^{j}} \\ & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \\ \hline 2 & \frac{1}{2} & & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \hline 3 & \frac{1}{3} & \frac{1}{2} & & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \hline 4 & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \hline 5 & \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & & \frac{1}{2} & \frac{1}{3} \\ \hline 6 & \frac{1}{6} & \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & & \frac{1}{2} \\ \hline 7 & \frac{1}{7} & \frac{1}{6} & \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & \\ \hline \end{array} \right.$$

$$\mathbf{E}[\text{number of comparisons}] = \sum_i \sum_{j \neq i} \mathbf{E}[A_{i,j}] = \sum_i \sum_{j \neq i} \frac{1}{|i-j|+1}$$

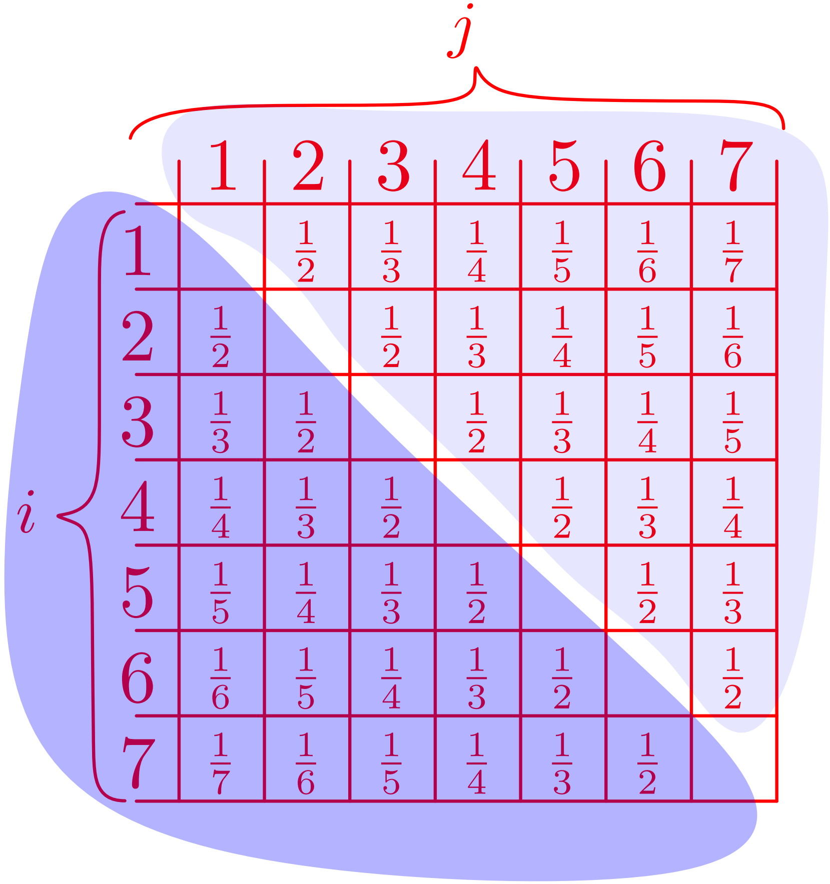
$$= 2 \cdot \sum_{i=1}^{n} \sum_{j=1}^{i-1} \frac{1}{i-j+1}$$

Change of indices: let's write $k := i - j + 1$. When j runs from 1 up to $i - 1$, k runs from i down to 2.

$$= 2 \cdot \sum_{i=1}^{n} \sum_{k=2}^{i} \frac{1}{k}$$

$$\leq 2 \cdot \sum_{i=1}^{n} \sum_{k=2}^{n} \frac{1}{k}$$

$$\leq 2\, n \sum_{k=2}^{n} \frac{1}{k} \ .$$



	1	2	3	4	5	6	7
1		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$
2	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$
3	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$
4	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
5	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$
6	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$
7	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	

$$\mathbf{E}[\text{number of comparisons}] = \sum_i \sum_{j \neq i} \mathbf{E}[A_{i,j}] = \sum_i \sum_{j \neq i} \frac{1}{|i-j|+1}$$

$$= 2 \cdot \sum_{i=1}^{n} \sum_{j=1}^{i-1} \frac{1}{i-j+1}$$

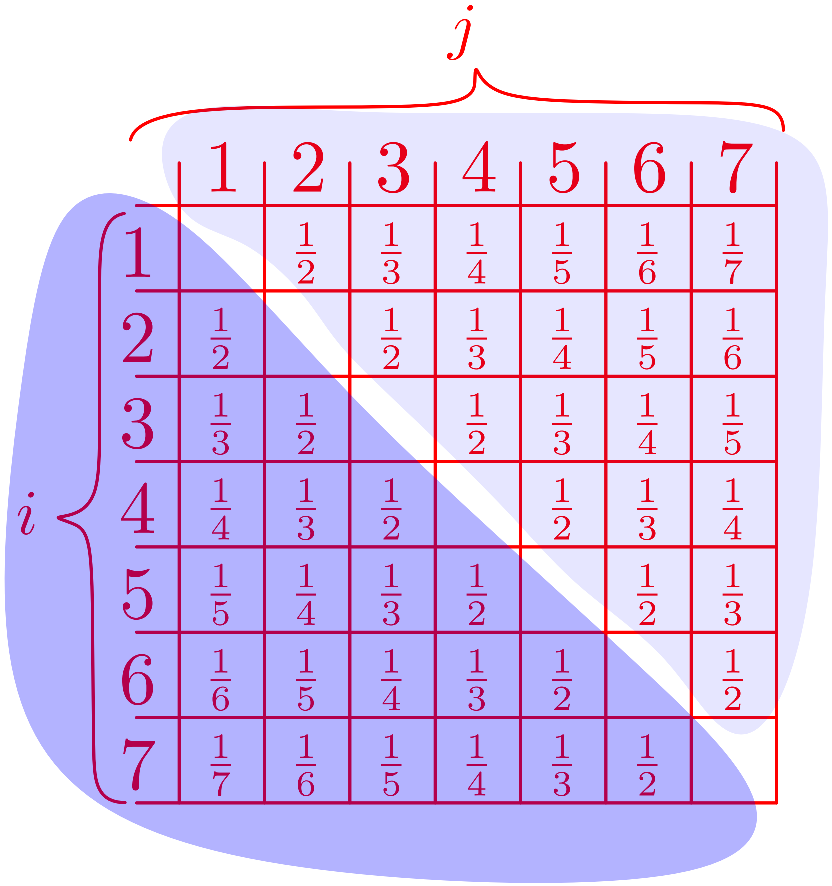
Change of indices: let's write $k := i - j + 1$. When j runs from 1 up to $i-1$, k runs from i down to 2.

$$= 2 \cdot \sum_{i=1}^{n} \sum_{k=2}^{i} \frac{1}{k}$$

$$\leq 2 \cdot \sum_{i=1}^{n} \sum_{k=2}^{n} \frac{1}{k}$$

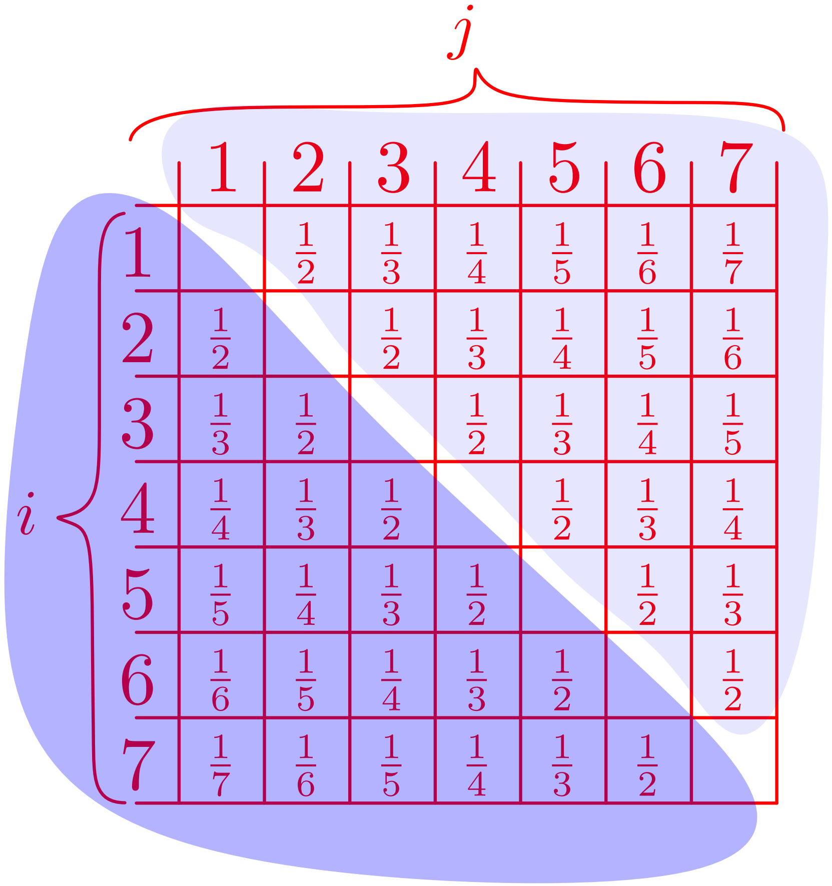
$$\leq 2\,n \sum_{k=2}^{n} \frac{1}{k} \, .$$

What is this?



	1	2	3	4	5	6	7
1		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$
2	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$
3	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$
4	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
5	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$
6	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$
7	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	

$$\mathbf{E}[\text{number of comparisons}] = \sum_i \sum_{j\neq i} \mathbf{E}[A_{i,j}] = \sum_i \sum_{j\neq i} \frac{1}{|i-j|+1}$$



$$\begin{array}{c|ccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7\\
\hline
1 & & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7}\\
2 & \frac{1}{2} & & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6}\\
3 & \frac{1}{3} & \frac{1}{2} & & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5}\\
4 & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & & \frac{1}{2} & \frac{1}{3} & \frac{1}{4}\\
5 & \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & & \frac{1}{2} & \frac{1}{3}\\
6 & \frac{1}{6} & \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & & \frac{1}{2}\\
7 & \frac{1}{7} & \frac{1}{6} & \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} &
\end{array}$$

$$= 2 \cdot \sum_{i=1}^{n} \sum_{j=1}^{i-1} \frac{1}{i-j+1}$$

Change of indices: let's write $k := i - j + 1$. When j runs from 1 up to $i-1$, k runs from i down to 2.

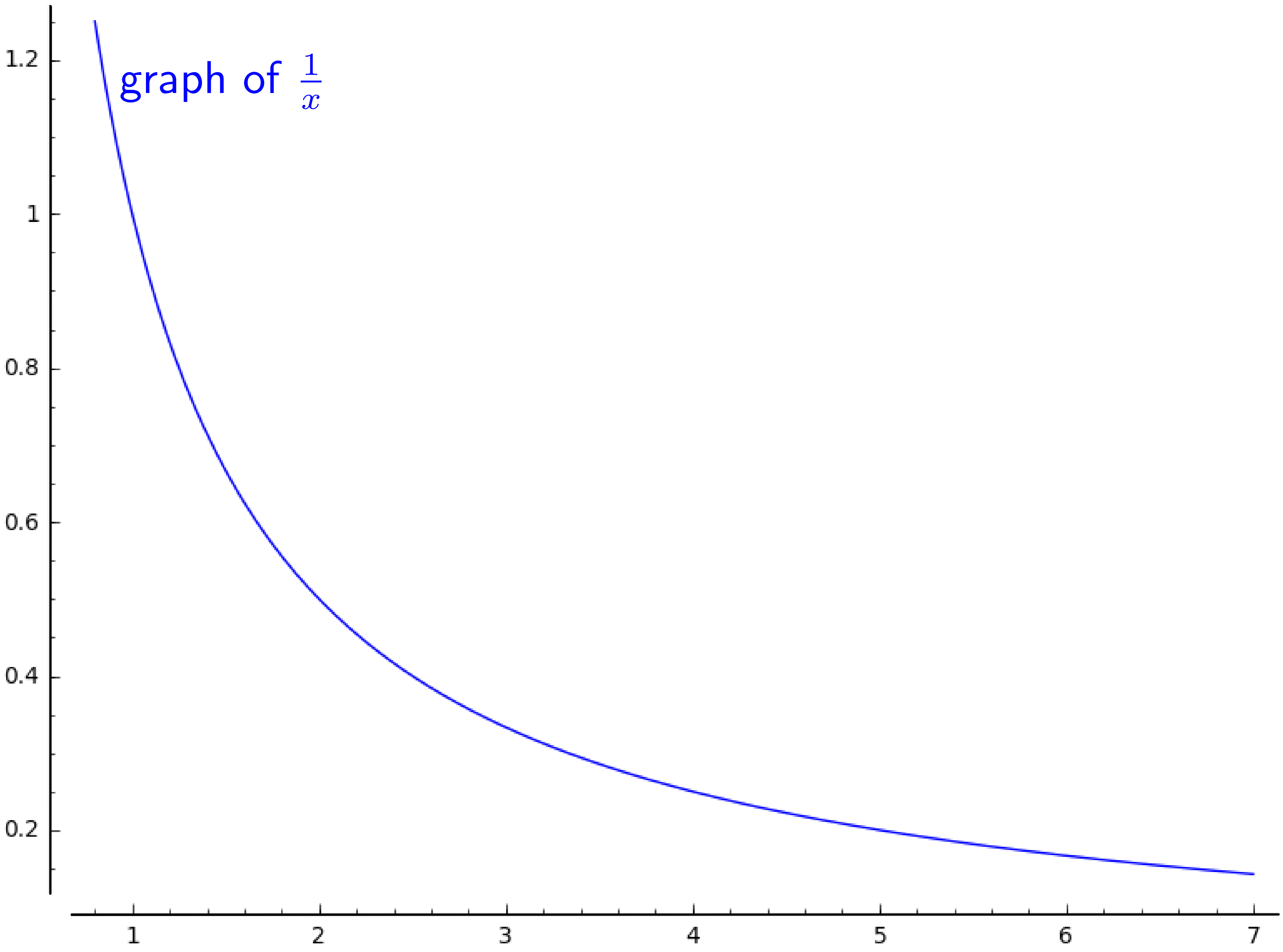
$$= 2 \cdot \sum_{i=1}^{n} \sum_{k=2}^{i} \frac{1}{k}$$

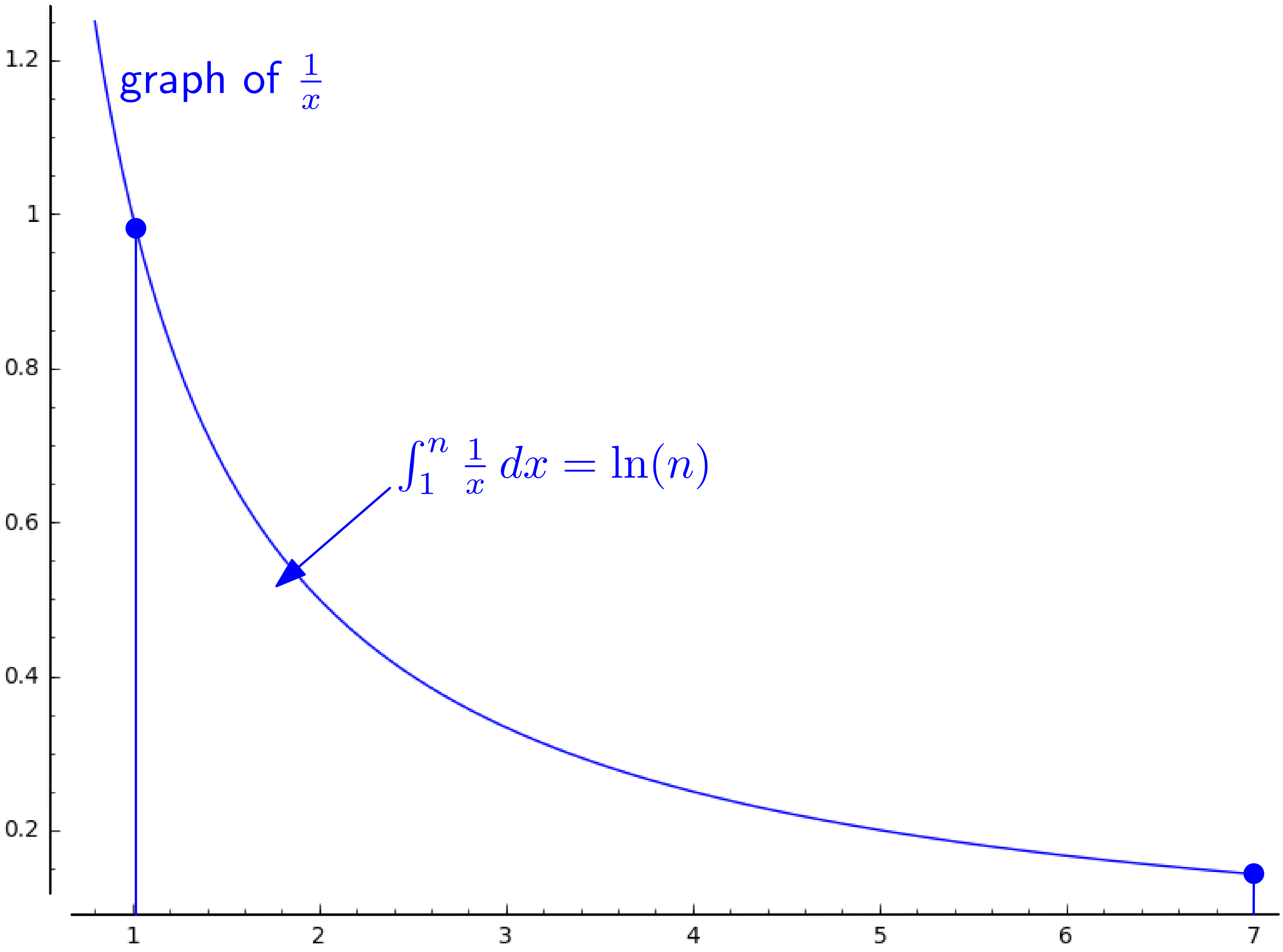
$$\le 2 \cdot \sum_{i=1}^{n} \sum_{k=2}^{n} \frac{1}{k}$$

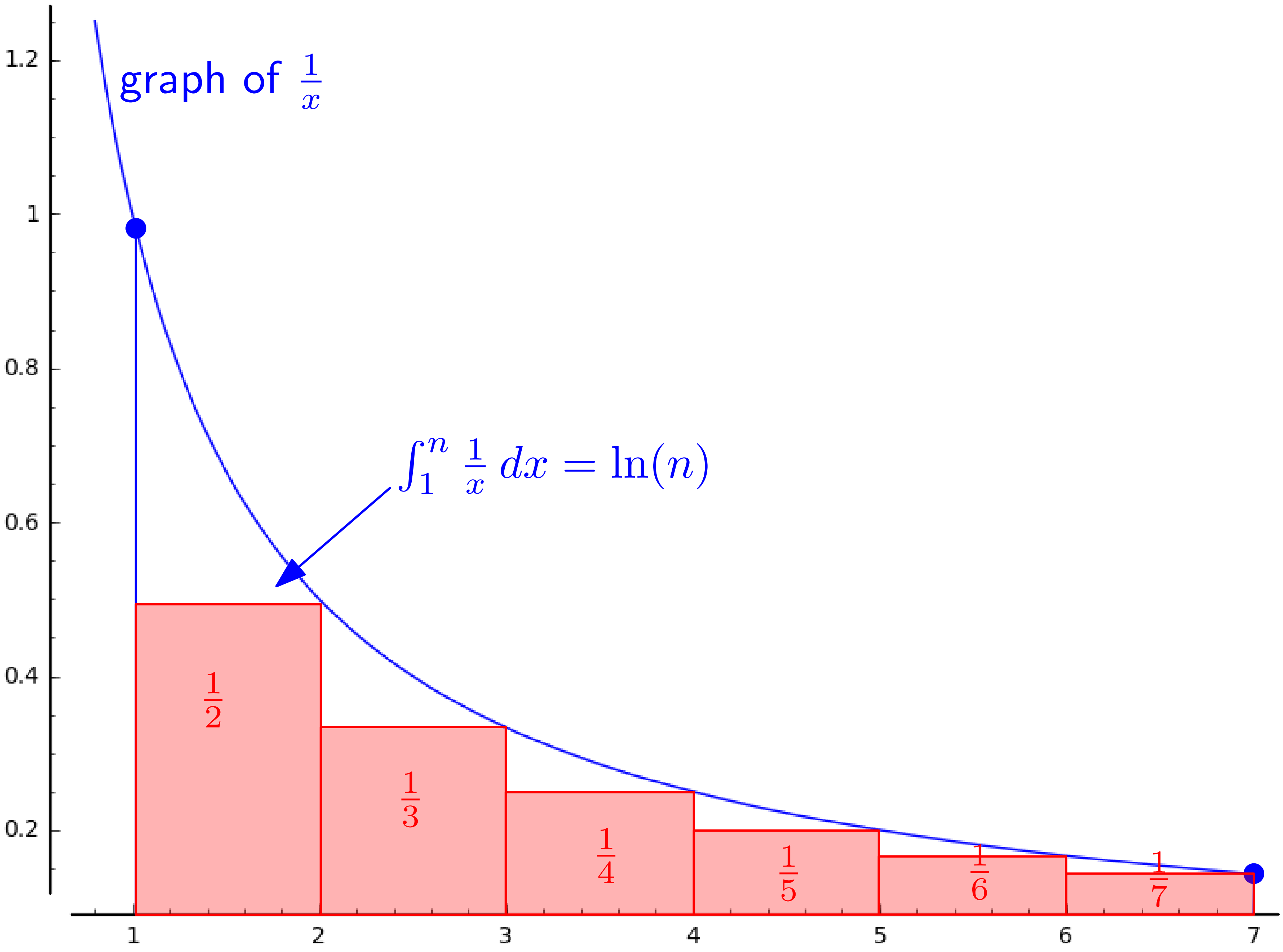
$$\le 2\,n \sum_{k=2}^{n} \frac{1}{k} \cdot$$

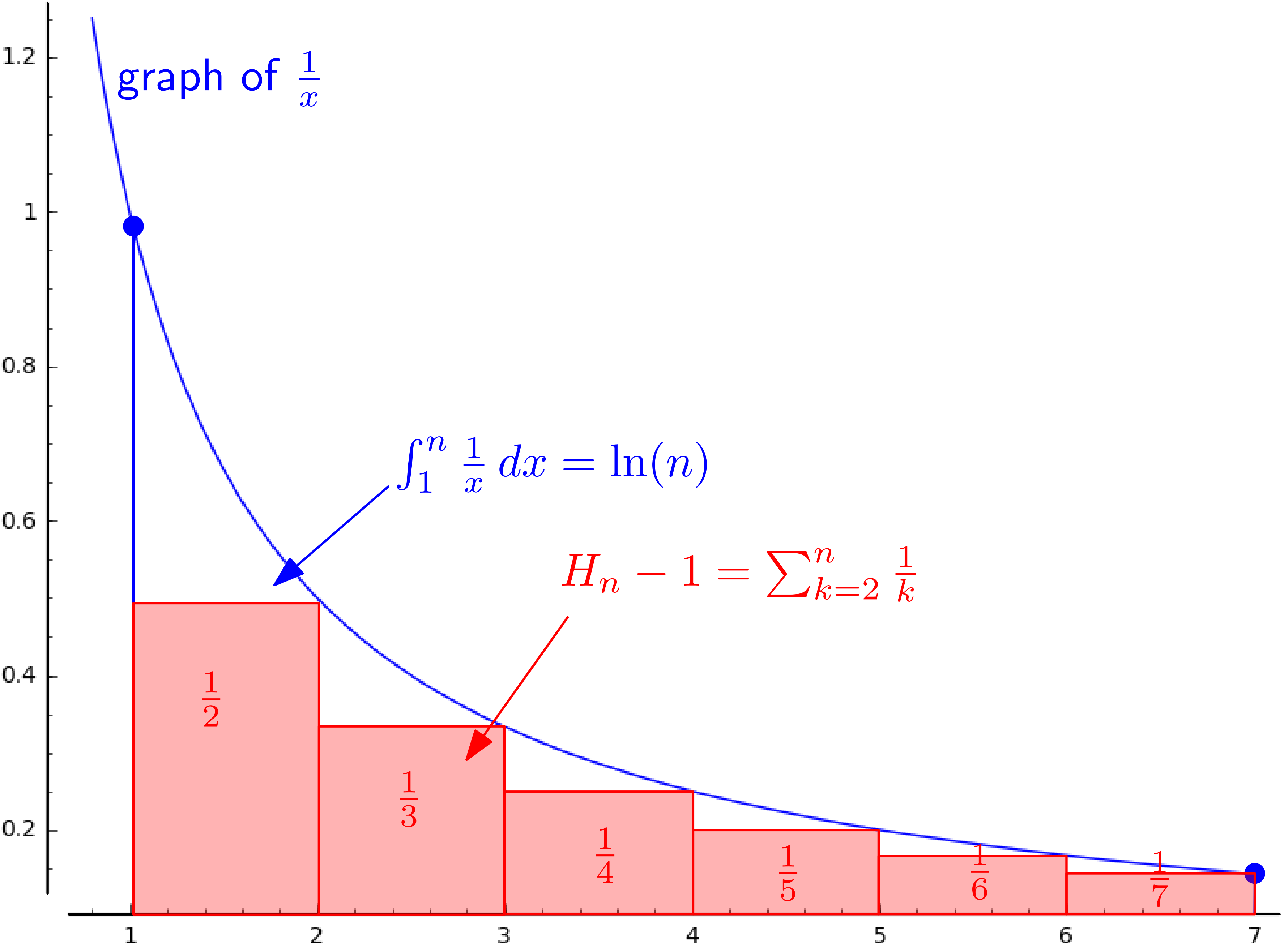
What is this?

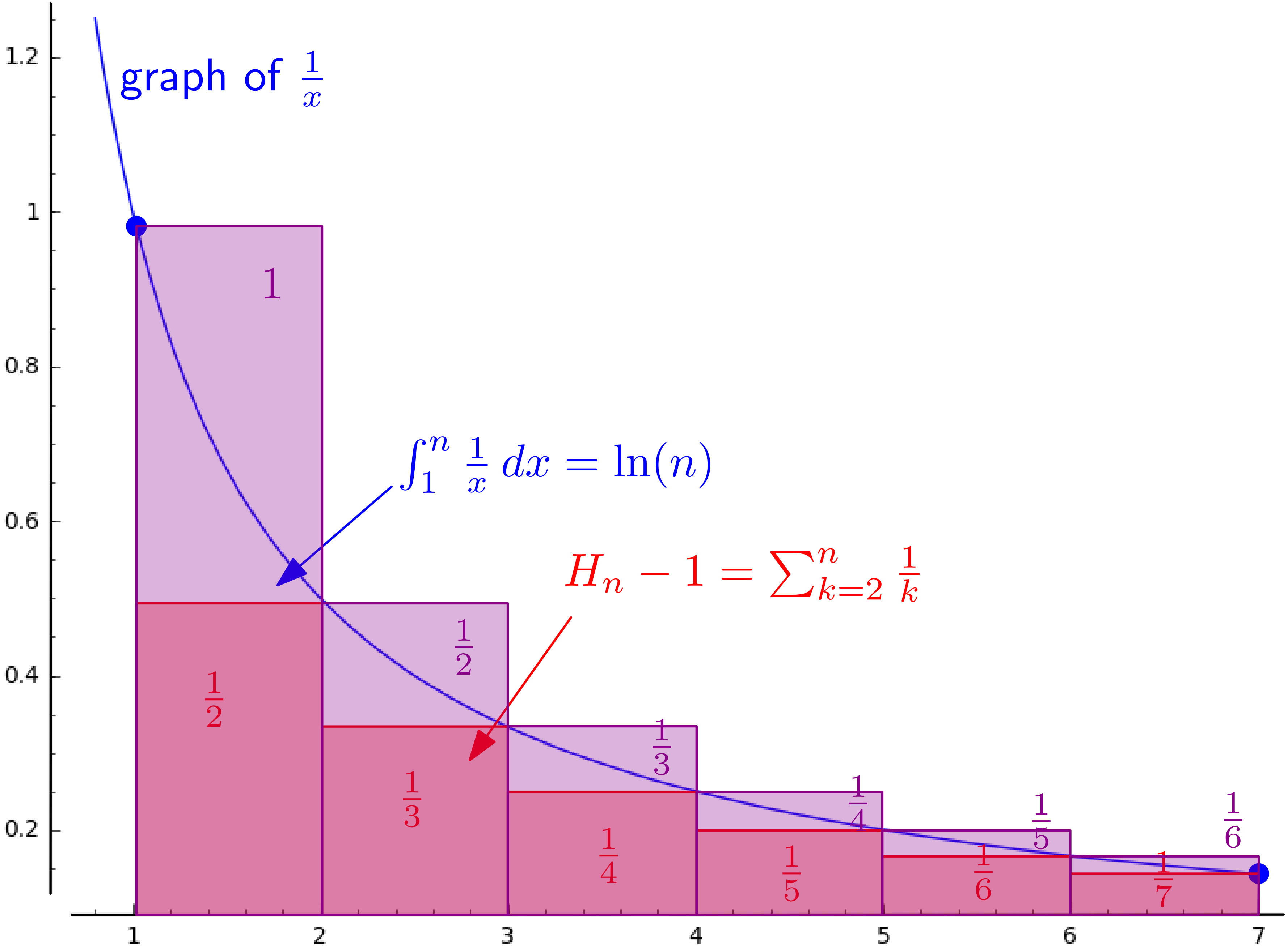
Google + Wikipedia: $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is the n^{th} *harmonic number*, and $H_n \approx \ln(n) + 0.5772156649$.


graph of $\frac{1}{x}$
1.2
1
0.8
0.6
0.4
0.2
1
2
3
4
5
6
7


graph of $\frac{1}{x}$
$\int_1^n \frac{1}{x}\,dx = \ln(n)$
1.2
1
0.8
0.6
0.4
0.2
1
2
3
4
5
6
7


graph of $\frac{1}{x}$
$\int_1^n \frac{1}{x}\,dx = \ln(n)$
$\frac{1}{2}$
$\frac{1}{3}$
$\frac{1}{4}$
$\frac{1}{5}$
$\frac{1}{6}$
$\frac{1}{7}$
1.2
1
0.8
0.6
0.4
0.2
1
2
3
4
5
6
7


graph of $\frac{1}{x}$
$\int_1^n \frac{1}{x}\,dx = \ln(n)$
$H_n - 1 = \sum_{k=2}^n \frac{1}{k}$
$\frac{1}{2}$
$\frac{1}{3}$
$\frac{1}{4}$
$\frac{1}{5}$
$\frac{1}{6}$
$\frac{1}{7}$
1.2
1
0.8
0.6
0.4
0.2
1
2
3
4
5
6
7


graph of $\frac{1}{x}$
$\int_1^n \frac{1}{x}\, dx = \ln(n)$
$H_n - 1 = \sum_{k=2}^{n} \frac{1}{k}$
1
$\frac{1}{2}$
$\frac{1}{3}$
$\frac{1}{4}$
$\frac{1}{5}$
$\frac{1}{6}$
$\frac{1}{2}$
$\frac{1}{3}$
$\frac{1}{4}$
$\frac{1}{5}$
$\frac{1}{6}$
$\frac{1}{7}$
1.2
1
0.8
0.6
0.4
0.2
1
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3
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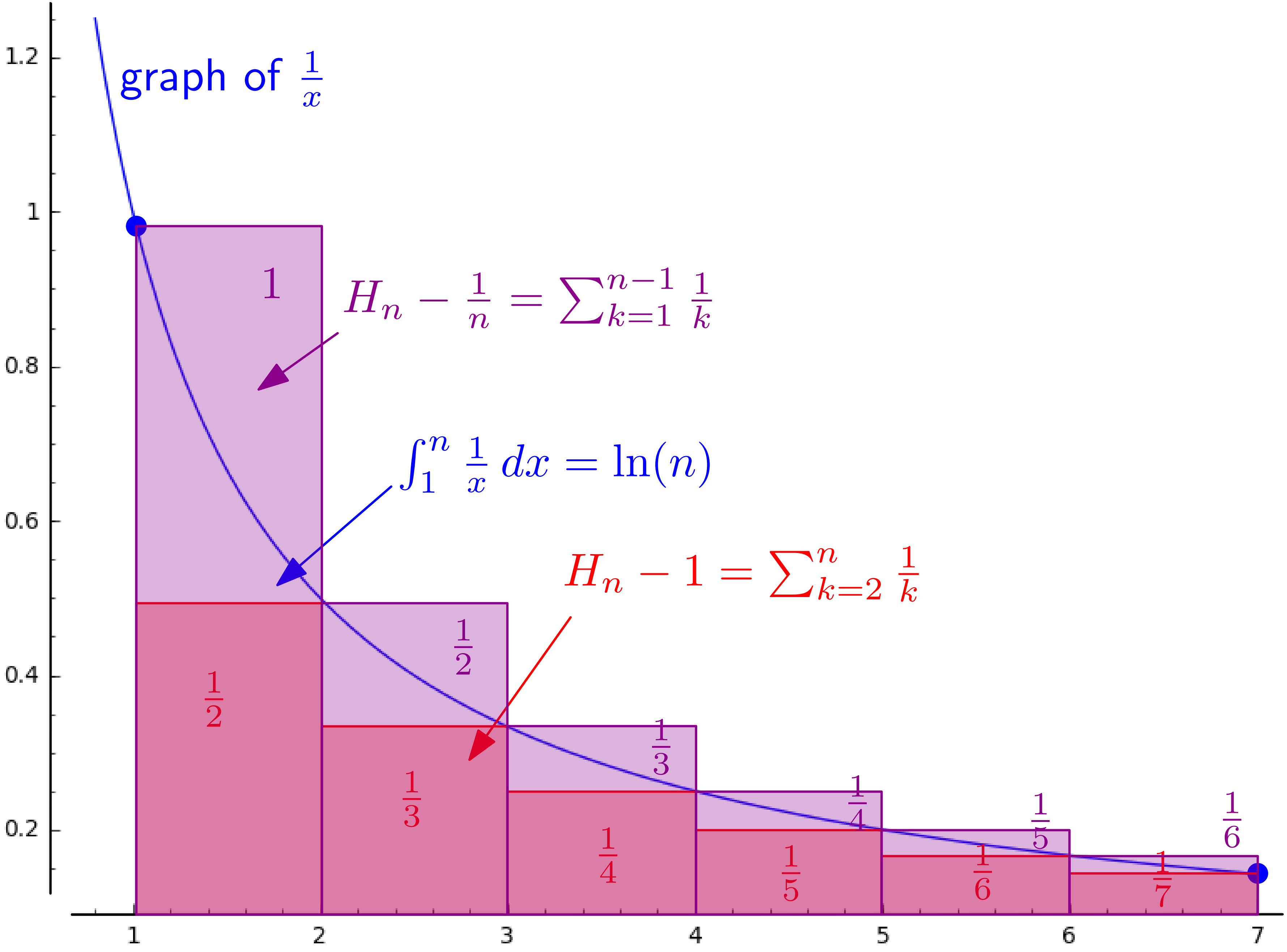
5
6
7


graph of $\frac{1}{x}$
$H_n - \frac{1}{n} = \sum_{k=1}^{n-1} \frac{1}{k}$
$\int_1^n \frac{1}{x}\,dx = \ln(n)$
$H_n - 1 = \sum_{k=2}^{n} \frac{1}{k}$
1
$\frac{1}{2}$
$\frac{1}{3}$
$\frac{1}{4}$
$\frac{1}{5}$
$\frac{1}{6}$
$\frac{1}{2}$
$\frac{1}{3}$
$\frac{1}{4}$
$\frac{1}{5}$
$\frac{1}{6}$
$\frac{1}{7}$
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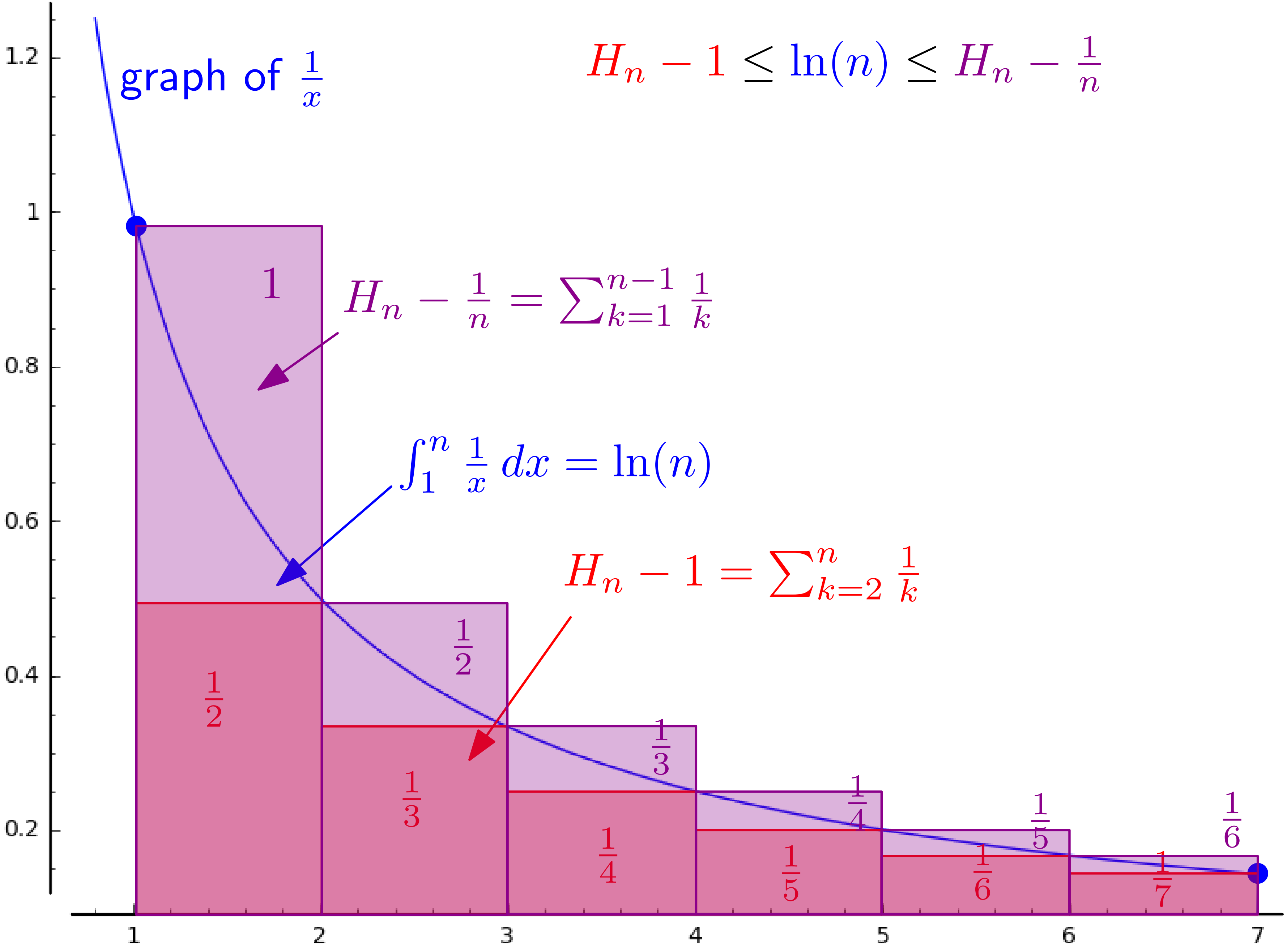
graph of $\frac{1}{x}$
$H_n - 1 \leq \ln(n) \leq H_n - \frac{1}{n}$
$H_n - \frac{1}{n} = \sum_{k=1}^{n-1} \frac{1}{k}$
$\int_1^n \frac{1}{x}\, dx = \ln(n)$
$H_n - 1 = \sum_{k=2}^{n} \frac{1}{k}$
1
$\frac{1}{2}$
$\frac{1}{3}$
$\frac{1}{4}$
$\frac{1}{5}$
$\frac{1}{6}$
$\frac{1}{2}$
$\frac{1}{3}$
$\frac{1}{4}$
$\frac{1}{5}$
$\frac{1}{6}$
$\frac{1}{7}$
1.2
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1
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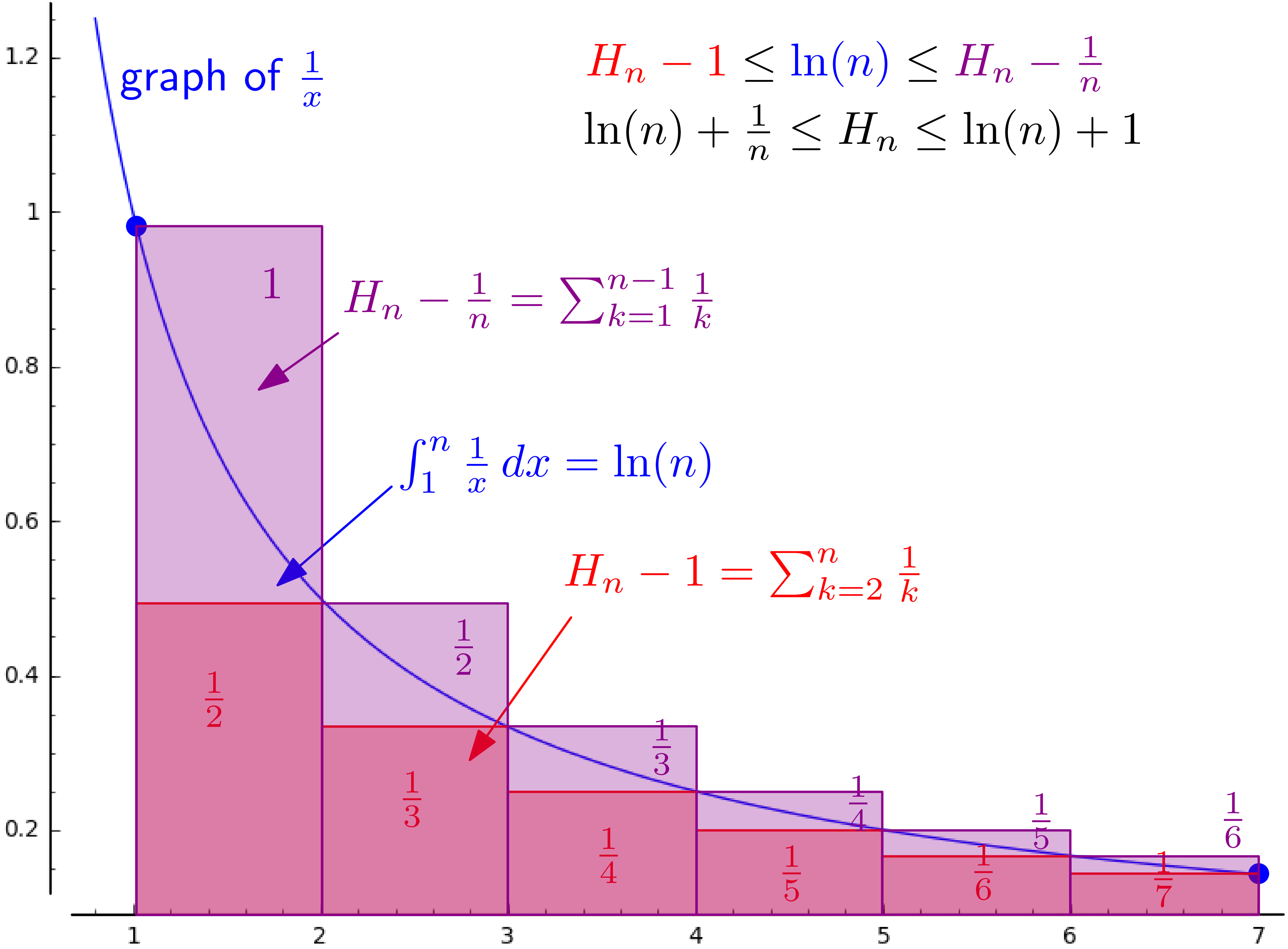
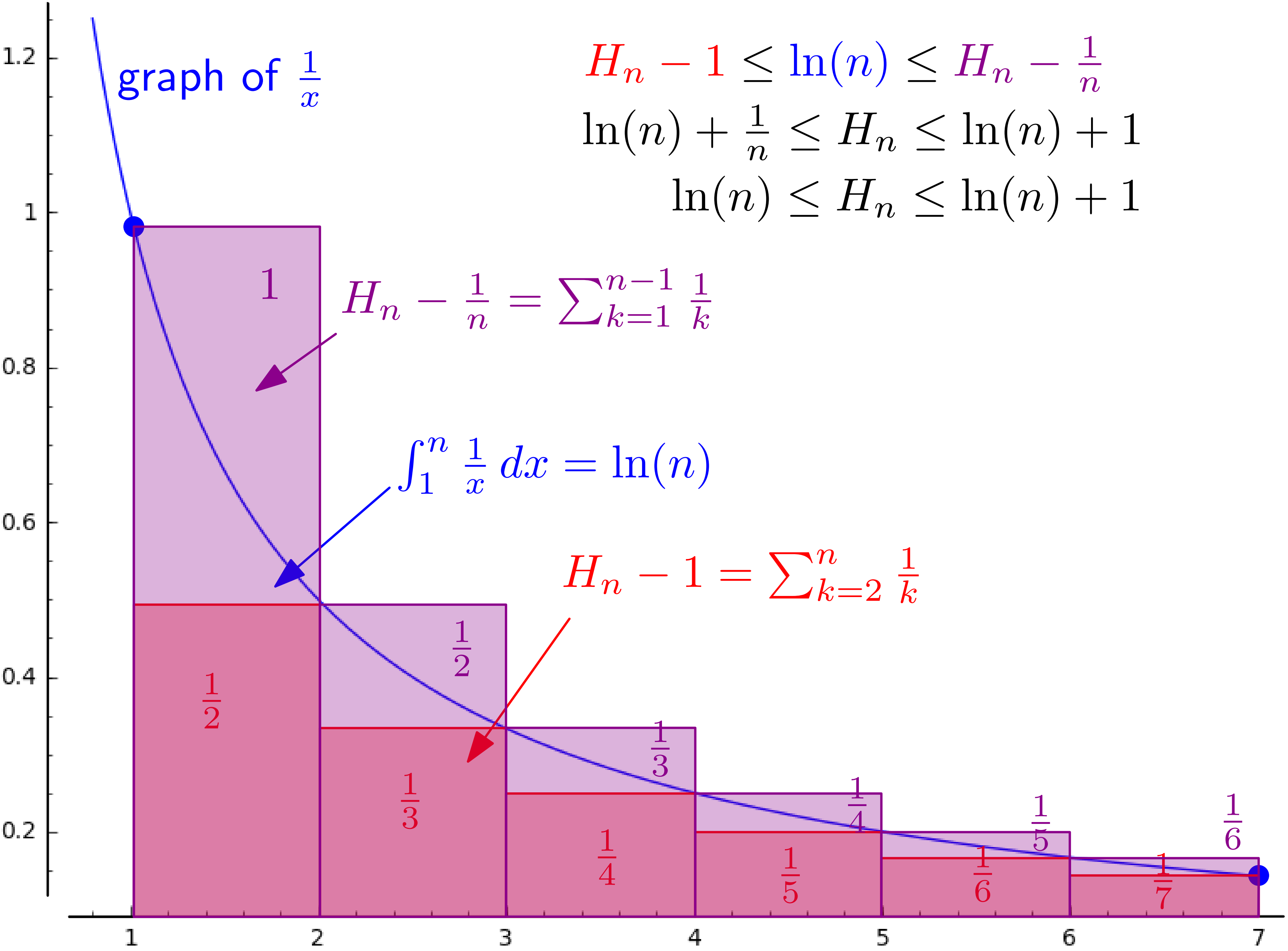
7


graph of $\frac{1}{x}$
$H_n - 1 \le \ln(n) \le H_n - \frac{1}{n}$
$\ln(n) + \frac{1}{n} \le H_n \le \ln(n) + 1$
$H_n - \frac{1}{n} = \sum_{k=1}^{n-1} \frac{1}{k}$
$\int_1^n \frac{1}{x}\, dx = \ln(n)$
$H_n - 1 = \sum_{k=2}^{n} \frac{1}{k}$
1
$\frac{1}{2}$
$\frac{1}{3}$
$\frac{1}{4}$
$\frac{1}{5}$
$\frac{1}{6}$
$\frac{1}{2}$
$\frac{1}{3}$
$\frac{1}{4}$
$\frac{1}{5}$
$\frac{1}{6}$
$\frac{1}{7}$
1.2
1
0.8
0.6
0.4
0.2
1
2
3
4
5
6
7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1
$\frac{1}{2}$
$\frac{1}{3}$
$\frac{1}{4}$
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$\frac{1}{6}$
$\frac{1}{2}$
$\frac{1}{3}$
$\frac{1}{4}$
$\frac{1}{5}$
$\frac{1}{6}$
$\frac{1}{7}$
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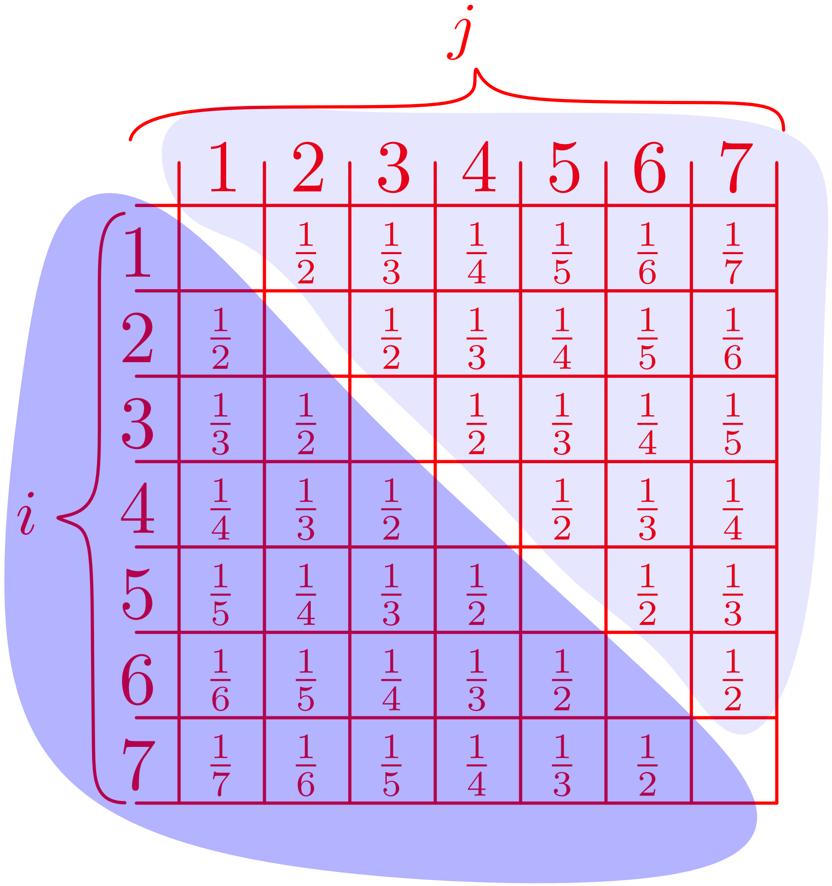

	1	2	3	4	5	6	7
1		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$
2	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$
3	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$
4	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
5	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$
6	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$
7	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	

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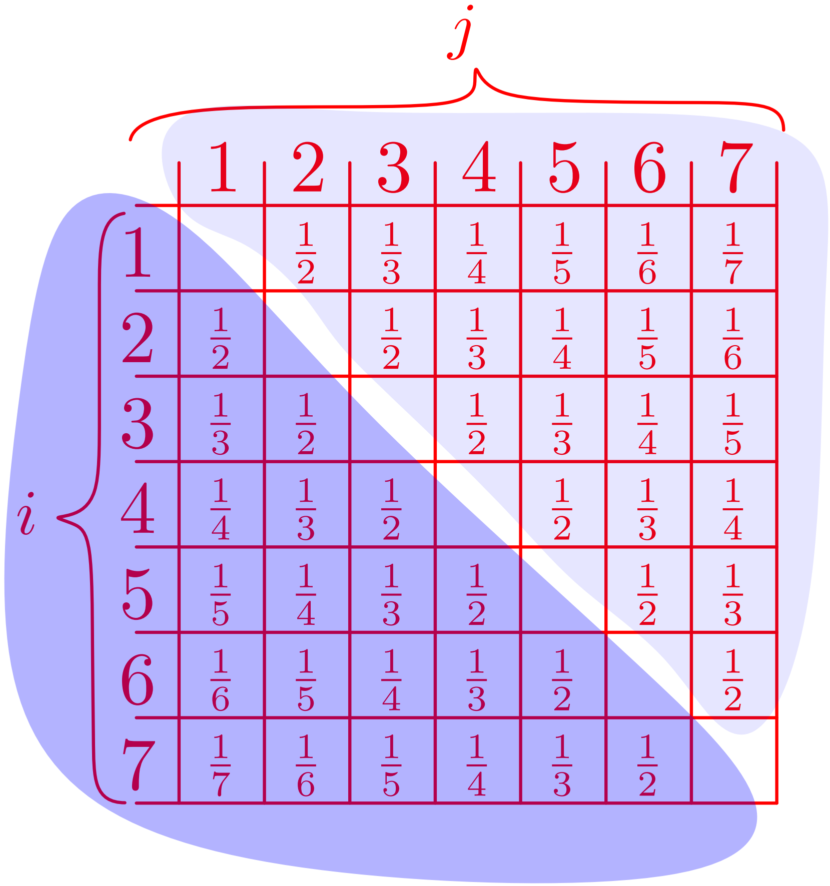

	1	2	3	4	5	6	7
1		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$
2	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$
3	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$
4	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
5	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$
6	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$
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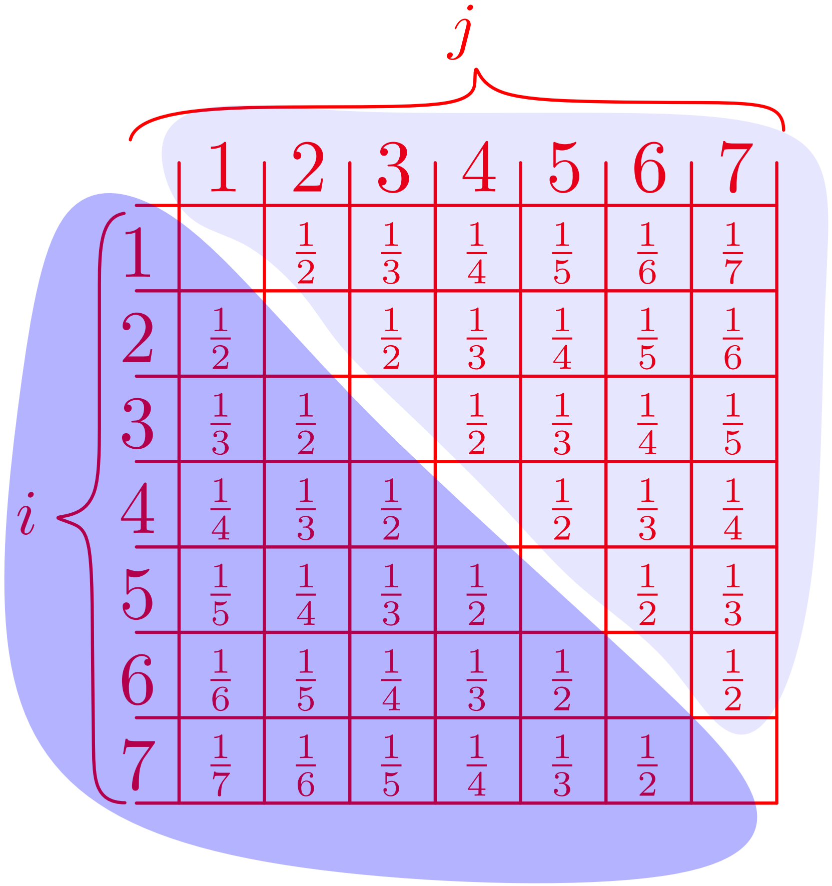

	1	2	3	4	5	6	7
1		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$
2	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$
3	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$
4	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
5	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$		$\frac{1}{2}$	$\frac{1}{3}$
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Theorem. The expected number of comparisons made by QuickSort on an array with n elements is at most $2\, n \ln(n)$.

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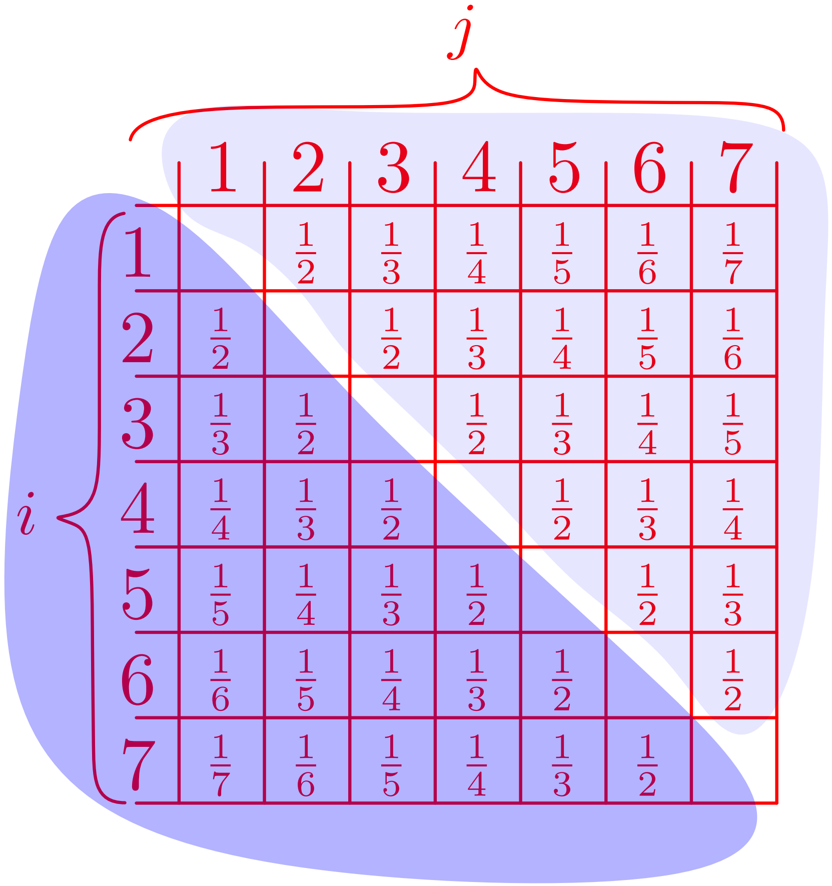

	1	2	3	4	5	6	7
1		$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$
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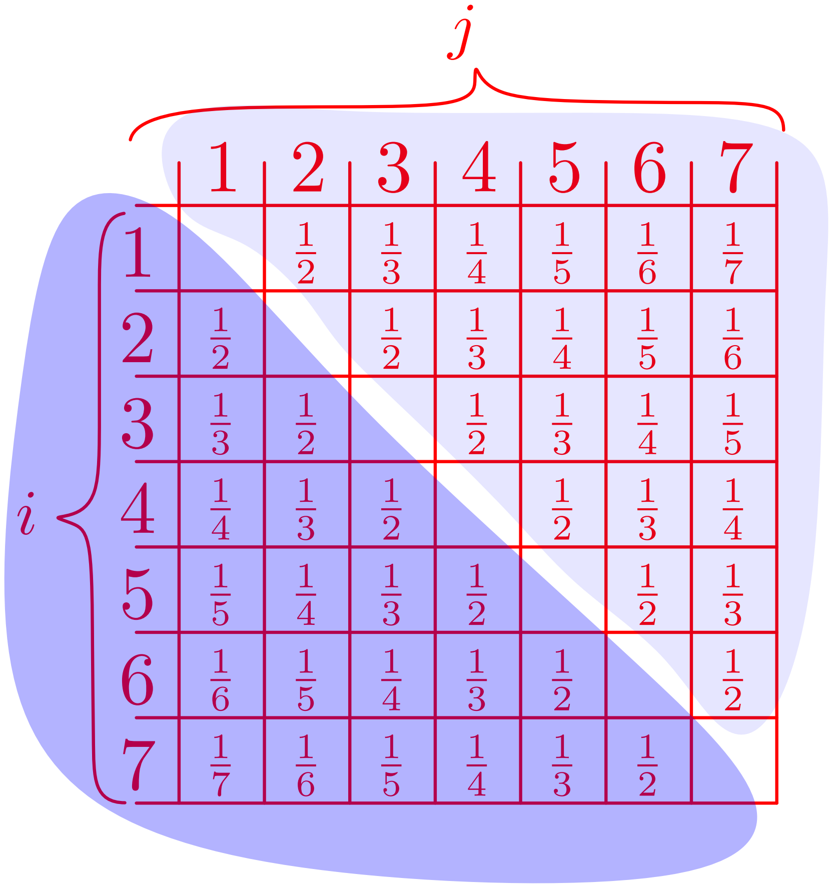

$$\begin{array}{c|ccccccc} i \backslash j & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \\ 2 & \frac{1}{2} & & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ 3 & \frac{1}{3} & \frac{1}{2} & & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ 4 & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ 5 & \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & & \frac{1}{2} & \frac{1}{3} \\ 6 & \frac{1}{6} & \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & & \frac{1}{2} \\ 7 & \frac{1}{7} & \frac{1}{6} & \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & \end{array}$$

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$$= 2 \cdot \sum_{i=1}^{n} \sum_{k=1}^{n} [k \geq i+1] \frac{1}{k}$$

Bracket notation: $[\texttt{cond}]$ is 1 when cond holds and 0 else.

Allows us to replace the “dynamic” start index $k = i+1$ by the “static” start index $k = 1$ because all in $[1..i]$ is killed by $[k \geq i+1]$ anyway.

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$$= 2 \cdot \sum_{k=1}^{n} \sum_{i=1}^{k-1} \frac{1}{k}$$

$$= 2 \cdot \sum_{k=1}^{n} \frac{k-1}{k} = 2 \cdot \sum_{k=1}^{n} \left(1 - \frac{1}{k}\right)$$

$$\mathbf{E}[\text{number of comparisons}] = \sum_i \sum_{j \neq i} \mathbf{E}[A_{i,j}] = \sum_i \sum_{j \neq i} \frac{1}{|i-j|+1}$$

$$= 2 \cdot \sum_{i=1}^{n} \sum_{k=2}^{i} \frac{1}{k}$$

$$= 2 \cdot \sum_{i=1}^{n} \left(\sum_{k=2}^{n} \frac{1}{k} - \sum_{k=i+1}^{n} \frac{1}{k} \right)$$

$$= 2 \cdot \sum_{i=1}^{n} \sum_{k=2}^{n} \frac{1}{k} - 2 \cdot \sum_{i=1}^{n} \sum_{k=i+1}^{n} \frac{1}{k}$$

$$= 2\, n (H_n - 1)$$

switch order of summation

$$= 2 \cdot \sum_{i=1}^{n} \sum_{k=1}^{n} [k \geq i+1] \frac{1}{k}$$

$$= 2 \cdot \sum_{k=1}^{n} \sum_{i=1}^{n} [k \geq i+1] \frac{1}{k}$$

$$= 2 \cdot \sum_{k=1}^{n} \sum_{i=1}^{n} [i \leq k-1] \frac{1}{k}$$

$$= 2 \cdot \sum_{k=1}^{n} \sum_{i=1}^{k-1} \frac{1}{k}$$

$$= 2 \cdot \sum_{k=1}^{n} \frac{k-1}{k} = 2 \cdot \sum_{k=1}^{n} \left(1 - \frac{1}{k} \right)$$

$$= 2n - 2 \sum_{k=1}^{n} \frac{1}{k}$$

$$\mathbf{E}[\text{number of comparisons}] = \sum_i \sum_{j\neq i} \mathbf{E}[A_{i,j}] = \sum_i \sum_{j\neq i} \frac{1}{|i-j|+1}$$

$$= 2 \cdot \sum_{i=1}^{n} \sum_{k=2}^{i} \frac{1}{k}$$

$$= 2 \cdot \sum_{i=1}^{n} \left(\sum_{k=2}^{n} \frac{1}{k} - \sum_{k=i+1}^{n} \frac{1}{k} \right)$$

$$= 2 \cdot \sum_{i=1}^{n} \sum_{k=2}^{n} \frac{1}{k} - 2 \cdot \sum_{i=1}^{n} \sum_{k=i+1}^{n} \frac{1}{k}$$

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switch order of summation

$$= 2 \cdot \sum_{i=1}^{n} \sum_{k=1}^{n} [k \geq i+1] \frac{1}{k}$$

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$$= 2 \cdot \sum_{k=1}^{n} \sum_{i=1}^{k-1} \frac{1}{k}$$

$$= 2 \cdot \sum_{k=1}^{n} \frac{k-1}{k} = 2 \cdot \sum_{k=1}^{n} \left(1 - \frac{1}{k}\right)$$

$$= 2n - 2 \sum_{k=1}^{n} \frac{1}{k} = 2n - 2\,H_n$$

$$\mathbf{E}[\text{number of comparisons}] = \sum_i \sum_{j \neq i} \mathbf{E}[A_{i,j}] = \sum_i \sum_{j \neq i} \frac{1}{|i-j|+1}$$

$$= 2 \cdot \sum_{i=1}^{n} \sum_{k=2}^{i} \frac{1}{k}$$

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$$= 2 \cdot \sum_{i=1}^{n} \sum_{k=2}^{n} \frac{1}{k} - 2 \cdot \sum_{i=1}^{n} \sum_{k=i+1}^{n} \frac{1}{k}$$

$$= 2n - 2\, H_n$$

$$= 2\, n(H_n - 1)$$

$$\mathbf{E}[\text{number of comparisons}] = \sum_i \sum_{j\neq i} \mathbf{E}[A_{i,j}] = \sum_i \sum_{j\neq i} \frac{1}{|i-j|+1}$$

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$$= 2 \cdot \sum_{i=1}^{n} \sum_{k=2}^{n} \frac{1}{k} - 2 \cdot \sum_{i=1}^{n} \sum_{k=i+1}^{n} \frac{1}{k}$$

$$= 2n - 2\, H_n$$

$$= 2\, n (H_n - 1)$$

$$= 2\, n (H_n - 1) - (2\, n - 2\, H_n)$$

$$\mathbf{E}[\text{number of comparisons}] = \sum_i \sum_{j \neq i} \mathbf{E}[A_{i,j}] = \sum_i \sum_{j \neq i} \frac{1}{|i-j|+1}$$

$$= 2 \cdot \sum_{i=1}^{n} \sum_{k=2}^{i} \frac{1}{k}$$

$$= 2 \cdot \sum_{i=1}^{n} \left(\sum_{k=2}^{n} \frac{1}{k} - \sum_{k=i+1}^{n} \frac{1}{k} \right)$$

$$= 2n - 2\,H_n$$

$$= 2 \cdot \sum_{i=1}^{n} \sum_{k=2}^{n} \frac{1}{k} - 2 \cdot \sum_{i=1}^{n} \sum_{k=i+1}^{n} \frac{1}{k}$$

$$= 2\,n(H_n - 1)$$

$$= 2\,n(H_n - 1) - (2\,n - 2\,H_n)$$

$$= 2\,nH_n - 4\,n + 2\,H_n$$

$$\mathbf{E}[\text{number of comparisons}] = \sum_i \sum_{j\neq i} \mathbf{E}[A_{i,j}] = \sum_i \sum_{j\neq i} \frac{1}{|i-j|+1}$$

$$= 2\cdot \sum_{i=1}^{n} \sum_{k=2}^{i} \frac{1}{k}$$

$$= 2\cdot \sum_{i=1}^{n} \left(\sum_{k=2}^{n} \frac{1}{k} - \sum_{k=i+1}^{n} \frac{1}{k}\right)$$

$$= 2\cdot \sum_{i=1}^{n} \sum_{k=2}^{n} \frac{1}{k} - 2\cdot \sum_{i=1}^{n} \sum_{k=i+1}^{n} \frac{1}{k}$$

$$= 2n - 2\,H_n$$

$$= 2\,n(H_n - 1)$$

$$= 2\,n(H_n - 1) - (2\,n - 2\,H_n)$$

$$= 2\,nH_n - 4\,n + 2\,H_n$$

Theorem. The expected number of comparisons made by QuickSort on an array with n elements is exactly $2\,nH_n - 4\,n + 2\,H_n$. [Provided the array has no duplicate elements, in which case there might be fewer.]

QuickSelect

```
def quickselect(array, k):
    n = len(array)
    if (n == 1):
        return array[0]

    pivot = array[0]

    left = [x for x in array if x < pivot]
    same = [x for x in array if x == pivot]
    right = [x for x in array if x > pivot]

    if (k <= len(left)):
        return quickselect(left, k)
    elif (k <= len(left) + len(same)):
        return pivot
    else:
        return quickselect(right, k - len(left) - len(sa
```

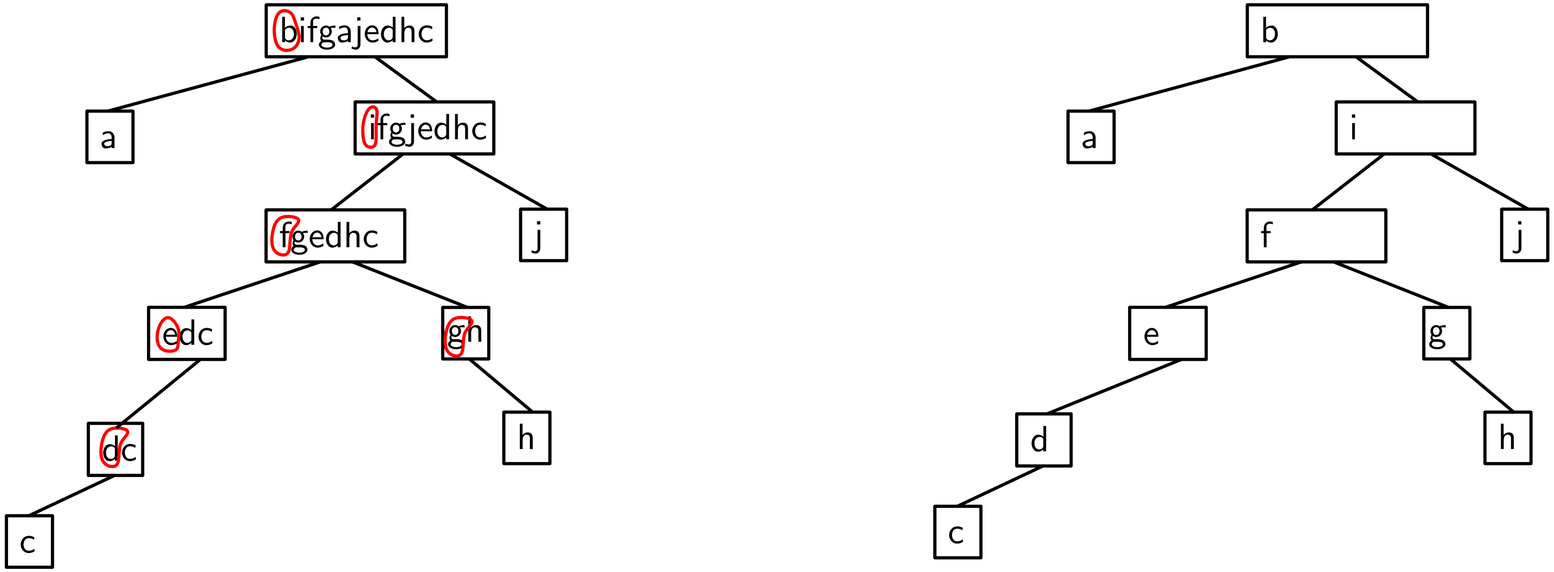
```
def quickselect(array, k):
    n = len(array)
    if (n == 1):
        return array[0]

    pivot = array[0]

    left = [x for x in array if x < pivot]
    same = [x for x in array if x == pivot]
    right = [x for x in array if x > pivot]

    if (k <= len(left)):
        return quickselect(left, k)
    elif (k <= len(left) + len(same)):
        return pivot
    else:
        return quickselect(right, k - len(left) - len(sa
```

Like Quicksort, but we only explore the part containing k


bifgajedhc
a
ifgjedhc
fgedhc
j
edc
gh
dc
h
c
b
a
i
f
j
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d
h
c