# Exercises to "Introduction to $\mathcal{D}$-modules"

1. Let $X = \mathbb{A}^r$ and $Y = \mathbb{A}^n$ with $n \geq r$. Let

$$\begin{array}{rcl} i : X & \hookrightarrow & Y \\ (x_1, \ldots, x_r) & \longmapsto & (x_1, \ldots, x_r, 0, \ldots, 0) =: (y_1, \ldots, y_n) \end{array}$$

be an embedding.

   (a) Check that we have an isomorphism of $(\mathcal{D}_X, i^{-1}\mathcal{D}_Y)$-bimodules

$$\mathcal{D}_{X\to Y} \cong \mathcal{D}_X[\partial_{y_{r+1}}, \ldots, \partial_{y_n}] \cong \frac{i^{-1}\mathcal{D}_Y}{(y_{r+1}, \ldots, y_n)i^{-1}\mathcal{D}_Y}$$

   and make the bimodule structure on the right hand sides explicit.

   (b) Similarly, check that

$$\mathcal{D}_{Y\leftarrow X} \cong \mathbb{C}[\partial_{y_{r+1}}, \ldots, \partial_{y_n}] \otimes_{\mathbb{C}} \mathcal{D}_X \cong \frac{i^{-1}\mathcal{D}_Y}{i^{-1}\mathcal{D}_Y(y_{r+1}, \ldots, y_n)}$$

   taking into account the $(i^{-1}\mathcal{D}_Y, \mathcal{D}_X)$-bimodule structure.

2. Consider now a projection from $X = \mathbb{A}^n$ to $Y = \mathbb{A}^r$ with $n \geq r$, that is:

$$\begin{array}{rcl} f : X & \twoheadrightarrow & Y \\ (x_1, \ldots, x_r, x_{r+1}, \ldots, x_n) & \longmapsto & (x_1, \ldots, x_r) =: (y_1, \ldots, y_r) \end{array}$$

   (a) Show again the that we have an isomorphism of $(\mathcal{D}_X, f^{-1}\mathcal{D}_Y)$-bimodules

$$\mathcal{D}_{X\to Y} \cong \mathcal{O}_X[\partial_{y_1}, \ldots, \partial_{y_r}] \cong \frac{\mathcal{D}_X}{\mathcal{D}_X(\partial_{x_{r+1}}, \ldots, \partial_{x_n})}.$$

   (b) Prove that

$$\mathcal{D}_{Y\leftarrow X} \cong \mathbb{C}[\partial_{y_1}, \ldots, \partial_{y_r}] \otimes \mathcal{O}_X \cong \frac{\mathcal{D}_X}{(\partial_{x_{r+1}}, \ldots, \partial_{x_n})\mathcal{D}_X}$$

   with its $(f^{-1}\mathcal{D}_Y, \mathcal{D}_X)$-bimodule structure.

3. Let $\mathcal{M}$ be a left $\mathcal{D}_X$-module and $f : X \to Y$ be a morphism. Show that there is an isomorphism

$$(\omega_X \otimes_{\mathcal{O}_X} \mathcal{M}) \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y} \cong (\omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_{X\to Y}) \otimes_{\mathcal{D}_X} \mathcal{M}$$

of right $f^{-1}\mathcal{D}_Y$-modules (make the right $f^{-1}\mathcal{D}_Y$-module structure explicit on both sides) given by

$$(\omega \otimes m) \otimes P \longmapsto (\omega \otimes P) \otimes m.$$

4. (a) Show that an affine variety is $\mathcal{D}$-affine.

   (b) Show that for a $\mathcal{D}$-affine variety $X$, the functor

$$\Gamma(X, -) : \mathcal{M}od_{q.c.}(\mathcal{D}_X) \longrightarrow \mathrm{Mod}(\Gamma(X, \mathcal{D}_X))$$

   of global sections yields an equivalence between the category of sheaves of $\mathcal{O}_X$-quasi-coherent $\mathcal{D}_X$-modules and the category of modules over the ring $\Gamma(X, \mathcal{D}_X)$.

5. Let us temporarily define an increasing filtration on a left $\mathcal{D}_X$-module $\mathcal{M}$ to be a sequence $F_k\mathcal{M}$ with $F_k\mathcal{M} \subset F_{k+1}\mathcal{M}$ and such that $\cup_{k\in\mathbb{Z}} F_k\mathcal{M} = \mathcal{M}$, such that $F_k\mathcal{M} = 0$ for $k \ll 0$ and such that

$$F_k\mathcal{D}_X \cdot F_k\mathcal{M} \subset F_{k+l}\mathcal{M},$$

and with a similar condition for right modules.

Now for a morphism $f : X \to Y$, put

$$F_k\mathcal{D}_{X\to Y} := \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}F_k\mathcal{D}_X,$$

then show that this defines a filtration of $\mathcal{D}_{X\to Y}$ as a left $\mathcal{D}_X$ and as a right $f^{-1}\mathcal{D}_Y$-module and that

$$gr^F\mathcal{D}_{X\to Y} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}gr^F\mathcal{D}_Y.$$

6. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms (between smooth algebraic varieties, as always).
   (a) Construct a canonical isomorphism

   $$\mathcal{D}_{X\to Y} \otimes_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{D}_{Y\to Z} \cong \mathcal{D}_{X\to Z}$$

   as right $(g \circ f)^{-1}\mathcal{D}_Z$-modules.
   (b) Show that this isomorphism is also left $\mathcal{D}_X$-linear (use the chain rule).
   (c) Endow the transfer modules with filtrations as in the last exercise, then show that the isomorphism just constructed is filtered.