Spectral theory v.s. metric geometry on Riemannian manifolds

Chemnitz - 2025

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CHAPTER 1

Abstract

In this series of lectures we give an introduction on how metric geometry influences the spectral theory of the Laplace-Beltrami operator on a Riemannian manifold. We discuss self-adjoint realizations, bounds for the bottom of the spectrum and some spectral theory on L^∞, with the latter being related to global properties of Brownian motion. The involved methods are so robust, that they generalize to large classes of less smooth spaces (e.g. metric measure spaces and discrete graphs).

The spectral theory of the Laplace-Beltrami-Operator (or rather a specific self-adjoint realization) on a Riemannian manifold is in close relation with the geometry of the manifold. In this course we discuss which features of the manifold viewed as a metric measure space influence spectral geometry.

It is impossible to cover such a vast subject in three lectures, in particular when it comes to technical details. We aim to explain proof techniques but keep technical details at a minimum. We refer to the excellent book [**4**] and to the survey [**3**] for more details.

CHAPTER 2

Setup and preliminaries

In this chapter we briefly introduce the setup used throughout this mini course. We mostly follow [**4**] in our presentation. The first two sections are mostly contained in [**4**, Chapter 4].

1. Weighted Riemannian manifolds

Let (M,g) be a smooth Riemannian manifold which is connected and without boundary. Charts are denoted by (U,ψ). Here, $U \subseteq M$ and $\psi(U) \subseteq \mathbb{R}^d$ are open and $\psi\colon U \to \psi(U)$ is a homeomorphism belonging to the chosen smooth structure. For $f\colon M \to A$, where A is any set, we use the notation $\bar{f} = f \circ \psi^{-1}\colon \psi(U) \to A$. In this sense, $\bar{f}$ is a local representation of f in the coordinates of (U,ψ).

Remark 2.1. For all results discussed in this text less smoothness is sufficient but we will not discuss this in detail.

The volume measure on the Borel-σ-algebra of M is denoted by vol and dx denotes the Lebesgue measure on (subsets of) $\mathbb{R}^d$. In a chart (U,ψ) we have

$$\mathrm{vol}(A) = \int_{\psi(A)} \sqrt{\det(\overline{g}_{ij})}dx$$

for all Borel sets $A \subseteq U$. Instead of the volume measure itself, we we consider the weighted measure $\mu = e^{-\Phi}\mathrm{vol}$ with $\Phi \in C^\infty(M)$. The triplet (M,g,μ) is called *weighted Riemannian manifold*.

For a piecewise smooth curve $\gamma\colon I \to M$ we define its length by

$$L(\gamma) = \int_I |\dot{\gamma}|_g dt.$$

The *geodesic distance* between $x,y \in M$ is defined by

$$\varrho(x,y) = \inf_\gamma L(\gamma),$$

where the infimum is taken over all piecewise smooth curves containing x and y. The function ϱ is a metric on M that induces the topology of M. The manifold M is called *complete* if (M,ϱ) is complete.

We write $C^\infty(M)$ for the smooth function $M \to \mathbb{R}$ and $C_c^\infty(M) = \mathcal{D}(M)$ for the smooth functions with compact support. Elements of $\mathcal{D}(M)$ are called *test functions*. We say that a sequence (φ_n) in $\mathcal{D}(M)$ converges to $\varphi \in \mathcal{D}(M)$ and write $\varphi_n \overset{\mathcal{D}}{\to} \varphi$ if the following holds:

(a) There exists $K \subseteq M$ such that $\operatorname{supp} \varphi_n \subseteq K$ for all $n \in \mathbb{N}$.
(b) $\partial^\alpha \varphi_n \to \partial^\alpha \varphi$ uniformly in each chart (∂^α is defined with respect to the chart).

We write $\mathfrak{X}(M)$ for the smooth vector fields and denote by $\vec{\mathcal{D}}(M)$ the smooth vector fields of compact support. We equip the latter space with a similar notion of convergence as in $\mathcal{D}(M)$ by requiring the convergence in (b) for each component of the vector fields in local coordinates.

A sequentially continuous linear functional $T\colon \mathcal{D}(M) \to \mathbb{R}$ is called distribution and a sequentially continuous linear functional $U\colon \vec{\mathcal{D}}(M) \to \mathbb{R}$ is called distributional vector field. We identify any $f \in L^1_{\text{loc}}(M)$ with the distribution

$$T_f\colon \mathcal{D}(M) \to \mathbb{R}, \quad T_f(\varphi) = \int_M f\varphi d\mu.$$

Similarly, we identify $X \in \vec{L}^1_{\text{loc}}(M)$ with the distributional vector field

$$U_X\colon \vec{\mathcal{D}}(M) \to \mathbb{R}, \quad U_X(\omega) = \int_M \langle X, \omega\rangle_g d\mu.$$

The space of all distributions is denoted by $\mathcal{D}'(M)$ and the space of all distributional vector fields is denoted by $\vec{\mathcal{D}}'(M)$.

For $f \in C^\infty(M)$ we denote by $\nabla f \in \mathfrak{X}(M)$ its gradient. It is the unique vector field such that

$$df(X) = X(f) = \langle \nabla f, X\rangle_g \text{ for all } X \in \mathfrak{X}(M).$$

In local coordinates it is given by

$$\nabla f = g^{ij} \frac{\partial f}{\partial x^j} \frac{\partial}{\partial x^i}.$$

Here, we use two conventions: The Einstein notation and that (g^{ij}) is the inverse of the matrix (g_{ij}).

For each $X \in \mathfrak{X}(M)$ there exists a unique function $\operatorname{div}_\mu X \in C^\infty(M)$ such that Green's formula

$$\int_M \langle \nabla\varphi, X\rangle_g d\mu = -\int_M \varphi \operatorname{div}_\mu X d\mu$$

holds for all $\varphi \in C_c^\infty(M)$.

In local coordinates with $X = X^i \frac{\partial}{\partial x^i}$ we have

$$\begin{aligned} \operatorname{div} X &= \frac{1}{\sqrt{\det g_{ij}}} \frac{\partial}{\partial x^k} \left(\sqrt{\det g_{ij}} X^k \right) \\ &= \frac{\partial}{\partial x^k} X^k + X^k \frac{\partial}{\partial x^k} \log \sqrt{\det g_{ij}}. \end{aligned}$$

It is readily verified that $\mathrm{div}_\mu X = \frac{1}{e^{-\Phi}}\mathrm{div}\left(e^{-\Phi}X\right)$. Using the product rule, we infer

$$\mathrm{div}_\mu X = \frac{1}{e^{-\Phi}}\mathrm{div}\left(e^{-\Phi}X\right) = \mathrm{div}X - \langle \nabla\Phi, X\rangle_g.$$

The weighted Laplace-Beltrami operator Δ_μ is defined by $\Delta_\mu = \mathrm{div}_\mu \circ \nabla$. For $f \in C^\infty(M)$ it is given by

$$\Delta_\mu f = \Delta_{\mathrm{vol}} f - \langle \nabla\Phi, \nabla f\rangle_g = \Delta_{\mathrm{vol}} f - \nabla\Phi(f).$$

In local coordinates it takes the form

$$\Delta_\mu f = \frac{1}{\sqrt{\det g_{ij}}}\frac{\partial}{\partial x^k}\left(\sqrt{\det g_{ij}}g^{kl}\frac{\partial}{\partial x^l}f\right) - g^{ij}\frac{\partial\Phi}{\partial x^j}\frac{\partial f}{\partial x^i}.$$

For $T \in \mathcal{D}'(M)$ we let

$$\nabla T\colon \vec{\mathcal{D}}(M) \to \mathbb{R}, \quad \nabla T(\omega) = -T(\mathrm{div}_\mu\omega)$$

and for $U \in \vec{\mathcal{D}}(M)'$ we let

$$\mathrm{div}_\mu U\colon \mathcal{D}(M) \to \mathbb{R}, \quad \mathrm{div}_\mu U(\varphi) = -U(\nabla\varphi).$$

By Green's formula these extensions to distributions are consistent with the definition on smooth functions. Moreover, as on functions we let $\Delta_\mu = \mathrm{div}_\mu \circ \nabla$. Hence, we obtain the Laplacian as an operator $\mathcal{D}'(M) \to \mathcal{D}'(M)$.

Below we ALWAYS work with the weighted Laplacian Δ_μ but for convenience we drop the subscript μ.

Example 2.2 (Standard example)**.** Let $M = \Omega \subseteq \mathbb{R}^d$ open and $A\colon \Omega \to \mathbb{R}^{d\times d}$ with the following properties:

(a) For all $x \in \Omega$ the matrix $A(x)$ is symmetric.
(b) $A_{ij} \in C^\infty(\Omega)$, $i, j = 1, \dots, d$.
(c) $\langle A(x)\xi, \xi\rangle > 0$ for all $x \in \Omega$ and $\xi \in \mathbb{R}^d$.

Using the canonical chart (Ω, id) we identify $T_x\Omega = \mathbb{R}^d$. Then

$$g_x(\xi, \eta) = \langle A(x)\xi, \eta\rangle, \quad x \in \Omega,\ \xi, \eta \in \mathbb{R}^d$$

defines a smooth Riemannian metric on Ω. In this case, the volume is given by

$$\mathrm{vol}(B) = \int_B \sqrt{\det A}dx$$

for all Borel $B \subseteq \Omega$.

The weighted Laplacian takes the form

$$\begin{aligned}\Delta_\mu f &= \frac{1}{\sqrt{\det A}}\mathrm{div}_e\left(\sqrt{\det A}A^{-1}\nabla_e f\right) - \langle A^{-1}\nabla_e\Phi, \nabla_e f\rangle \\ &= \mathrm{div}_e\left(A^{-1}\nabla_e f\right) - \langle X, \nabla_e f\rangle.\end{aligned}$$

Here, div_e and ∇_e denote the standard divergence and gradient on Euclidean space and $X \in C^\infty(\Omega, \mathbb{R}^n)$ is a suitable vector space. This shows that Δ_μ is an 'elliptic operator in divergence form with drift term'.

If we choose $\Phi = \log \sqrt{\det A}$, then $d\mathrm{vol} = dx$ and

$$\Delta_\mu f = \mathrm{div}_e \left(A^{-1} \nabla_e f \right).$$

If one wants to avoid inverting A, one can modify the examples and start with A^{-1} instead of A.

Example 2.3 (Model manifolds)**.** We call a d-dimensional Riemannian manifold (M, g) (with $d \geq 2$) a model manifold if the following holds:

(a) For some $R \in (0, \infty]$ we have $M = \{x \in \mathbb{R}^d \mid |x| < R\}$.

(b) There exists $\psi \colon (0, R) \to (0, \infty)$ such that the map

$$M \setminus \{0\} \to (0, R) \times \mathbb{S}^{d-1}, \quad x \mapsto (|x|, |x|^{-1} x)$$

is a Riemannian isometry, where on $(0, R) \times \mathbb{S}^{d-1}$ the tangent space

$$T_{r,\theta}((0, R) \times \mathbb{S}^{d-1}) = \mathbb{R} \oplus T_\theta \mathbb{S}^{d-1}$$

is equipped with the metric $g_{\mathbb{R}} \oplus \psi(r)^2 g_{\mathbb{S}^{d-1},\theta}$.

The function ψ is called scaling function of the model.

In this case, we have

$$\mathrm{vol}(B_r(0)) = \omega_{d-1} \int_0^r \psi(t)^{d-1} dt,$$

where ω_{d-1} is the $(d-1)$-dimensional volume of $\mathbb{S}^{d-1}$.

If $f \in C^\infty(M)$ with $f(x) = h(|x|)$ for $x \neq 0$ and some $h \in C^\infty((0, \infty))$, then

$$\Delta_{\mathrm{vol}} f(x) = h''(r) + (d-1) \frac{\psi'(r)}{\psi(r)} h'(r),$$

where $r = |x|$ and $x \neq 0$.

Example 2.4. (a) $\mathbb{R}^d$ is a model manifold with $R = \infty$ and scaling function $\psi(r) = r$.

(b) The sphere without one point $\mathbb{S}^d \setminus \{p\}$ is isometric to a model manifold with $R = \pi$ and scaling function $\psi(r) = \sin r$.

(c) Hyperbolic space $\mathbb{H}^d$ is isometric to a model manifold with $R = \infty$ and scaling function $\psi(r) = \sinh r$.

2. Sobolev spaces and a self-adjoint Laplacian

In this section we introduce several function spaces on the weighted manifold. Some depend on the choice of the density function Φ in the measure μ and some do not. This is reflected in our notation.

The first order Sobolev space on (M, g, μ) is defined by

$$W^1(M, \mu) = \{f \in L^2(M, \mu) \mid \nabla f \in \vec{L}^2(M, \mu)\}.$$

Equipped with the norm

$$\|\cdot\|_{W^1} \colon W^1(M, \mu) \to [0, \infty), \quad \|f\|_{W^1} = \left(\|f\|_2^2 + \|\nabla f\|_2^2 \right)^{1/2},$$

which is clearly induced by an inner product $\langle\cdot,\cdot\rangle_{W^1}$, it is a Hilbert space. By a celebrated result of Meyers and Serrin [**6**] $C^\infty(M)\cap W^1(M,\mu)$ is dense in $W^1(M,\mu)$. By $W^1_0(M,\mu)$ we denote the closure of $C_c^\infty(M)$ in $W^1(M,\mu)$. We write $W^1_c(M)$ for the functions in $W^1(M,\mu)$ with compact support. It can be inferred as a consequence to the Meyers Serrin theorem that $W^1_c(M)\subseteq W^1_0(M,\mu)$.

We let

$$W^1_{\mathrm{loc}}(M)=\{f\in L^2_{\mathrm{loc}}(M)\mid \nabla f\in L^2_{\mathrm{loc}}\vec{(M)}\}.$$

Then $f\in W^1_{\mathrm{loc}}(M)$ if and only if for all relatively compact open $\Omega\subseteq M$ there exists $g\in W^1(M,\mu)$ with $f=g$ a.s. on Ω. It follows from the discussed properties that $W^1_c(M)$ and $W^1_{\mathrm{loc}}(M)$ are independent of the choice of μ and every function in $W^1_{\mathrm{loc}}(M)$ with compact support belongs to $W^1_c(M)$.

The following result is the key tool in our analysis.

Theorem 2.5 (Rademacher)**.** *A function $f\colon M\to\mathbb{R}$ is Lipschitz (with respect to ϱ) if and only if $f\in L^1_{\mathrm{loc}}(M)$ and $\nabla f\in L^\infty(M)$. In this case, $f\in W^1_{\mathrm{loc}}(M)$ and*

$$\|\nabla f\|_\infty=\sup_{x\neq y}\frac{|f(x)-f(y)|}{\varrho(x,y)}=:\mathrm{Lip}(f).$$

In particular, Lipschitz functions with compact support belong to $W^1_c(M)$.

Proof. The if statement and the inequality $\|\nabla f\|_\infty\le\mathrm{Lip}(f)$ is contained in [**4**, Theorem 11.3]. For the reverse inequality and the only if statement see [**2**, Theorem 4.5] and its proof. □

Corollary 2.6. *For all $x,y\in M$ we have*

$$\varrho(x,y)=\sup\{|f(x)-f(y)|\mid f\in L^1_{\mathrm{loc}}(M)\ \text{with}\ \|\nabla f\|_\infty\le 1\}.$$

Remark 2.7. Rademacher's theorem and its corollary have two consequences, which make extensions of the presented results to non-smooth spaces possible.

(a) We define the local Lipschitz constant of $f\colon M\to\mathbb{R}$ at $x\in M$ by

$$\mathrm{Lip}_x(f):=\inf_U\sup_{x\neq y,x,y\in U}\frac{|f(x)-f(y)|}{\varrho(x,y)},$$

where the infimum is taken over all open neighborhoods of x. From localizing the equality in Rademacher's theorem, we infer

$$|\nabla f|(x)=\mathrm{Lip}_x(f)$$

for $f\in C^1(M)$-functions. Hence, local Lipschitz constants determine $|\nabla f|$. Since local Lipschitz constants are available in metric spaces (without differentiable structure and Riemannian metric), this paves the way towards an analysis on metric measure spaces by considering Cheeger energies and the induced Laplacians, see e.g. [**1**].

(b) The previous corollary shows that ϱ can be recovered from the knowledge of all functions with $\|\nabla f\|_\infty \leq 1$. Hence, if in some space one can give a meaning to the inequality $\|\nabla f\|_\infty \leq 1$ (e.g. this is possible in regular Dirichlet spaces), one can define a metric d through the formula

$$d(x,y) = \sup\{|f(x) - f(y)| \mid f \in L^1_{\mathrm{loc}}(M) \text{ with } \|\nabla f\|_\infty \leq 1\}.$$

Under relatively mild assumptions this metric satisfies Rademacher's theorem (or at least part of it), namely $\|\nabla f\|_\infty \leq \mathrm{Lip}(f)$. We refer to [**8, 9, 7**], where this Ansatz is used in the analysis of strongly local Dirichlet spaces, and to [**5**], which discusses discrete spaces.

Corollary 2.8 (Existence of cut-off functions)**.** *For $0 < r < R$ and $o \in M$ consider the function*

$$\varphi = \varphi_{r,R} \colon M \to \mathbb{R}, \quad \varphi(x) = (1 - \varrho(x, B_r(o))/(R-r))_+.$$

Then $\varphi = 1$ on $B_r(o)$, $\varphi = 0$ on $M \setminus B_R(o)$ and $\mathrm{Lip}(\varphi) \leq 1/(R-r)$. In particular, $f \in W^1_{\mathrm{loc}}(M)$ and $\|\nabla f\|_\infty \leq 1/(R-r)$. If, moreover, (M, ϱ) is complete, then $\varphi \in W^1_c(M)$.

Proof. All assertions are more or less trivial or consequences of Rademacher's theorem except 'moreover'-statement. It follows from the Hopf-Rinow theorem, which states that (M, ϱ) is complete if and only if $B_r(x)$ is compact for all $r > 0$ and $x \in M$. □

Remark 2.9. Instead of requiring completeness of M, for many applications the existence of a sequence of cut-off functions (φ_n) with $0 \leq \varphi_n \leq 1$, $\varphi_n \to 1$ pointwise and $\|\nabla \varphi_n\|_\infty \to 0$ is sufficient. However, it turns out that the existence of such a sequence is equivalent to completeness.

The following proposition summarizes several operations on $W^1_{\mathrm{loc}}(M)$ that will be used below.

Proposition 2.10 (Product- and chain rules on $W^1_{\mathrm{loc}}(M)$)**.** *(a) (Chain rule) Let (C_n) in $C^1(\mathbb{R})$ such that $\sup_n \|C_n'\|_\infty < \infty$ and let $C, D \colon \mathbb{R} \to \mathbb{R}$ such that $C_n \to C$ and $C_n' \to D$ pointwise. Then for any $f \in W^1_{\mathrm{loc}}(M)$ we have $C \circ f \in W^1_{\mathrm{loc}}(M)$ and*

$$\nabla(C \circ f) = (D \circ f)\nabla f.$$

(b) (Truncation property) For all $f, h \in W^1_{\mathrm{loc}}(M)$ we have $f \wedge h \in W^1_{\mathrm{loc}}(M)$ and

$$\nabla(f \wedge h) = 1_{\{f \leq h\}} \nabla f + 1_{\{f > h\}} \nabla h.$$

(c) (Product rule) Let $f, h \in W^1_{\mathrm{loc}}(M)$ such that $fh \in W^1_{\mathrm{loc}}(M)$. Then

$$\nabla(fh) = f\nabla h + h \nabla f.$$

(d) (Ideal property) For all $\varphi \in \mathrm{Lip}_c(M)$ and $f \in W^1_{\mathrm{loc}}(M)$ we have $\varphi f \in W^1_c(M, \mu)$.

Proof. All of the results except (b) are clear for smooth functions, other functions have to be approximated (e.g. by using the Meyers-Serrin theorem). To prove (b) one has to use (a) , $f \wedge g = (f+g-|f-g|)/2$ and an approximation of $|\cdot|$ by smooth functions. The details can be found somewhat scattered in [**4**]. □

We finish this section by constructing a non-negative self-adjoint Laplacian on $L^2(M,\mu)$.

Theorem 2.11 (The Friedrichs extension)**.** *The operator Δ_F defined by*

$$D(\Delta_F) = \{f \in W_0^1(M,\mu) \mid \Delta_\mu f \in L^2(M,\mu)\}$$

and

$$\Delta_F = \Delta_\mu f$$

is a non-positive self-adjoint operator on $L^2(M,\mu)$ (i.e. $\sigma(\Delta_F) \subseteq (-\infty,0]$). It is the Friedrichs extension of the restriction of Δ_μ to $C_c^\infty(M)$. In particular,

$$\begin{aligned}\inf(\sigma(-\Delta_F)) &= \inf\{\langle -\Delta_F\varphi,\varphi\rangle \mid \varphi \in C_c^\infty(M), \|\varphi\|_2 = 1\}\\ &= \inf\{\int_M |\nabla f|_g^2 d\mu \mid f \in W_0^1(M,\mu), \|\varphi\|_2 = 1\}.\end{aligned}$$

For all $\alpha > 0$ its resolvent $(\alpha - \Delta_F)^{-1}$ is positivity improving ($f \gneq 0$ implies $(\alpha-\Delta_F)^{-1}f > 0$ a.s.) and Markovian ($f \leq 1$ implies $\alpha(\alpha-\Delta_F)^{-1}f \leq 1$).

Proof. It follows from Green's formula that the restriction of Δ_μ to $C_c^\infty(M)$ is non-positive and symmetric. Hence, it has a Friedrichs extension. The formula for Δ_F then follows directly from the abstract construction of the Friedrichs extension and the obsercation that the adjoint of the restriction of Δ_μ to $C_c^\infty(M)$ is the restriction of Δ_μ to $\{f \in L^2(M,\mu) \mid \Delta_\mu f \in L^2(M,\mu)$.

For the other properties we refer to [**4**, Chapter 5]. □

3. Local regularity results and a local Harnack inequality

Theorem 2.12 (Hypoellipticity)**.** *(a) Let $\alpha \in \mathbb{R}$ and $1 < p < \infty$. Every $u \in L^p_{\mathrm{loc}}(M)$ with $-\Delta u + \alpha u \in C^\infty(M)$ satisfies $u \in C^\infty(M)$.*
(b) If $u \in \mathcal{D}'((0,\infty)\times M)$ satisfies $\partial_t u = \Delta u$, then $u \in C^\infty(M)$.

Proof. See [**4**, Chapter 7]. □

Theorem 2.13 (Local Harnack inequality)**.** *Let $\lambda \in \mathbb{R}$ and let $K \subseteq M$ compact. There exists a constant $C = C(K,\lambda) \geq 0$ such that for all nonnegative $u \in C^\infty(M)$ with $-\Delta u - \lambda u = 0$ the following inequality holds:*

$$\sup_K u \leq C \inf_K u.$$

The constants can be chosen such that $\mathbb{R} \to (0,\infty)$, $\lambda \mapsto C(K,\lambda)$ is increasing.

Proof. This is contained in [**4**, Theorem 13.11] under the assumption that there exists $\lambda' \geq \lambda$ and $h > 0$, $h \in C^\infty(M)$, such that $-\Delta h - \lambda' h = 0$. However, using the following ingredients this assumption can be completely removed:

(a) The Agmon-Allegretto-Piepenbrink theorem on relatively compact open subsets of M, which does not rely on Harnack inequalities but on Sobolev embedding theorems.
(b) A domain monotonicity argument.
(c) The implication of the Agmon-Allegretto-Piepenbrink theorem, which does not use Harnack inequalities.

Below in our proof of the Agmon-Allegretto-Piepenbrink theorem we cheat a bit, because we do not prove (a) and (b) separately, but use the Harnack inequality directly. □

CHAPTER 3

Liouville theorems and essential self-adjointness

Theorem 3.1 (Karp's Liouville theorem)**.** *Assume that M is complete and let $1 < p < \infty$. Let $f \in W^1_{\text{loc}}(M) \cap L^\infty_{\text{loc}}(M)$ satisfy $f \geq 0$ and $\Delta f \geq 0$ (in the sense of distributions). If for some (all) $o \in M$ and some (all) $r_0 > 0$*

$$\int_{r_0}^{\infty} \frac{r}{\|f 1_{B_r(o)}\|_p^p} dr = \infty,$$

then f is constant.

Remark 3.2. Using local regularity theory for subharmonic functions it can be proven that any $f \in W^1_{\text{loc}}(M)$ with $f \geq 0$ and $\Delta f \geq 0$ automatically belongs to $L^\infty_{\text{loc}}(M)$. Such regularity is complicated and beyond the scope of this lecture.

Before establishing Karp's theorem we need one further result, which is interesting on its own right.

Theorem 3.3 (Caccioppoli inequalities)**.** *Let $1 < p < \infty$. Let $f \in W^1_{\text{loc}}(M) \cap L^\infty_{\text{loc}}(M)$ with $f \geq 0$ and $\Delta f \geq 0$. For any $\varphi \in \text{Lip}_c(M)$ and $n \in \mathbb{N}$ we have*

$$\int_M f^{p-2} \varphi^2 |\nabla f|_g^2 \, d\mu \leq -\frac{2}{p-1} \int_M f^{p-1} \varphi \langle \nabla f, \nabla \varphi \rangle_g d\mu$$

and

$$\int_M f^{p-2} \varphi^2 |\nabla f|_g^2 \, d\mu \leq \frac{4}{(p-1)^2} \int_M f^p |\nabla \varphi|_g^2 d\mu.$$

Proof. Let $n \in \mathbb{N}$. Instead of f we consider $f_n = f \vee n^{-1}$. Then $f_n \in W^1_{\text{loc}}(M) \cap L^\infty_{\text{loc}}(M)$, $f_n \geq n^{-1}$ and $C \colon (0, \infty) \to \mathbb{R}$, $x \mapsto x^{p-1}$ being locally Lipschitz imply $f_n^{p-1} \in W^1_{\text{loc}}(M)$.

For $\varphi \in \text{Lip}_c(M)$ the product rule yields $\varphi^2 f_n^{p-1} \in W^1_c(M) \cap L^\infty(M)$. The inequality $\Delta f \geq 0$ in the sense of distributions yields

$$0 \geq \int_M \langle \nabla f, \nabla(\varphi^2 f_n^{p-1}) \rangle_g d\mu,$$

after we approximate $\varphi^2 f_n^{p-1} \in W^1_c(M)$ by nonnegative functions in $C^\infty_c(M)$[1]. The chain rule, the product rule and the truncation property

[1]This is not entirely trivial.

yield

$$\begin{aligned}
&\int_M \langle \nabla f, \nabla(\varphi^2 f_n^{p-1})\rangle_g d\mu \\
&= (p-1)\int_M \varphi^2 f_n^{p-2}\langle \nabla f, \nabla f_n\rangle_g d\mu + 2\int_M f_n^{p-1}\varphi\langle \nabla f, \nabla\varphi\rangle_g d\mu \\
&= (p-1)\int_{\{f\geq 1/n\}} \varphi^2 f^{p-2}|\nabla f|_g^2 d\mu + 2\int_M f_n^{p-1}\varphi\langle \nabla f, \nabla\varphi\rangle_g d\mu
\end{aligned}$$

Combining both inequalities and letting $n \to \infty$ yields

$$\int_M \varphi^2 f^{p-2}|\nabla f|_g^2 d\mu \leq -\frac{2}{p-1}\int_M f^{p-1}\varphi\langle \nabla f, \nabla\varphi\rangle_g d\mu.$$

For the existence of both of the integrals (which is important for the convergence as $n \to \infty$) we used $f \in L^\infty_{\mathrm{loc}}(M)$.

Next we use the elementary inequality $|ab| \leq \varepsilon a^2 + 1/(4\varepsilon)b^2$ to estimate

$$\int_M |f^{p-1}\varphi\langle \nabla f, \nabla\varphi\rangle_g| d\mu \leq \varepsilon\int_M \varphi^2 f^{p-2}|\nabla f|_g^2 d\mu + \frac{1}{4\varepsilon}\int_M f^p|\nabla\varphi|_g^2 d\mu.$$

Combining the two previous inequalities yields

$$(1-\frac{2\varepsilon}{p-1})\int_M \varphi^2 f^{p-2}|\nabla f|_g^2 d\mu \leq \frac{2}{4\varepsilon(p-1)}\int_M f^p|\nabla\varphi|_g^2 d\mu.$$

We obtain the statement by letting $\varepsilon = (p-1)/4$. □

Remark 3.4. (a) The Caccioppoli inequality does not use the completeness of the manifold.

(b) Instead of assuming $f \in L^\infty_{\mathrm{loc}}(M)$, it would be sufficient to require

$$\int_M f^{p-1}|\varphi||\langle \nabla f, \nabla\varphi\rangle_g| d\mu < \infty,$$

which is used in one step of our proof. By the inequality $2|ab| \leq a^2 + b^2$, we have

$$\begin{aligned}
&\int_M f^{p-1}|\varphi||\langle \nabla f, \nabla\varphi\rangle_g| d\mu \\
&\leq \int_M f^{2p-2}|\nabla\varphi|^2 d\mu + \int_M |\varphi|^2|\nabla f|_g^2 d\mu.
\end{aligned}$$

Hence, our proof works under the assumption $f \in L^{2p-2}_{\mathrm{loc}}(M)$. It is automatically satisfied for $f \in W^1_{\mathrm{loc}}(M)$, as long as $1 \leq p \leq 2$.

To simplify notation we let $B_r = B_r(o)$ and denote by

$$A_{r,R} = B_R \setminus B_r = \{x \in M \mid r < \varrho(x,o) \leq R\}$$

the annulus with the radii $0 < r < R$ around o.

Corollary 3.5 (Key estimate)**.** *Assume M is complete, let $1 < p < \infty$ and $0 < r < R$.Then for any $f \in W^1_{\mathrm{loc}}(M) \cap L^\infty_{\mathrm{loc}}(M)$ with $f \geq 0$ and $\Delta f \geq 0$ the following inequalities hold:*

$$\left(\int_M f^{p-2}\varphi^2|\nabla f|_g^2 d\mu\right)^2 \leq \frac{4}{(p-1)^2}\frac{1}{(R-r)^2}\|f1_{A_{r,R}}\|_p^p \int_{A_{r,R}} f^{p-2}\varphi^2|\nabla f|_g^2\, d\mu$$

and

$$\int_{B_r} f^{p-2}|\nabla f|_g^2 d\mu \leq \frac{4}{(p-1)^2}\frac{1}{(R-r)^2}\|f1_{A_{r,R}}\|_p^p.$$

Here, $\varphi = \varphi_{r,R}$ is the cut-off function of Corollary 2.8.

Proof. Let $\varepsilon > 0$ such that $r+\varepsilon < R$ and set $\tilde{r} = r+\varepsilon$. Since M is complete, the cut-off function $\varphi_{r,R}$ discussed in Corollary 2.8 belongs to $\mathrm{Lip}_c(M)$. It is constant on the open sets $U_{\tilde{r}}$ and $M \setminus B_R$ and therefore satisfies $\nabla\varphi_{\tilde{r},R} = 0$ on $U_{\tilde{r}} \cup (M \setminus B_R)$. Moreover, by Rademacher's theorem $|\nabla\varphi_{\tilde{r},R}| \leq \mathrm{Lip}(\varphi_{\tilde{r},R}) \leq 1/(R-\tilde{r})$. Using these estimates, $0 \leq \varphi_{\tilde{r},R} \leq 1$ and $\varphi_{\tilde{r},R} = 1$ on $B_{\tilde{r}}$, the Caccioppoli inequalities imply the desired result (after another application of Cauchy-Schwarz for the first inequality) with $A_{r,R}$ replaced by $B_R \setminus U_{\tilde{r}}$. With this at hand, the statement follows after letting $\varepsilon \searrow 0$. □

Karp's theorem. We choose $R > 0$ such that $f1_{B_R} \neq 0$ and let $R_n = 2^n R$. We consider the quantities

$$Q_n = Q_n(R) = \int_{B_{R_n}} f^{p-2}|\nabla f|_g^2\, d\mu$$

and

$$V_n = \|f1_{A_{R_{n-1},R_n}}\|_p^p.$$

We show $Q_1 = Q_1(R) = 0$ for all $R > 0$, which implies

$$\int_M f^{p-2}|\nabla f|_g^2\, d\mu = 0,$$

that is $f^{p-2}|\nabla f|_g^2 = 0$ a.s. Since $f \geq 0$, we have $f = f_+$ and the truncation property yields

$$\nabla f = \nabla f_+ = 1_{\{f>0\}}\nabla f.$$

Hence, $f^{p-2}|\nabla f|_g^2 = 0$ a.s. implies $\nabla f = 0$ a.s. and we infer that f is constant from Rademacher's theorem.

The assumption

$$\int_{r_0}^\infty \frac{r}{\|f1_{B_r(o)}\|_p^p} dr = \infty$$

implies

$$\sum_{n=1}^\infty \frac{R_n^2}{V_n} = \infty.$$

We let $C = 4/(p-1)^2$ and $\varphi_n = \varphi_{R_{n-1},R_n}$ (the cut-off function from Corollary 2.8). Our key estimate shows

$$\begin{aligned}
&\left(Q_{n-1} + \int_{A_{R_{n-1},R_n}} \varphi_n^2 f^{p-2} |\nabla f|_g^2 d\mu\right)^2 \\
&= \left(\int_M \varphi_n^2 f^{p-2} |\nabla f|_g^2 d\mu\right)^2 \\
&\leq \frac{CV_n}{(R_n - R_{n-1})^2} \int_{A_{R_{n-1},R_n}} \varphi_n^2 f^{p-2} |\nabla f|_g^2 d\mu \\
&= \frac{4CV_n}{R_n^2} \int_{A_{R_{n-1},R_n}} \varphi_n^2 f^{p-2} |\nabla f|_g^2 d\mu.
\end{aligned}$$

Now assume that $Q_1 > 0$ and hence $Q_n > 0$ for all $n \in \mathbb{N}$. If we write $K_n = \int_{A_{R_{n-1},R_n}} \varphi_n^2 f^{p-2} |\nabla f|_g^2 d\mu$, the previous inequality and $K_n \leq Q_n - Q_{n-1}$ (which follows from $0 \leq \varphi_n \leq 1$) yield

$$\begin{aligned}
\frac{4CV_n}{R_n^2} &\geq \frac{(Q_{n-1} + K_n)^2}{K_n} \geq Q_{n-1} \frac{(Q_{n-1} + K_n)}{K_n} \\
&\geq Q_{n-1} \left(\frac{Q_{n-1}}{Q_n - Q_{n-1}} + 1\right) = \frac{Q_{n-1} Q_n}{Q_n - Q_{n-1}}.
\end{aligned}$$

Rearranging this inequality leads to

$$\frac{R_n^2}{V_n} \leq 4C \left(\frac{1}{Q_{n-1}} - \frac{1}{Q_n}\right)$$

and summing it up shows

$$\sum_{n=2}^{\infty} \frac{R_n^2}{V_n} \leq \frac{4C}{Q_1} < \infty,$$

a contradiction. □

Corollary 3.6 (Yau's Liouville theorem)**.** *Assume M that is complete and $1 < p < \infty$. Any $f \in L^p(M, \mu)$ with $-\Delta f + \alpha f = 0$ for some $\alpha \geq 0$ is constant. If $\alpha > 0$, then $f = 0$.*

Proof. By local regularity $f \in C^\infty(M) \subseteq W^1_{\mathrm{loc}}(M) \cap L^\infty_{\mathrm{loc}}(M)$. We choose a sequence of increasing convex functions (C_n) in $C^2(\mathbb{R})$ with $\|C_n'\|_\infty \leq 1$, $C_n = 0$ on $(-\infty, 0]$ and $C_n(x) \to x 1_{[0,\infty)}(x)$ for all $x \in \mathbb{R}$. For example, the functions C_n defined by

$$C_n(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } x \geq 1/n \\ x^3(3n^4x^2 - 8n^3x + 6n^2) & \text{if } 0 < x < 1/n \end{cases}$$

do the job. The Chain rule for the Laplacian implies

$$\Delta C_n(f) = C_n''(f)|\nabla f|_g^2 + C_n'(f)\Delta f = C_n''(f)|\nabla f|_g^2 + \alpha C_n'(f) f.$$

We infer $(\Delta C_n(f), \varphi) \geq 0$ for all nonnegative $\varphi \in \mathcal{D}(M)$ and, taking the limit $n \to \infty$, arrive at $(\Delta f_+, \varphi) \geq 0$ for all nonnegative $\varphi \in \mathcal{D}(M)$. With this at hand, f_+ being constant follows from Karp's theorem. Applying the result also to $-f$ instead of f gives that f is constant.

Clearly, if $\alpha > 0$, the only constant function f satisfying $\Delta f = \alpha f$ is $f = 0$. □

Corollary 3.7 (Gaffney's theorem)**.** *Assume that M is complete. Then the restriction of Δ to $C_c^\infty(M)$ is essentially self-adjoint on $L^2(M, \mu)$. Its unique self-adjoint extension is Δ_F and*

$$D(\Delta_F) = \{f \in L^2(M, \mu) \mid \Delta f \in L^2(M, \mu)\}$$

Moreover, $W_0^1(M, \mu) = W^1(M, \mu)$, i.e. $C_c^\infty(M)$ is dense in $H^1(M, \mu)$.

Proof. Let Δ_c denote the restriction of Δ to $C_c^\infty(M)$. By definition we have to show that $(\Delta_c)^*$ is self-adjoint.

It follows directly from the distributional definition of Δ that

$$D((\Delta_c)^*) = \{f \in L^2(M, \mu) \mid \Delta f \in L^2(M, \mu)\}$$

and that $(\Delta_c)^* f = \Delta f$.

Clearly, $\Delta_c \subseteq \Delta_F$, and by closedness of Δ_F, also $\overline{\Delta_c} \subseteq \Delta_F$. We show $(\Delta_c)^* \subseteq \Delta_F$, which then implies

$$\Delta_F = (\Delta_F)^* \subseteq (\Delta_c)^{**} = \overline{\Delta}_c \subseteq \Delta_F.$$

Let $f \in D((\Delta_c)^*)$ and for $\alpha > 0$ let $g = (\alpha - \Delta_F)^{-1}(\alpha - \Delta)f$. Then $g \in D(\Delta_F) \subseteq W_0^1(M, \mu)$ and

$$(\alpha - \Delta)g = (\alpha - \Delta)(\alpha - \Delta_F)^{-1}(\alpha - \Delta)f = (\alpha - \Delta)f.$$

Hence, $(\alpha - \Delta)(g - f) = 0$ and Yau's Liouville theorem implies $f = g \in D(\Delta_F)$.

Recall that we equipped $W^1(M, \mu)$ with the complete inner product

$$\langle f, h \rangle_{W^1} = \int_M \langle \nabla f, \nabla h \rangle_g d\mu + \int_M fh d\mu.$$

It suffices to show $C_c^\infty(M)^\perp = \{0\}$. Assume that $\langle h, \varphi \rangle_{W^1} = 0$ for all $\varphi \in C_c^\infty(M)$. Then

$$(-\Delta h + h, \varphi) = \langle h, \varphi \rangle_{W^1} = 0$$

for all $\varphi \in C_c^\infty(M)$, i.e., $-\Delta h + h = 0$ in the sense of distributions. We infer $h = 0$ from Yau's Liouville theorem. □

CHAPTER 4

The bottom of the spectrum

We study the bottom of the spectrum of the self-adjoint operator $-\Delta_F$ on $L^2(M,\mu)$. We let $\lambda_0(M) = \inf \sigma(-\Delta_F)$.

1. Hardy inequalities and the Agmon-Allegretto-Piepenbrink theorem

We start with a description in terms of positive solutions and ground states.

Theorem 4.1 (Agmon-Allegretto-Piepenbrink)**.** *Let M be complete. For $\lambda \in \mathbb{R}$ the following assertions are equivalent.*

(i) $\lambda \leq \lambda_0(M)$.
(ii) There exists $f \in C^\infty(M)$ *such that* $f > 0$ *on* M *and* $-\Delta f \geq \lambda f$.
(iii) There exists a nonnegative $f \in W^1_{\mathrm{loc}}(M)$ *with* $1/f \in L^\infty_{\mathrm{loc}}(M)$ *and* $-\Delta f \geq \lambda f$.

If M is not compact, the function in (ii) can be chosen to satisfy $-\Delta f = \lambda f$

The proof will require a weighted Hardy inequality, which is interesting on its own right.

Theorem 4.2 (A weighted Hardy inequality)**.** *Let $h \in W^1_{\mathrm{loc}}(M)$ nonnegative with $1/h \in L^\infty_{\mathrm{loc}}(M)$ such that $-\Delta h \geq w$ for some $w \in L^1_{\mathrm{loc}}(M)$. Then*

$$\int_M |\nabla \varphi|_g^2 d\mu \geq \int_M |\varphi|^2 \frac{w}{h} d\mu$$

for all $\varphi \in C_c^\infty(M)$. If M is complete, this inequality extends to all $\varphi \in H^1(M)$.

Proof. Let $\psi \in W^1_{\mathrm{loc}}(M)$ such that $\psi^2, h\psi, h\psi^2 \in W^1_c(M)$. The product rule yields

$$\langle \nabla(h\psi), \nabla(h\psi)\rangle_g - \langle \nabla(h\psi^2), \nabla h\rangle_g = h^2 |\nabla \psi|^2.$$

Clearly, this implies

$$\int_M |\nabla(h\psi)|_g^2 d\mu \geq \int_M \langle \nabla(h\psi^2), \nabla h\rangle_g d\mu.$$

Approximating $h\psi^2$ by nonnegative test functions and using $-\Delta h \geq w$ leads to

$$\int_M \langle \nabla(h\psi^2), \nabla h\rangle_g d\mu \geq \int_M \psi^2 h w d\mu.$$

To finish the proof it suffices to show that for $\varphi \in C_c^\infty(M)$ the function $\psi = \varphi/h$ satisfies the assumptions leading to the previous inequality.

Since $1/h \in L^\infty_{\mathrm{loc}}(M)$, the local Lipschitz continuity of $(0,\infty) \to (0,\infty)$, $x \mapsto 1/x$ and $(0,\infty) \to (0,\infty)$, $x \mapsto 1/x^2$ yields $1/h, 1/h^2 \in W^1_{\mathrm{loc}}(M)$. For $\varphi \in C_c^\infty(M)$ we infer $\varphi/h, \varphi^2/h^2, \varphi/h^2 \in W^1_c(M)$. □

Remark 4.3. (a) The proof yields the more precise estimate

$$\int_M |\nabla\varphi|_g^2 d\mu \geq \int_M |\varphi|^2 \frac{w}{h} d\mu + \int_M h^2 |\nabla \frac{\varphi}{h}|_g^2 d\mu, \quad \varphi \in C_c^\infty(M).$$

If $-\Delta h = w$, it is an equality.

(b) Using fine properties of functions $0 \neq h \in W^1_{\mathrm{loc}}(M)$ with $h \geq 0$ and $-\Delta h \geq 0$, the assumption $1/h \in L^\infty_{\mathrm{loc}}(M)$ can be dropped if $w \geq 0$.

Proof. (i) ⇒ (ii): If $\lambda < \lambda_0(M)$, we can choose $f = (-\Delta_F - \lambda)^{-1}\varphi$ with $\varphi \in C^\infty(M) \cap L^2(M)$ and $\varphi > 0$. Then $(-\Delta - \lambda)f = \varphi \in C^\infty(M)$ implies $f \in C^\infty(M)$. Since the resolvent is positivity improving, we obtain $f > 0$.

Case 1: M is compact. In this case, $1 \in L^2(M)$ and $-\Delta 1 = 0 \in L^2(M)$. Since compact manifolds are complete, this implies $1 \in D(-\Delta_F)$ and hence $0 \in \sigma(-\Delta_F)$, showing $\lambda_0(M) = 0$.

Case 2: M is not compact. We choose an increasing sequence $\emptyset \neq \Omega_n \subseteq M$ of relatively compact open connected sets such that $\overline{\Omega_n} \subseteq \Omega_{n+1}$ and an increasing sequence $\lambda_n < \lambda$ with $\lambda_n \nearrow \lambda$. We further choose nonnegative $\varphi_n \in C_c^\infty(M)$ with $\operatorname{supp}\varphi_n \subseteq M \setminus \overline{\Omega_n}$ and let $g_n = (-\Delta_F - \lambda_n)^{-1}\varphi_n$. As a consequence to the hypoellipticity we have $g_n \in C^\infty(M)$.

The condition on the support of φ_n implies $(-\Delta_F - \lambda_n)g_n = 0$ on Ω_n and since the resolvent is positivity improving and g_n is smooth, we have $g_n(x) > 0$ all $x \in M$.

By the local Harnack inequality applied to the compact set $\overline{\Omega_n}$ in the manifold Ω_{n+1}, there exist constants $C_n > 0$ such that

$$\sup_{\overline{\Omega_n}} g_k \leq C_n \inf_{\overline{\Omega_n}} g_k, \quad k \geq n+1.$$

We let $K_n = \inf_{\overline{\Omega_1}} g_n$ and $f_n = (1/K_n)g_n$. From the local Harnack inequality we infer $0 \leq f_k \leq C_n$ on $\overline{\Omega_n}$ for all $k \geq n+1$. Hence, (f_k) is bounded in $L^2(\Omega_n, \mu)$. Using weak compactness of balls in L^2-spaces and an exhaustion argument we infer that there exists a subsequence (f_{n_k}) and $f \in L^2_{\mathrm{loc}}(M)$ such that $f_{n_k} \to f$ weakly in $L^2_{\mathrm{loc}}(M)$. For

$\varphi \in C_c^\infty(M)$ this implies

$$\begin{aligned}((-\Delta-\lambda)f,\varphi) &= (f,(-\Delta-\lambda)\varphi)\\ &= \lim_{k\to\infty}(f_{n_k},(-\Delta-\lambda_{n_k})\varphi)\\ &= \lim_{k\to\infty}\langle(-\Delta-\lambda_{n_k})f_{n_k},\varphi\rangle\\ &= \lim_{k\to\infty}\frac{1}{K_{n_k}}\langle\varphi_{n_k},\varphi\rangle\\ &= 0,\end{aligned}$$

where for the last equality we use $\operatorname{supp}\varphi \subseteq \Omega_n$ for some large n and $\operatorname{supp}\varphi_k \subseteq M\setminus\overline{\Omega_n}$ for all $k \geq n$. Since $\inf_{\Omega_1} f_n = 1$, we also infer $f \neq 0$. The strict positivity of f follows from the local Harnack inequality.

(ii) $\Rightarrow$ (iii): This is obvious.

(iii) $\Rightarrow$ (i): Using $-\Delta f \geq \lambda f$, the weighted Hardy inequality yields

$$\int_M |\nabla\varphi|_g^2 d\mu \geq \lambda \int_M |\varphi|^2 d\mu$$

for all $\varphi \in C_c^\infty(M)$. This implies $\lambda_0(M) \geq \lambda$. □

2. The theorem of Brooks

In this section we seek for upper and lower bounds of the spectrum that involve Lipschitz functions.

Theorem 4.4. *Let f be 1-Lipschitz with $\Delta f \geq \alpha$. Then $\lambda_0(M) \geq \frac{\alpha^2}{4}$.*

Proof. For $\varphi \in C_c^\infty(M)$ we obtain

$$(\Delta f, \varphi^2) \geq \alpha \int_M \varphi^2 d\mu.$$

Using $\|\nabla f\|_\infty \leq \operatorname{Lip}(f) \leq 1$, we infer $f \in W^1_{\mathrm{loc}}(M)$ and

$$\begin{aligned}(\Delta f,\varphi^2) &= -\int_M \langle\nabla f,\nabla\varphi^2\rangle_g d\mu = -2\int_M \langle\nabla f,\nabla\varphi\rangle_g \varphi d\mu\\ &\leq 2\left(\int_M |\nabla\varphi|_g^2 d\mu\right)^{1/2}\left(\int_M \varphi^2 d\mu\right)^{1/2}.\end{aligned}$$

This implies the claim. □

Theorem 4.5. *Assume that M is complete. If there exists a 1-Lipschitz function $f\colon M \to \mathbb{R}$ and $\beta > 0$ such that $e^{-\beta f} \in L^1(M,\mu)$, then $\lambda_0(M) \leq \frac{\beta^2}{4}$.*

Proof. We consider the function $h = e^{-\frac{1}{2}\beta f}$. It belongs to $L^2(M,\mu)$ and satisfies

$$\nabla h = \frac{1}{2}\beta h \nabla f.$$

Using $\|\nabla f\|_\infty \leq 1$, we infer

$$\int_M |\nabla h|^2 d\mu = \frac{\beta^2}{4} \int_M h^2 |\nabla f|_g^2 d\mu \leq \frac{\beta^2}{4} \int_M h^2 d\mu.$$

This shows $h \in W^1(M, \mu)$. Using the completeness of M, we infer $h \in W_0^1(M, \mu)$ and arrive at $\lambda_0(M) \leq \beta^2/4$. □

Remark 4.6. Instead of using completeness to deduce $h \in W_0^1(M, \mu)$, one can also require $\lim_{x\to\infty} f(x) = \infty$.

Corollary 4.7 (Brooks)**.** *Assume that M is complete and let*

$$\beta = \limsup_{r\to\infty} \frac{\log \mu(B_r(o))}{r}.$$

Then $\lambda_0(M) \leq \frac{\beta^2}{4}$. In particular, if M has polynomial volume growth, then $\lambda_0(M) = 0$.

Proof. The condition implies that for given $\varepsilon > 0$ and $R = R(\varepsilon) > 0$ large enough we have

$$\mu(B_r(o)) \leq e^{(\beta+\varepsilon)r}, \quad r \geq R.$$

For $\alpha > \beta + \varepsilon$ consider the function $f = e^{-\alpha\varrho(o,\cdot)}$. By completeness of M balls are relatively compact and we obtain that f is integrable over any ball. For $N > R$ we infer

$$\begin{aligned}\int_{M\setminus B_N(o)} e^{-\alpha\varrho(o,x)} d\mu(x) &= \sum_{k=N}^{\infty} \int_{B_{k+1}(o)\setminus B_k(o)} e^{-\alpha\varrho(o,x)} d\mu(x) \\ &\leq \sum_{k=N}^{\infty} e^{-\alpha k} \mu(B_{k+1}(o)) \\ &\leq \sum_{k=N}^{\infty} e^{-\alpha k} e^{(k+1)(\beta+\varepsilon)} \\ &= e^{\beta+\varepsilon} \sum_{k=N}^{\infty} e^{(\beta+\varepsilon-\alpha)k} < \infty.\end{aligned}$$

With this at hand the statement follows from the previous theorem. □

Remark 4.8. If $\mu(M) = \infty$, the stronger inequality

$$\inf \sigma_{\text{ess}}(-\Delta_F) \leq \frac{\beta^2}{4}$$

holds.

Example 4.9 (Hyperbolic space)**.** We consider the hyperbolic space $\mathbb{H}^d$, $d \geq 2$, and $\mu = \text{vol}$. We claim $\lambda_0(\mathbb{H}^d) = (d-1)^2/4$.

As discussed above, $\mathbb{H}^d$ is isometrically isomorphic to a d-dimensional model manifold with scaling $\psi(r) = \sinh r$. The formula for the volume of balls around 0 in model manifolds implies

$$\mathrm{vol}(B_r(o)) = \int_0^r \psi^{d-1}(t)dt \leq Ce^{(d-1)r},$$

for some constant $C > 0$ and all r large enough. This shows

$$\limsup_{r\to\infty} \frac{\log \mathrm{vol}(B_r(o))}{r} \leq d-1$$

and we infer $\lambda_0(\mathbb{H}^d) \leq (d-1)^2/4$ from Brook's theorem.

We now identify $\mathbb{H}^d$ with the model manifold $(\mathbb{R}^d, g_\psi)$ with scaling $\psi = \sinh$ and the appropriate metric g_ψ.

We consider the function $f\colon \mathbb{R}^d \to \mathbb{R}$, $f(x) = |x|$, i.e., $f(x) = h(|x|)$ with $h(r) = r$, $r > 0$. Our formula for Δ on radially symmetric functions implies

$$\Delta f(x) = h''(r) + (d-1)\frac{\psi'(r)}{\psi(r)}h'(r) = (d-1)\coth(r) \geq (d-1)$$

for all $x \neq 0$ and $r = |x|$. Since in a model manifold $|x| = \rho(x, 0)$, the function f is also 1-Lipschitz. Our lower estimate on λ_0 yields $\lambda_0(\mathbb{H}^d \setminus \{o\}) = \lambda_0((\mathbb{R}^d \setminus \{0\}, g_\psi)) \geq (d-1)^2/4$, where o is the point that gets mapped to 0 under the isometry between $\mathbb{H}^d$ and $(\mathbb{R}^d, g_\psi)$. Now there are two possibilities to show that the same estimate also holds for $\lambda_0(\mathbb{H}^d)$.

1. In a Riemannian manifold (M, g) of dimension at least 2 we have $\lambda_0(M) = \lambda_0(M \setminus \{o\})$ for any $o \in M$. This follows from the fact that $\{\varphi \in C_c^\infty(M) \mid \mathrm{supp}\,\varphi \subseteq M \setminus \{o\}\}$ is dense in $W_0^1(M)$ in dimension $d \geq 2$, which can relatively easily be verified using local charts.

2. For any $o \in \mathbb{H}^d$ one can use polar coordinates starting in o to obtain an isometry $\Phi_o\colon \mathbb{H}^d \to (\mathbb{R}^d, g_\psi)$ with $\Phi_o(o) = 0$. Hence, our discussion shows $\lambda_0(\mathbb{H}^d \setminus \{o\}) \geq (d-1)^2/4$ for all $o \in \mathbb{H}^d$.

Let $\varphi \in C_c^\infty(M)$ and choose $o \in \mathbb{H}^d \setminus \mathrm{supp}\,\varphi$. Then its restriction satisfies

$$\varphi|_{\mathbb{H}^d\setminus\{o\}} \in C_c^\infty(\mathbb{H}^d \setminus \{o\})$$

and we obtain

$$\int_{\mathbb{H}^d} |\nabla\varphi|_g^2 d\mathrm{vol} = \int_{\mathbb{H}^d\setminus\{o\}} |\nabla\varphi|_g^2 d\mathrm{vol} \geq \lambda_0(\mathbb{H}^d\setminus\{o\})\|\varphi\|_2^2 \geq \frac{(d-1)^2}{4}\|\varphi\|_2^2.$$

This shows $\lambda_0(\mathbb{H}^d) \geq (d-1)^2/4$.

CHAPTER 5

Probabilistic properties

The Laplace operator $-\Delta_F$ generates an operator semigroup (P_t) through the formula

$$P_t = e^{t\Delta_F} = \lim_{n\to\infty}\left(1 + \frac{t}{n}\Delta_F\right)^{-n}.$$

The semigroup and the resolvent are Markovian (i.e. $0 \le f \le 1$ implies $0 \le P_t f \le 1$ and $0 \le (1+\alpha\Delta_F)^{-1} f \le 1$). For measurable $f \ge 0$ we choose $f_n \ge 0$ in $L^2(M,\mu)$ with $f_n \nearrow f$ a.s. Then

$$P_t f := \lim_{n\to\infty} P_t f_n$$

and

$$(1+\alpha\Delta_F)^{-1} f := \lim_{n\to\infty} (1+\alpha\Delta_F)^{-1} f_n$$

exist (as an a.e. defined $[0,\infty]$-valued function) and are independent of the chosen sequence (f_n).

If $f \in L^\infty(M)_+$, then $P_t f \in L^\infty(M)_+$ by the Markov property. We extend P_t to a linear operator on $L^\infty(M)$ by letting $P_t f = P_t f_+ - P_t f_-$. It can be proven that for $f \in L^2(M,\mu) + L^\infty(M,\mu)$ the function $u\colon (0,\infty)\times M \to \mathbb{R}$, $u(t,\cdot) = P_t f$ satisfies the heat equation

$$\partial_t u = \Delta u$$

in the sense of distributions. By the hypoellipticity of the heat operator, this implies $u \in C^\infty((0,\infty)\times M)$ and u solves the heat equation in the classical sense. Moreover, it satisfies the initial condition $u(0,\cdot) = f$ in the sense of $L^2_{\mathrm{loc}}(M)$, i.e.,

$$\lim_{t\to 0+} u(t,\cdot) = f$$

in $L^2_{\mathrm{loc}}(M)$.

There is a deep connection between (minimal) Brownian motion and the operator $-\Delta_F$. More precisely, they are related through the Feynman-Kac formula

$$P_t f(x) = \mathbb{E}_x[1_{\{t<\tau\}} f(B_t)],$$

which holds for all $f \in L^\infty(M,\mu)$ and $x \in M$. Here, $\mathbb{E}_x$ denotes the expectation with respect to $\mathbb{P}(\cdot \mid B_0 = x)$ and τ denotes the lifetime of (B_t).

There are two interesting fundamental properties of Brownian motion that depend on the global geometry of M:

(1) How much time does (B_t) spend in bounded regions of M (more precisely open relatively compact subsets). We call the weighted manifold *recurrent* if the expectation of this time is infinite for all relatively compact open sets and *transient* if it is finite for all relatively compact open sets.
(2) Does $1 = P_t 1 = \mathbb{P}(B_t \in M \mid B_0 = \cdot)$ hold for all $t > 0$? If this is the case, Brownian motion does not leave M in finite time and we call the weighted manifold *stochastically complete*. Otherwise, it is called *stochastically incomplete*.

1. Recurrence

We start by investigating the average time spent by Brownian motion in a relatively compact open set $\Omega \subseteq M$. To this end, we compute

$$\mathbb{E}_x \int_0^\infty 1_{\{s\in\mathbb{R} \mid B_s \in \Omega\}}(t)dt = \int_0^\infty \mathbb{E}_x 1_\Omega(B_t)dt$$
$$= \int_0^\infty P_t 1_\Omega(x)dt$$

The map $G\colon L^1(M,\mu)_+ \to L^+(M,\mu)$ defined by

$$Gf = \int_0^\infty P_t f dt$$

is called *Green operator*. With the help of the spectral theorem, it can be proven that

$$Gf = \lim_{\alpha\to 0+} (\alpha - \Delta_F)^{-1} f,$$

and this limit is monotone.

The weighted manifold is called *recurrent* if $\mu(\{0 < Gf < \infty\}) = 0$ for all $f \in L^1(M,\mu)_+$. In contrast, it is called transient if for all $f \in L^1(M,\mu)_+$ we have $Gf < \infty$ a.s. (note that $f > 0$ a.s. implies $Gf > 0$ a.s.). First we give an analytic criterion for recurrence in terms of a Liouville type result.

Theorem 5.1 (A characterization of recurrence)**.** *The weighted manifold (M, g, μ) is recurrent if and only if every $f \in W^1_{\mathrm{loc}}(M) \cap L^\infty(M)$ with $\Delta f \leq 0$ is constant.*

Proof. Here we only show the if part.

Idea: If $Gf < \infty$ a.s., we have $\Delta Gf = -f \leq 0$. Hence, if $Gf \in L^\infty(M)$, our assumption implies that Gf is constant and we arrive at

$$0 = \Delta Gf = -f \leq 0,$$

showing $f = 0$. Since in general Gf will not be essentially bounded and $Gf \neq \infty$ is weaker than $Gf < \infty$ a.s., several approximations are needed to make this argument rigorous.

We use the notation $G_\alpha = (\alpha - \Delta_F)^{-1}$.

Let $f \in L^1(X,\mu)_+$ with $Gf \neq \infty$. We show $f = 0$.

We first prove $Gf < \infty$ a.s. The resolvent identity implies that for $\alpha, \beta > 0$ we have

$$G_\alpha f = G_\beta f + (\beta - \alpha) G_\beta G_\alpha f.$$

Letting $\alpha \to 0+$, we infer (using monotonicity in α)

$$Gf = G_\beta f + \beta G_\beta G f.$$

Assume that there exists $A \subseteq M$ with $\mu(A) > 0$ and $Gf = \infty$ a.s. on A. Then $f \neq 0$. Since G_β is positivity improving, we have $G_\beta f > 0$ a.s. and $G_\beta G f \geq n G_\beta 1_A$ for all $n \geq 1$ with $G_\beta 1_A > 0$ a.s. Together with the previous identity, this implies

$$Gf \geq n G_\beta 1_A \to \infty \text{ a.s., as } n \to \infty.$$

Hence, $Gf = \infty$ a.s., which contradicts our assumption.

Without loss of generality we can assume $f \in L^1(M, \mu) \cap L^2(M, \mu)$ (else consider $f \wedge 1$ and use $\{f = 0\} = \{f \wedge 1 = 0\}$). Moreover, we can assume $fGf \in L^1(M, \mu)_+ \cap L^2(M, \mu)$ (else consider $g = f/(Gf \vee 1)$ and use

$$gGg \leq \frac{f}{Gf \vee 1} Gf \leq f \in L^1(M, \mu) \cap L^2(M, \mu)$$

and $\{f = 0\} = \{g = 0\}$).

The L^2-lower semicontinuity of the energy (which follows from $H_0^1(M, \mu)$ being a Hilbert space) yields

$$\begin{aligned}
\int_M |\nabla(Gf)|_g^2 d\mu &\leq \liminf_{\alpha \to 0+} \int_M |\nabla(G_\alpha f)|_g^2 d\mu \\
&= \liminf_{\alpha \to 0+} \langle -\Delta G_\alpha f, G_\alpha f \rangle \\
&= \liminf_{\alpha \to 0+} \left(\langle f, G_\alpha f \rangle - \alpha \langle G_\alpha f, G_\alpha f \rangle \right) \\
&\leq \int_M f G f d\mu < \infty.
\end{aligned}$$

For the second equality we used $G_\alpha f \in W_0^1(M, \mu)$, which allows the application of Green's formula. This implies $Gf \in W^1_{\mathrm{loc}}(M)$ (actually one also has to prove $Gf \in L^2_{\mathrm{loc}}(M)$, but this is a consequence of local Poincaré inequalities - we refrain from giving details).

From $Gf \in L^2_{\mathrm{loc}}(M)$, we infer $G_\alpha f \to Gf$ in $L^2_{\mathrm{loc}}(M)$. Using the continuity of Δ with respect to the weak-$*$-topology on distributions, we infer

$$\Delta G f = \lim_{\alpha \to 0+} \Delta G_\alpha f = \lim_{\alpha \to 0+} (-f + \alpha G_\alpha f) = -f,$$

where we used $Gf < \infty$ a.s. for the last identity.

Let $\alpha > 0$. We choose a C^2-function $C\colon \mathbb{R} \to \mathbb{R}$ with the following properties:

(1) $C_n(t) = t$ for $t \leq \alpha - 2/n$ and $C_n(t) = \alpha - 1/n$ for $t \geq \alpha - 1/n$.
(2) C_n is 1-Lipschitz.

(3) C_n is increasing and concave.

Then $C_n(t) \to t \wedge \alpha$, as $n \to \infty$, and $C_n'(t) \to D(t)$, as $n \to \infty$, with $D(t) = 1$ if $t < \alpha$ and $D(t) = 0$ if $t \geq \alpha$. Using our chain rules, we obtain $C_n(Gf) \in W^1_{\mathrm{loc}}(M)$ and

$$\begin{aligned}\Delta C_n(Gf) &= C_n''(Gf)|\nabla f|^2 + C_n'(Gf)\Delta Gf \\ &= C_n''(Gf)|\nabla f|^2 - C_n'(Gf)f.\end{aligned}$$

Letting $n \to \infty$ and using $C_n'' \leq 0$, we infer

$$\Delta(Gf \wedge \alpha) \leq -D(Gf)f \leq 0.$$

Hence, $Gf \wedge \alpha \in W^1_{\mathrm{loc}}(M) \cap L^\infty(M)$ with $\Delta(Gf \wedge \alpha) \leq 0$. Our assumption implies that $Gf \wedge \alpha$ is constant, leading to

$$0 = \Delta(Gf \wedge \alpha) \leq -D(Gf)f \leq 0.$$

Since $D(Gf) = 1$ on $\{Gf < \alpha\}$, $Gf < \infty$ a.s. and $\alpha > 0$ was arbitrary, we infer $f = 0$ a.s. □

Theorem 5.2 (Karp's volume growth test for recurrence)**.** *Assume that M is complete and that for some $o \in M$ and $r_0 > 0$ we have*

$$\int_{r_0}^\infty \frac{r}{\mu(B_r(o))} dr = \infty.$$

Then (M, g, μ) is recurrent.

Proof. Let $f \in W^1_{\mathrm{loc}}(M) \cap L^\infty(M)$ with $\Delta f \leq 0$. Without loss of generality we assume $\|f\|_\infty \leq 1$. Then $h = 1 - f \in W^1_{\mathrm{loc}}(M) \cap L^\infty_{\mathrm{loc}}(M)$, $h \geq 0$ and $\Delta h \geq 0$. Moreover,

$$\|1_{B_r(o)} f\|_2^2 \leq \|f\|_\infty^2 \mu(B_r(o)).$$

Hence, our volume growth assumption yields

$$\int_{r_0}^\infty \frac{r}{\|1_{B_r(o)} f\|_2^2} dr = \infty.$$

We obtain that f is constant from Karp's Liouville theorem. □

Remark 5.3. The volume growth test is satisfied if there are $R, C > 0$ such that $\mu(B_r(o)) \leq Cr^2 \log r$ for all $r \geq R$.

2. Stochastic completeness

In this section we study stochastic completeness of (M, g, μ). Here, (M, g, μ) is called *stochastically complete* if $P_t 1 = 1$ for all $t > 0$.

Remark 5.4. It is quite remarkable that there are complete but stochastically incomplete manifolds (e.g. model manifolds of a certain volume growth, see [**3**]). In contrast, in the incomplete case it is clear that stochastic incompleteness can occur. On open relatively compact domains $\Omega \subseteq \mathbb{R}^d$ the minimal Brownian motion that we consider is Brownian motion that is killed upon hitting the boundary $\partial\Omega$. Since Brownian motion will will hit $\partial\Omega$ with probability one, the probability that it stays in Ω for all times is strictly less than one.

In order to obtain criteria for stochastic completeness we consider $w\colon (0,\infty)\times M \to \mathbb{R}$, $w(t,\cdot) = 1 - P_t 1$. By our earlier discussion $w \in C^\infty((0,\infty)\times M)$ and

$$\begin{cases} \partial_t w = \Delta w \\ w(0,\cdot) = 0 \quad \text{in the sense of } L^2_{\mathrm{loc}}(M) \end{cases}.$$

In order to show $w = 0$, it suffices to prove that the previous equation has unique bounded solutions.

Theorem 5.5 (Grigor'yan's uniqueness class)**.** *Let M be complete and let $T > 0$. Assume that $u \in C^\infty((0,T)\times M)$ solves*

$$\begin{cases} \partial_t u = \Delta u \\ \lim\limits_{t\to 0+} u(t,\cdot) = 0 \quad \textit{in } L^2_{\mathrm{loc}}(M) \end{cases}.$$

Further assume that there is a monotone increasing $f\colon (0,\infty) \to (0,\infty)$ with

$$\int^\infty \frac{r}{f(r)} dr = \infty,$$

such that for some $o \in M$ and all $R > 0$ we have

$$\int_0^T \int_{B_R(o)} |u(t,x)|^2 d\mu dt \le \exp(f(R)).$$

Then $u = 0$ on $(0,T)\times M$.

Corollary 5.6 (Grigor'yan's volume growth test)**.** *Assume that M is complete and that for some $o \in M$ and $r_0 > 0$ we have*

$$\int_{r_0}^\infty \frac{r}{\log^\sharp \mu(B_r(o))} dr = \infty.$$

Then (M,g,μ) is stochastically complete. Here, $\log^\sharp = \max\{\log, 1\}$*.*

Remark 5.7. Stochastic completeness holds if $\mu(B_r(o)) \le Ce^{Dr^2 \log r}$ for some $C, D > 0$ and all r large enough. If the Ricci tensor of (M,g) is bounded from below, then by standard volume comparison results $\mathrm{vol}(B_r(o)) \le e^{Dr}$ for some $D > 0$ and all $r > 0$. Hence, in this case (M,g,vol) is stochastically complete. This observation is due to Yau.

Proof. According to our previous discussion, it suffices to show that any $u \in C_b^\infty((0,\infty)\times M)$ with

$$\begin{cases} \partial_t u = \Delta u \\ \lim\limits_{t\to 0+} u(t,\cdot) = 0 \quad \text{in } L^2_{\mathrm{loc}}(M) \end{cases}$$

satisfies $u = 0$. Let

$$S := \sup_{t>0,\, x\in M} |u(t,x)|$$

and

$$f(r) := \log^\sharp(S^2 T\mu(B_r(o))).$$

Our assumption yields

$$\int^\infty \frac{r}{f(r)} dr = \infty,$$

and the choice of S and f implies

$$\int_0^T \int_{B_R(o)} |u(t,x)|^2 d\mu dt \leq S^2 T\mu(B_r(o)) \leq \exp(f(R)).$$

With this at hand, $u = 0$ follows directly from Grigor'yan's uniqueness class. □

Next we establish Grigor'yan's uniqueness class. The proof is taken from ...

Lemma 5.8 (A priori estimate)**.** *Assume that M is complete. Let $0 \leq a < b$ and $u \in C^\infty((a,b) \times M)$ satisfy $\partial_t u = \Delta u$ and assume that the limits*

$$u(a,\cdot) := \lim_{t\to a+} u(t,\cdot) \text{ and } u(b,\cdot) := \lim_{t\to b-} u(t,\cdot)$$

exist in $L^2_{\mathrm{loc}}(\mu)$. If $f\colon (0,\infty) \to (0,\infty)$ is increasing such that

$$\int_a^b \int_{B_R(o)} |u(t,x)|^2 dmu(x) dt \leq \exp(f(R)) \text{ for all } R > 0,$$

then

$$\int_{B_R(o)} |u(b,x)|^2 d\mu(x) \leq \int_{B_{4R}(o)} |u(a,x)|^2 d\mu(x) + \frac{4}{R^2}, \qquad (\heartsuit)$$

as long as

$$b - a \leq \frac{R^2}{8f(4R)}. \qquad (\diamondsuit)$$

Remark 5.9. This lemma says the following: Up to a small error one can compare the size of solutions to the heat equation on a ball at a later time with its size on a larger ball at an earlier time. The allowed time difference depends on the sizes of the balls and the function f.

Proof. In this proof we use ρ for a 1-Lipschitz function (to be specified later) and, as above, ϱ for the path metric on M.

Using the continuity of u and compactness of balls, we can assume that u is smooth on an open neighborhood of $[a,b] \times M$. For a 1-Lipschitz function $\rho\colon M \to \mathbb{R}$ and $s \notin [a,b]$ we define $\xi\colon [a,b]\times M \to \mathbb{R}$

by

$$\xi(t,x) := \frac{\rho(x)^2}{4(t-s)}.$$

We will choose ρ and s later. Rademacher's theorem and the chain rule yield

$$|\nabla \xi(t,\cdot)| \leq \frac{|\rho|}{2|t-s|}.$$

Moreover,

$$\partial_t \xi(t,x) = -\frac{\rho^2(x)}{4(t-s)^2},$$

which yields

$$\partial_t \xi + |\nabla \xi|^2 \leq 0.$$

For given $R > 0$ we define $\varphi\colon M \to \mathbb{R}$ by

$$\varphi(x) := (3 - \rho(x,o)/R)_+ \wedge 1.$$

It has the following properties:

- $0 \leq \varphi \leq 1$ on M,
- $\varphi = 1$ on $B_{2R}(o)$,
- $\varphi = 0$ on $M \setminus B_{3R}(o)$,
- φ is $1/R$-Lipschitz.

Using the completeness of M we obtain $\varphi \in \mathrm{Lip}_c(M) \subseteq W_c^1(M,\mu)$. For fixed t the function $u\varphi^2 e^\xi$ is locally Lipschitz. Since φ has compact support, we obtain $u\varphi^2 e^\xi \in \mathrm{Lip}_c(M)$.

Multiplying the heat equation

$$\partial_t u = \Delta u$$

by $u\varphi^2 e^\xi$ and integrating the result over $[a,b] \times M$ yields

$$\int_a^b \int_M (\partial_t u) u\varphi^2 e^\xi d\mu dt = \int_a^b \int_M (\Delta u) u \varphi^2 e^\xi d\mu dt.$$

Since u and ξ are smooth in a neighborhood of $[a,b]$, we obtain

$$\begin{aligned}\int_a^b (\partial_t u) u \varphi^2 e^\xi dt &= \frac{1}{2}\int_a^b \partial_t(u^2)\varphi^2 e^\xi dt \\ &= \frac{1}{2}\, u^2\varphi^2 e^\xi\big|_a^b - \frac{1}{2}\int_a^b (\partial_t \xi)\varphi^2 u^2 e^\xi dt.\end{aligned}$$

We evaluate the integral over M with the help of Green's formula (use $u(t,\cdot) \in C^\infty(M)$ and approximate $u(t,\cdot)\varphi^2 e^{\xi(t,\cdot)} \in W_c^1(M,\mu)$ by smooth compactly supported functions)

$$\int_M (\Delta u) u \varphi^2 e^\xi d\mu = -\int_M \langle \nabla u, \nabla(u\varphi^2 e^\xi)\rangle d\mu.$$

The chain- and product rule yield

$$\begin{aligned}
-\langle \nabla u, \nabla(u\varphi^2 e^\xi)\rangle &= -|\nabla u|^2\varphi^2 e^\xi - \langle \nabla u, \nabla \xi\rangle u\varphi^2 e^\xi - 2\langle \nabla u, \nabla \varphi\rangle u\varphi e^\xi \\
&\le -|\nabla u|^2\varphi^2 e^\xi + |\nabla u||\nabla \xi||u|\varphi^2 e^\xi \\
&\quad + \left(\frac{1}{2}|\nabla u|^2\varphi^2 + 2|\nabla\varphi|^2 u^2\right) e^\xi \\
&= \left(-\frac{1}{2}|\nabla u|^2 + |\nabla u||\nabla \xi||u|\right)\varphi^2 e^\xi + 2|\nabla \varphi|^2 u^2 e^\xi.
\end{aligned}$$

Combining these inequalities and $\partial_t \xi + |\nabla \xi|^2 \le 0$, we infer

$$\begin{aligned}
\int_M u^2\varphi^2 e^\xi d\mu\Big|_a^b &= \int_a^b \int_M (\partial_t \xi)\varphi^2 u^2 e^\xi d\mu dt + 2\int_a^b\int_M (\Delta u) u\varphi^2 e^\xi d\mu dt \\
&\le \int_a^b \int_M \left(-|\nabla\xi|^2 u^2 - |\nabla u|^2 + 2|\nabla u||\nabla \xi||u|\right)\varphi^2 e^\xi d\mu dt \\
&\quad + 4\int_a^b\int_M |\nabla\varphi|^2 u^2 e^\xi d\mu dt \\
&= -\int_a^b\int_M \left(|\nabla u| - |\nabla\xi||u|\right)^2 d\mu dt + 4\int_a^b\int_M |\nabla \varphi|^2 u^2 e^\xi d\mu dt.
\end{aligned}$$

Hence,

$$\int_M u^2\varphi^2 e^\xi d\mu\Big|_a^b \le 4\int_a^b\int_M |\nabla\varphi|^2 u^2 e^\xi d\mu dt.$$

Rademacher's theorem yields $|\nabla\varphi|^2 \le 1/R^2$. Using also the other properties of φ, we obtain

$$\begin{aligned}
\int_{B_R(o)} u(b,x)^2 e^{\xi(x,b)} d\mu(x) &\le \int_{B_{4R}(o)} u(a,x)^2 e^{\xi(x,a)} d\mu(x) \\
&\quad + \frac{4}{R^2}\int_a^b \int_{B_{4R}(o)\setminus U_{2R}(o)} u^2 e^\xi d\mu dt.
\end{aligned}$$

Now we choose ρ and s. Let $\rho\colon M \to \mathbb{R}$, $\rho(x) := (\varrho(x,o) - R)_+$ and $s := 2b - a$. For all $t \in [a,b]$ we obtain

$$b - a \le s - t \le 2(b-a)$$

and, hence,

$$\xi(t,x) = -\frac{\rho(x)^2}{4(s-t)} \le -\frac{\rho(x)^2}{8(b-a)} \le 0.$$

Moreover, $\xi = 0$ on $B_R(o)$ and $\rho \ge R$ on $B_{4R}(o)\setminus U_{2R}(o)$, which implies

$$\xi \le -\frac{R^2}{8(b-a)} \text{ on } [a,b]\times B_{4R}(o)\setminus U_{2R}(o).$$

Plugged into the last integral inequality

$$\int_{B_R(o)} u(b,x)^2 d\mu(x) \le \int_{B_{4R}(o)} u(a,x)^2 d\mu(x) + \frac{4}{R^2}\exp\left(-\frac{R^2}{8(b-a)}\right)\int_a^b \int_{B_{4R}(o)} u^2 d\mu dt.$$

According to the assumptions u satisfies the growth bound

$$\int_a^b \int_{B_{4R}(o)} u^2 d\mu dt \le \exp(f(4R))$$

and our inequality further simplifies to

$$\int_{B_R(o)} u(b,x)^2 d\mu(x) \le \int_{B_{4R}(o)} u(a,x)^2 d\mu(x) + \frac{4}{R^2}\exp\left(-\frac{R^2}{8(b-a)} + f(4R)\right).$$

If (♢) holds, then

$$-\frac{R^2}{8(b-a)} + f(4R) \le 0.$$

This shows the a priori estimate. □

Proof of Grigor'yan's uniqueness class. Let $u \in C^\infty((0,T)\times M)$ solve

$$\begin{cases} \partial_t u = \Delta u \\ \lim_{t\to 0+} u(t,\cdot) = 0 \quad \text{in } L^2_{\text{loc}}(M) \end{cases}.$$

We define $u(0,x) := 0$ for all $x \in M$.

Let $R > 0$ and $0 < t < T$. We set $R_k := 4^k R$, $\tau_0 = 0$ and

$$\tau_k := \frac{1}{128}\frac{R_k^2}{f(R_k)},\ k \in \mathbb{N}.$$

We define the times

$$t_k := \begin{cases} t - \sum_{l=0}^k \tau_l & \text{if } t - \sum_{l=1}^k \tau_l > 0, \\ 0 & \text{else.} \end{cases}$$

If $t_{k-1} \neq 0$, then $0 \le t_k < t_{k-1}$ and

$$t_{k-1} - t_k \le \tau_k = \frac{1}{128}\frac{R_k^2}{f(R_k)} = \frac{1}{128}\frac{16R_{k-1}^2}{f(4R_{k-1})} = \frac{R_{k-1}^2}{8f(4R_{k-1})}.$$

Hence, the difference satisfies the condition (♢) of the previous Lemma. Inequality (♡) yields

$$\int_{B_{R_{k-1}}(o)} u(t_{k-1},x)^2 d\mu(x) \le \int_{B_{R_k}(o)} u(t_k,x)^2 d\mu(x) + \frac{4}{R_{k-1}^2}.$$

Iterating this inequality yields

$$\begin{aligned}\int_{B_R(o)} u(t,x)^2 d\mu(x) &\leq \int_{B_{R_k}(o)} u(t_k,x)^2 d\mu(x) + \sum_{l=1}^{k} \frac{4}{R_{l-1}^2} \\ &\leq \int_{B_{R_k}(o)} u(t_k,x)^2 d\mu(x) + \frac{C}{R^2}.\end{aligned}$$

Here, $C > 0$ is a constant that is independent of k. If we show $t_k = 0$ for some $k \in \mathbb{N}$, then the claim follows from this inequality using $u(t_k, \cdot) = u(0, \cdot) = 0$ and letting $R \to \infty$.

Since f is monotone increasing, the assume growth condition yields

$$\infty = \int_R^\infty \frac{r}{f(r)} dr \leq \sum_{k=0}^\infty \int_{R_k}^{R_{k+1}} \frac{r}{f(r)} dr \leq \sum_{k=0}^\infty \frac{R_{k+1}^2}{f(R_k)}.$$

From this we obtain

$$\sum_{k=0}^\infty \tau_k = \infty,$$

i.e., $t - \sum_{k=0}^N \tau_k \leq 0$ for some N big enough, which leads to $t_N = 0$. □

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