Discrete Optimization in a Nutshell

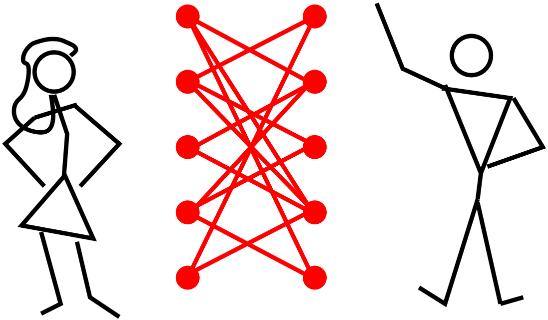
Christoph Helmberg

Integer Optimization

Contents

Contents

1.1 Application: The Marriage Problem (Bipartite Matching)



Find a maximum number of pairs!

men ↔ women, worker ↔ mashines, students ↔ positions, . . . (also weighted versions)

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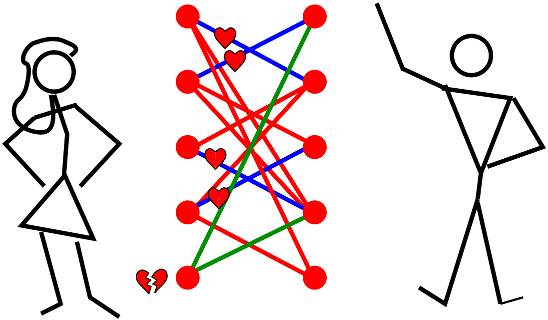
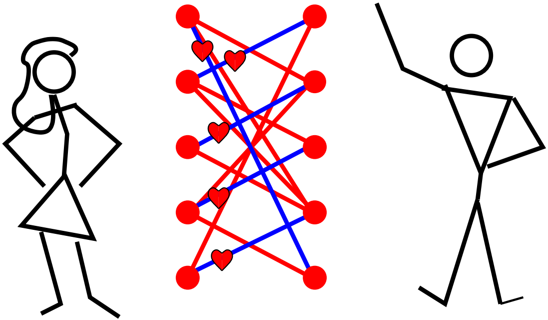


Find a maximum number of pairs!
Maximal (cannot be increased), but not of maximum cardinality

men ↔ women, worker ↔ mashines, students ↔ positions, . . . (also weighted versions)

1.1 Application: The Marriage Problem (Bipartite Matching)

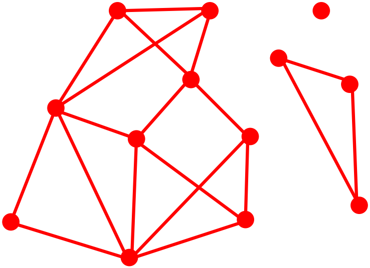


Find a maximum number of pairs!
Maximum Cardinality Matching (even perfect)

men ↔ women, worker ↔ mashines, students ↔ positions, . . . (also weighted versions)

Bipartite Matching

- An (undirected) graph $G = (V, E)$ is a pair consisting of a node/vertex set V and an edge set $E \subseteq \{\{u, v\} : u, v \in V, u \neq v\}$.
- Two nodes $u, v \in V$ are adjacent/neighbors, if $\{u, v\} \in E$.
- A node $v \in V$ and an edge $e \in E$ are incident, if $v \in e$.
- Two edges $e, f \in V$ are incident, if $e \cap f \neq \emptyset$.



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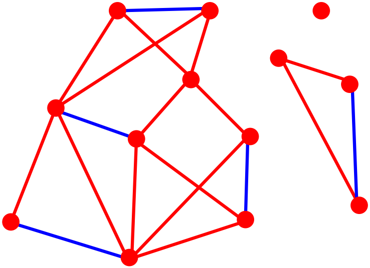
- An edge set $M \subseteq E$ is a matching/1-factor, if for $e, f \in M$ with $e \neq f$ there holds $e \cap f = \emptyset$. The matching is perfect, if $|V| = 2|M|$.



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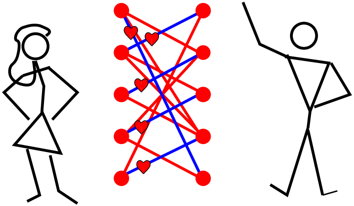
- $G = (V, E)$ is bipartite, if $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \emptyset$ and $E \subseteq \{\{u, v\} : u \in V_1, v \in V_2\}$.



Model: Maximum Cardinatliy Bipartite Matching

given: $G = (V_1 \dot{\cup} V_2, E)$ bipartite
find: matching $M \subseteq E$ with $|M|$ maximal

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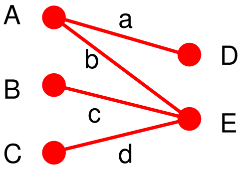
variables: $x \in \{0,1\}^E$ with $x_e = \begin{cases} 1 & \text{if } e \in M \\ 0 & \text{otherwise.} \end{cases} \quad (e \in E)$

(it represents the incidence/characteristic vector of M w.r.t. E)

constraints: $Ax \leq \mathbf{1}$,

where $A \in \{0,1\}^{V \times E}$ Node-Edge-Incidencematrix of G:

$$A_{v,e} = \begin{cases} 1 & \text{if } v \in e \\ 0 & \text{otherwise.} \end{cases} \quad (v \in V, e \in E)$$



$$A = \begin{array}{c} \\ (A) \\ (B) \\ (C) \\ (D) \\ (E) \end{array} \left[\begin{array}{cccc} (a) & (b) & (c) & (d) \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{array} \right]$$

Model: Maximum Cardinatliy Bipartite Matching

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optimization problem:
$$\begin{array}{ll} \max & \mathbf{1}^T x \\ \text{s.t.} & Ax \leq \mathbf{1} \\ & x \in \{0,1\}^E \end{array}$$

This is no LP! Enlarge $x \in \{0,1\}^E$ to $x \in [0,1]^E \to$ LP

For G bipartite, Simplex always delivers an optimal solution $x^* \in \{0,1\}^E$!
(this does not hold i.g. for general graphs G!)

Contents

1.2 Integeral Polyhedra

Simplex automatically yields an integral solution, if all vertices of the feasible set are integral.

$$\min c^T x \quad \text{s.t.} \quad x \in \mathcal{X} := \{x \geq 0 : Ax = b\}$$

Are all vertices of $\mathcal{X}$ integral?

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Are all vertices of $\mathcal{X}$ integral? Almost never!
But there is an important class of matrices $A \in \mathbb{Z}^{m \times n}$, for which $\mathcal{X}$ has only integral vertices for any (!) $b \in \mathbb{Z}^m$:

A vertex is integral $\Leftrightarrow$ basic solution $x_B = A_B^{-1} b \in \mathbb{Z}^m$.
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A matrix $A \in \mathbb{Z}^{m \times n}$ of full row rank is unimodular, if $|\det(A_B)| = 1$ holds for each basis B.

Theorem

$A \in \mathbb{Z}^{m \times n}$ is unimodular if and only if for each $b \in \mathbb{Z}^m$ all vertices of the polyhedron $\mathcal{X} := \{x \geq 0 : Ax = b\}$ are integral.

Does this also hold for $\mathcal{X} := \{x \geq 0 : Ax \geq b\}$?

Totally Unimodular Matrices

$$\{x \geq 0 : Ax \geq b\} \quad \rightarrow \quad \left\{ \begin{bmatrix} x \\ s \end{bmatrix} \geq 0 : [A, I] \begin{bmatrix} x \\ s \end{bmatrix} = b \right\}$$

Certainly integral, if $\bar{A} = [A, I]$ is unimodular.
Laplace development of the determinant for each basis $B \rightarrow$

A matrix A is totally unimodular if for each square submatrix of A the determinant has value 0, 1 or -1. (requires $A \in \{0, 1, -1\}^{m \times n}$)

Theorem (Hoffmann und Kruskal)

$A \in \mathbb{Z}^{m \times n}$ is totally unimodular if and only if for each $b \in \mathbb{Z}^m$ all vertices of the polyhedron $\mathcal{X} := \{x \geq 0 : Ax \geq b\}$ are integral.

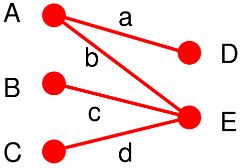
Note: A tot. unimod. $\Leftrightarrow$ A^T resp. $[A, -A, I, -I]$ tot. unimod.
consequence: dual LP, variants with equality constraints, etc. are integral

Recognizing Totally Unimodular Matrices

Theorem (Heller and Tompkins)

Let $A \in \{0, 1, -1\}^{m\times n}$ have at most two nonzero entries per column.
A is totally unimodular $\Leftrightarrow$ the rows A can be partitioned int two classes so that
(i) rows with one +1 and one -1 entry in the same column belong to the same class,
(ii) rows with two nonzeros of equal sign in the same column belong to distinct classes.

Example 1: the node-edge-incidencematrix of a bipartite graph



$$A = \left[\begin{array}{c|cccc} & (a) & (b) & (c) & (d) \\ (A) & 1 & 1 & 0 & 0 \\ (B) & 0 & 0 & 1 & 0 \\ (C) & 0 & 0 & 0 & 1 \\ \hline (D) & 1 & 0 & 0 & 0 \\ (E) & 0 & 1 & 1 & 1 \end{array}\right]$$

Example 1: Bipartite Graphs

A ... node-edge incidence matrix of $G = (V_1 \dot\cup V_2, E)$ bipartite

---

Bipartite Matching of Maximum Cardinality:

$$\max \mathbf{1}^T x \quad \text{s.t.} \quad Ax \le \mathbf{1},\ x \ge 0$$

Because A tot. unimod. the veritces of the feasilbe set are all integral
$\Rightarrow$ Simplex delivers optimal solution $x^* \in \{0,1\}^E$.

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The dual is also integral, because A^T tot. unimod.:

$$\min \mathbf{1}^T y \quad \text{s.t.} \quad A^T y \geq \mathbf{1},\ y \geq 0$$

Interpretation: $y^* \in \{0,1\}^V$ is the incidence vector of a smallest node set $V' \subseteq V$, so that $\forall e \in E : e \cap V' \neq \emptyset$ (**Minimum Vertex Cover**)

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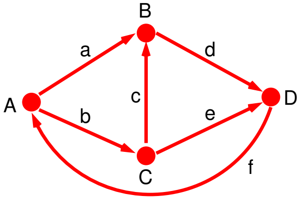
The **Assignment Problem**: $|V_1| = |V_2| = n$, complete bipartite:
$E = \{\{u,v\} : u \in V_1, v \in V_2\}$; edge weights $c \in \mathbb{R}^E$
Find a perfect matching of minimum total weight:

$$\min c^T x \quad \text{s.t.} \quad Ax = \mathbf{1},\ x \geq 0$$

is integral, too, because $[A; -A]$ tot. unimod.

Example 2: Node-Arc Incidence Matrix of Digraphs

- A digraph/directed graph $D = (V, E)$ is a pair consisting of a node set V and a (multi-)set of directed edges/arcs $E \subseteq \{(u, v) : u, v \in V, u \neq v\}$. [multiple arcs are allowed!]
- For $e = (u, v) \in E$, u is the tail and v the head of e.



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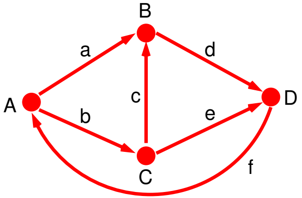
- For $e = (u, v) \in E$, u is the tail and v the head of e.
- The node-arc incidence matrix $A \in \{0, 1, -1\}^{V \times E}$ of D has entries

$$A_{v,e} = \begin{cases} -1 & v \text{ is the tail of } e \\ 1 & v \text{ is the head of } e \\ 0 & \text{otherwise} \end{cases} \qquad (v \in V, e \in E).$$

the node-arc incidence matrix of a digraph is totally unimodular.



$$A = \begin{bmatrix} & (a) & (b) & (c) & (d) & (e) & (f) \\ (A) & -1 & -1 & 0 & 0 & 0 & 1 \\ (B) & 1 & 0 & 1 & -1 & 0 & 0 \\ (C) & 0 & 1 & -1 & 0 & -1 & 0 \\ (D) & 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}$$

Contents

1.3 Application: Networkflow

Modelling tool: transport problems, evacuation planning, scheduling, (internet) traffic planning, ...

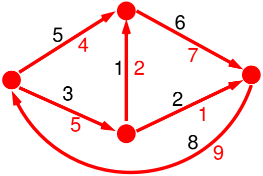
- A network (D, w) consists of a digraph $D = (V, E)$ and (arc-)capacities $w \in \mathbb{R}^E_+$.
- A vector $x \in \mathbb{R}^E$ is a flow on (D, w), if it satisfies the

flow conservation constraints $$\sum_{e=(u,v)\in E} x_e = \sum_{e=(v,u)\in E} x_e \quad (v \in V)$$

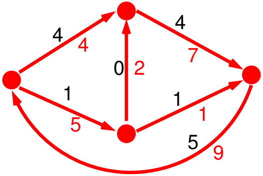
[$\Leftrightarrow$ $Ax = 0$ for node-arc incidence matrix A]

- A flow $x \in \mathbb{R}^E$ on (D,w) is feasible, if $0 \le x \le w$

[lower bounds would also be possible]



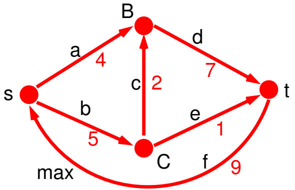
flow



feasible flow

Maximal s-t-Flows, Minimal s-t-Cuts

Given a source $s \in V$ and a sink $t \in V$ with $(t, s) \in E$, find a feasible flow $x \in \mathbb{R}^E$ on (D, w) with maximum flow value $x_{(t,s)}$.



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LP: $\max \ x_{(t,s)}$ s.t. $Ax = 0, \ 0 \le x \le w,$

If $w \in \mathbb{Z}^E$, the OS of Simplex is $x^* \in \mathbb{Z}^E$, because $[A; -A; I]$ tot. unimod.

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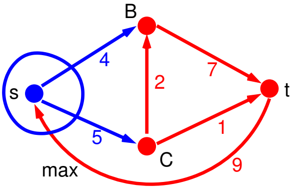


$S = \{s\}$,
$\delta^+(S) = \{(s, B), (s, C)\}$,
$w(\delta^+(S)) = 4 + 5 = 9$.

LP: $\max\ x_{(t,s)}$ s.t. $Ax = 0,\ 0 \le x \le w$,

If $w \in \mathbb{Z}^E$, the OS of Simplex is $x^* \in \mathbb{Z}^E$, because $[A; -A; I]$ tot. unimod.

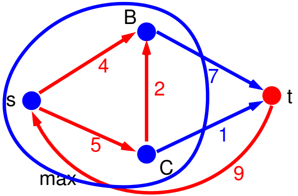
Each $S \subseteq V$ with $s \in S$ and $t \notin S$ defines an s-t-cut

$$\delta^+(S) := \{(u, v) \in E : u \in S, v \notin S\},$$

the out-flow is at most $w(\delta^+(S)) := \sum_{e \in \delta^+(S)} w_e$, the value of the cut.

Maximal s-t-Flows, Minimal s-t-Cuts

Given a source $s \in V$ and a sink $t \in V$ with $(t, s) \in E$, find a feasible flow $x \in \mathbb{R}^E$ on (D, w) with maximum flow value $x_{(t,s)}$.



$S = \{s, B, C\}$,
$\delta^+(S) = \{(B, t), (C, t)\}$,
$w(\delta^+(S)) = 7 + 1 = 8$.

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Theorem (Max-Flow Min-Cut Theorem of Ford and Fulkerson)

The maximum s-t-flow value equals the minimum s-t cut value.

[both computable by Simplex]

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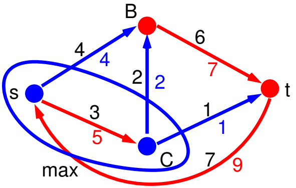

$S = \{s, C\}$,
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$w(\delta^+(S)) = 4+2+1 = 7 = x^*_{(t,s)}$

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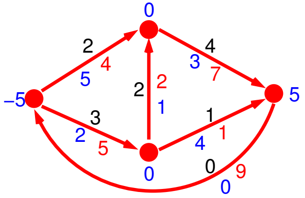
Minimum Cost Flow (Min-Cost-Flow)

The flow value is now prescribed by balances $b \in \mathbb{R}^V$ ($\mathbf{1}^T b = 0$) on the nodes; each unit of flow induces arc costs $c \in \mathbb{R}^E$.
Find the cheapest flow.



$$\begin{array}{ll} \min & \begin{bmatrix} 5 & 2 & 1 & 3 & 4 & 0 \end{bmatrix} x \\ \text{s.t.} & \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix} x = \begin{bmatrix} 5 \\ 0 \\ 0 \\ -5 \end{bmatrix} \\ & 0 \le x \le w \end{array}$$

LP: $\min\ c^T x \quad \text{s.t.} \quad Ax = b,\ 0 \le x \le w,$

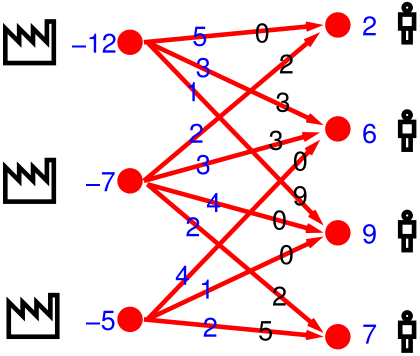
For b, c and w integr., Simplex gives OS $x^* \in \mathbb{Z}^E$, as $[A; -A; I]$ tot. unimod.
[also works for lower bounds on arcs: $u \le x \le w$!]

For LPs $\min\ c^T x \quad \text{s.t.} \quad Ax = b,\ u \le x \le w,$ A node-arc inc.
there is a particularly efficient simplex variant, the network simplex,
it only needs addtions, subtractions and comparisons!

Extremly broad scope of applications, popular modelling tool

Example: Transportation Problem

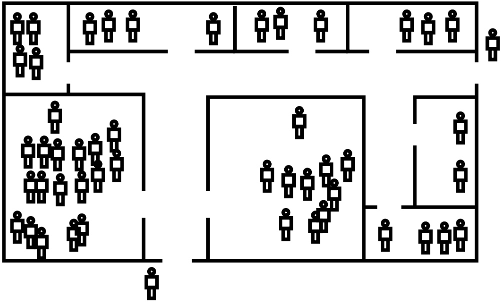
A company with several production sites has to serve several customers.
How is this best done in view of transportation costs?



Note: only one product!

Example: Evacuation Planning

Determine for each room a flight path, so that the building is cleared as fast as possible. Per aisle capacity and crossing times are known.



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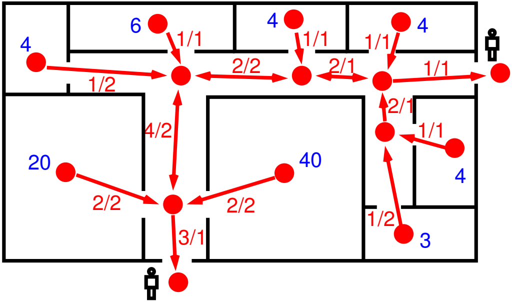

Arcs with capacities and crossing times

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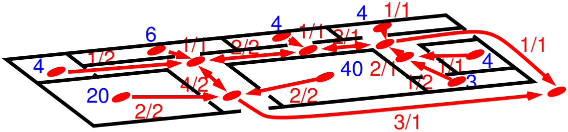

geometry is not important, simplify

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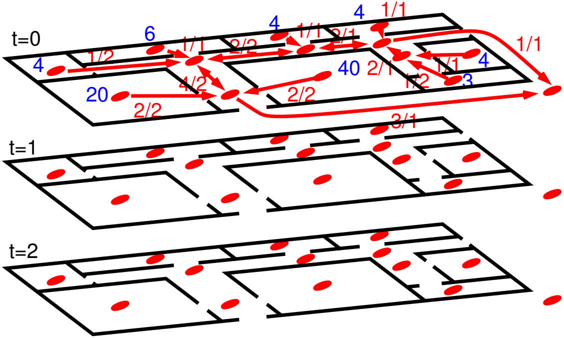

discretize time, one level per time step

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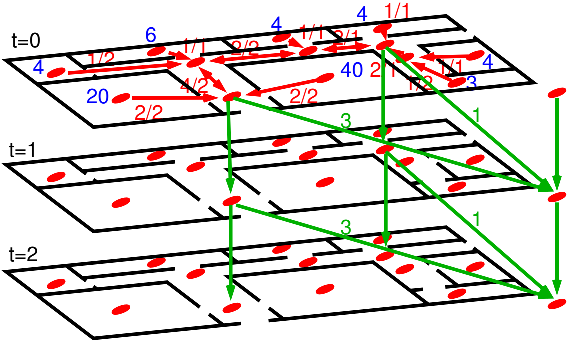

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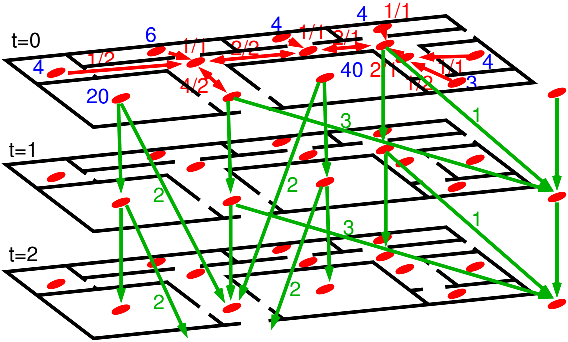

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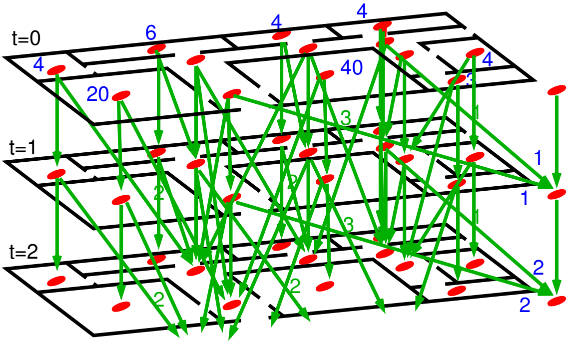

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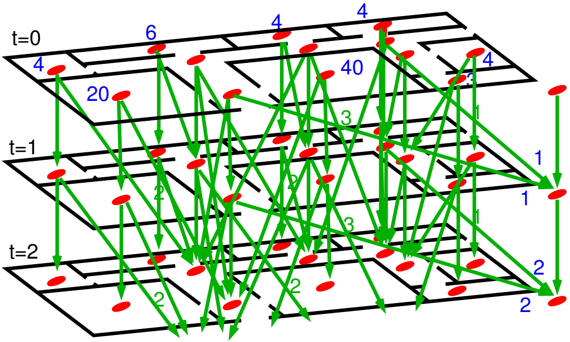


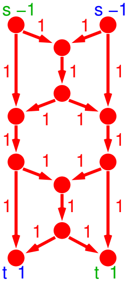
rising costs on the exit arcs to encourage fast exits

Approach only ok if persons do not need to be discerned!

Contents

1.4 Multi-Commodity Flow Problems

Given a network (D, w) and several different commodities $K = \{1, \ldots, k\}$ with sources/sinks (s_i, t_i) and flow values f_i, $i \in K$ find feasible flows $x^{(i)} \in \mathbb{R}^E$, so that in sum capacities are observed.



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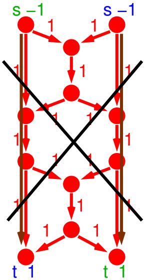

mixing is forbidden!

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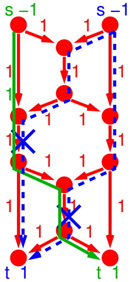

integr. is impossible

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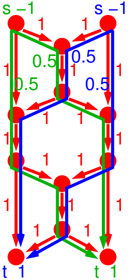

fractional works

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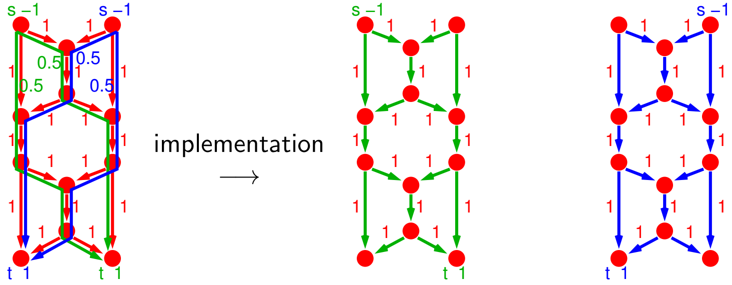

fractional works copy 1 copy 2

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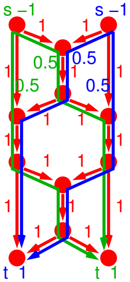
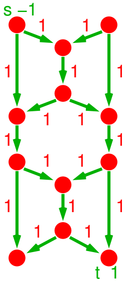


fractional works

implementation $\longrightarrow$



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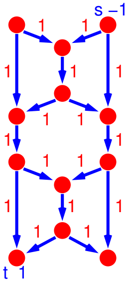

copy 2

$$\begin{array}{lcccccc}
\min & {c^{(1)}}^T x^{(1)} & + & {c^{(2)}}^T x^{(2)} & & \\
\text{s.t.} & A x^{(1)} & & & = & b^{(1)} \\
 & & & A x^{(2)} & = & b^{(2)} \\
 & I x^{(1)} & + & I x^{(2)} & \leq & w \\
 & x^{(1)} \geq 0, & & x^{(2)} \geq 0. & &
\end{array}$$

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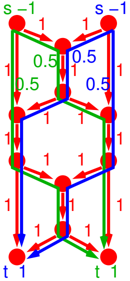

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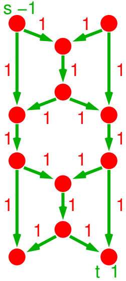

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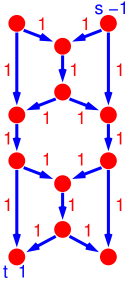

copy 2

$$\begin{bmatrix} A & 0 \\ 0 & A \\ I & I \end{bmatrix}$$

i.g. not tot.unimod.!

$$\begin{array}{llcllcl}
\min & {c^{(1)}}^T x^{(1)} & + & {c^{(2)}}^T x^{(2)} & & & \\
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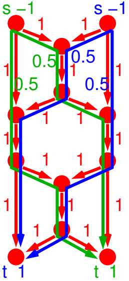

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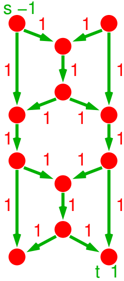

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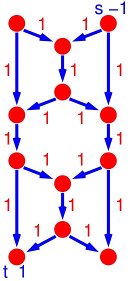

copy 2

fractionally solvable,
integral VERY difficult!

$$
\begin{array}{lccccc}
\min & {c^{(1)}}^T x^{(1)} & + & {c^{(2)}}^T x^{(2)} & & \\
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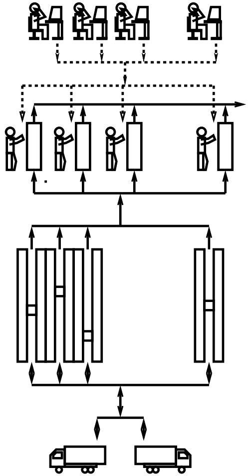
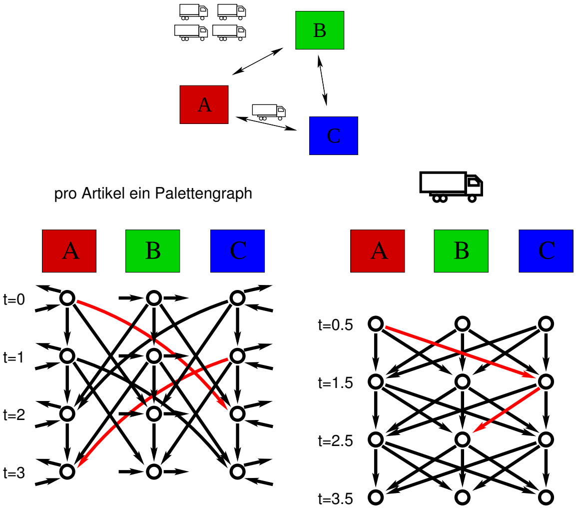
Example: Logistics

cover demand by shifting pallets by trucks between warehouses





Further Application Areas

$\left.\begin{array}{l}\bullet\ \text{fractional: capacity planning}\\ \bullet\ \text{integral: time discretized routing and scheduling}\end{array}\right\}$ for

- fractional: capacity planning
- integral: time discretized routing and scheduling
  - street traffic
  - trains
  - internet
  - logistics (bottleneck analysis/steering)
  - production (mashine loads/-assignment)

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---

network-design:
installed capacities should satisfy as many demands as possible also in case of failures [“robust” variants are extremly difficult!]

Multi-commodity flow is often used as basic model that is combined with further constraints.

Contents

1.5 Integer Optimization (Integer Programming)

mainly: linear programs with exclusively integer variables
(otherw. mixed integer programming)

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax \le b \\ & x \in \mathbb{Z}^n \end{array}$$

Typically contains many binary variables ($\{0,1\}$) for yes/no decisions

---

Difficulty: i.g. not solvable “efficiently”, complexity class *NP*
$\Rightarrow$ exact solutions rely heavily on enumeration (systematic exploration)

Exact solutions by combining the following techniques:

- (upper) bounds by linear/convex relaxation
  improved by cutting plane approaches
- feasible solutions (lower bounds) by rounding- and
  local search heuristics
- enumerate by branch&bound, branch&cut

Combinatorial Optimization

Mathematically: Given a finite ground set Ω, a set of feasible subsets $\mathcal{F} \subseteq 2^{\Omega}$[power set, set of all subsets] and a goal function $c : \mathcal{F} \to \mathbb{Q}$, determine

$$\max\{c(F) : F \in \mathcal{F}\} \quad \text{or} \quad F \in \text{Argmax}\{c(F) : F \in \mathcal{F}\}$$

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Ex1: maximum cardinality matching for $G = (V, E)$:
$\Omega = E$, $\mathcal{F} = \{M \subseteq E : M \text{ matching in } G\}$, $c = \mathbf{1}$

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formulation as a binary program:
Notation: incidence-/characteristic vector $\chi_{\Omega}(F) \in \{0, 1\}^{\Omega}$ for $F \subseteq \Omega$
[in short $\chi(F)$, satisfies $[\chi(F)]_e = 1 \Leftrightarrow e \in F$]
A linear program $\max\{c^T x : Ax \leq b, x \in [0, 1]^{\Omega}\}$ is a formulation of the combinatorial optimization problem, if

$$\{x \in \{0, 1\}^{\Omega} : Ax \leq b\} = \{\chi(F) : F \in \mathcal{F}\}.$$

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An instance I of a problem is a concrete assignment of values to the problem data; its size $|I|$ is the encoding length, i.e. the number of symbols in a description string according to a reasonalbe encoding scheme.

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Ex1: lineare optimization is in P, BUT Simplex is not polynomial.
Ex2: maximum cardinality matching in general graphs is in P.
Ex3: minimum vertex cover in bipartite graphs is in P.

Decision Problems and the Class *NP*

In a decision problem each instance is a question, that must be answered by either "yes" or "no".

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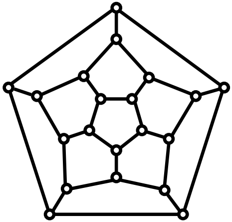

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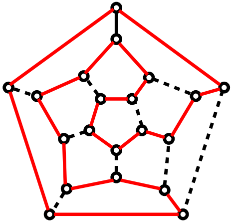

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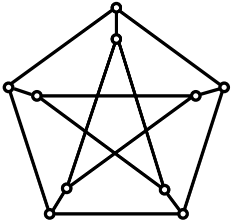

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NP-complete Problems

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There are voluminous collections of NP-complete problems; examples:

- integer optimization (in its decision version)
- integer multi-commodity flow
- Hamiltonian Cycle
- Minimum Vertex Cover on general graphs
- Knapsack (for big numbers)

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If there is an efficient algorithm for one NP-complete problem, all are solvable efficiently. For years the assumption is: $P \neq NP$.

If one wants to solve all instances of a problem, partial enumeration seems to be unavoidable.

A problem is NP-hard, if it would allow to solve an NP-complete problem.

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1.6 Branch-and-Bound

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$$\max\ c^T x \quad \text{s.t.} \quad a^T x \leq b,\ x \in \{0,1\}^n$$

upper bound: max $c^T x$ s.t. $a^T x \leq b,\ x \in [0,1]^n$ [LP-relaxation]
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algorithmic scheme (for maximization problems):
M ... set of open problems, initially $M = \{\text{orig. problem}\}$
$\underline{f}$ ... value of best known solution, initially $\underline{f} = -\infty$
1. if $M = \emptyset$ STOP, else choose $P \in M$, $M \leftarrow M \setminus \{P\}$
2. compute upper bound $\bar{f}(P)$.
3. if $\bar{f}(P) < \underline{f}$ (P contains no OS), goto 1.
4. compute feasible solutions $\hat{f}(P)$ for P (lower bound).
5. if $\hat{f}(P) > \underline{f}$ (new best solution), put $\underline{f} \leftarrow \hat{f}(P)$
6. if $\bar{f}(P) = \hat{\underline{f}}(P)$ (no better solution in P), goto 1.
7. split P into "smaller" subproblem P_i, $M \leftarrow M \cup \{P_1, \ldots, P_k\}$
8. goto 1.

Example: $\{0,1\}$-Knapsack problem

Item	A	B	C	D	E	F	capacity
weight (a)	9	7	6	4	4	3	14
profit (c)	18	6	18	7	6	5	

sorted profit/weight: $C > A > D > F > E > B$.

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P_5: $x_A = x_E = 0$
UB: $C + D + F + \frac{1}{7}B = 30\frac{6}{7}$
LB $< 31 \Rightarrow$ no OS $\square$

The branch&bound tree will get huge whenever many solutions are almost optimal.

For successful branch&bound we need to answer

How can we obtain good upper and lower bounds?

Contents

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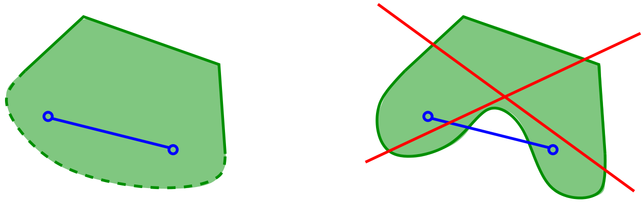
A set $C \subseteq \mathbb{R}^n$ is convex, if for all $x, y \in C$ the straight line segment $[x, y] := \{\alpha x + (1-\alpha)y : \alpha \in [0,1]\}$ lies in C.



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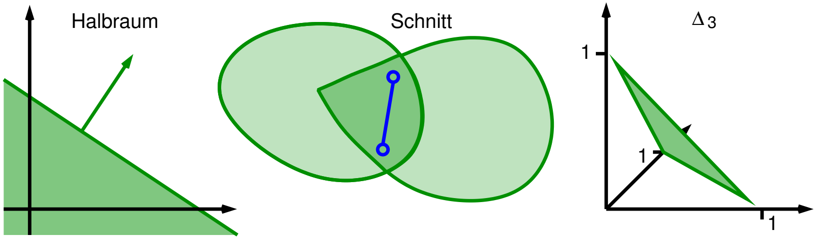

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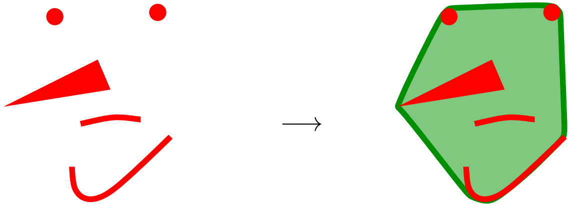

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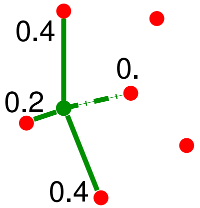

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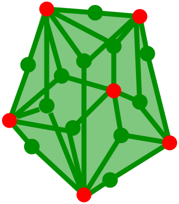
$\operatorname{conv} S$ is the set of all convex combinations of finitely many points in S, $\operatorname{conv} S = \{\sum_{i=1}^k \alpha_i x^{(i)} : x^{(i)} \in S, i = 1, \dots, k \in \mathbb{N}, \alpha \in \Delta_k\}$.



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Theorem

If $A \in \mathbb{Q}^{m \times n}$ and $b \in \mathbb{Q}^m$, the integer hull P_I of polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is itself a polyhedron.

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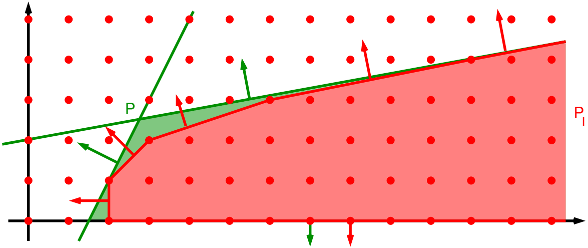

[exception e.g. for A tot. unimod., $b \in \mathbb{Z}^n$, then $P = P_I$]

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If the integer hull can be given explicitly, the integer optimization problem can be solved by the simplex method:

Theorem

Suppose the integer hull of $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is given by $P_I = \{x \in \mathbb{R}^n : A_I x \leq b_I\}$, then:

$$\begin{aligned} \sup\{c^T x : A_I x \leq b_I, x \in \mathbb{R}^n\} &= \sup\{c^T x : Ax \leq b, x \in \mathbb{Z}^n\}, \\ \operatorname{Argmin}\{c^T x : A_I x \leq b_I, x \in \mathbb{R}^n\} &= \operatorname{conv} \operatorname{Argmin}\{c^T x : Ax \leq b, x \in \mathbb{Z}^n\}. \end{aligned}$$

Convex Functions

A function $f : \mathbb{R}^n \to \bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ is convex if $f(\alpha x + (1-\alpha)y) \le \alpha f(x) + (1-\alpha)f(y)$ for $x, y \in \mathbb{R}^n$ and $\alpha \in [0,1]$.

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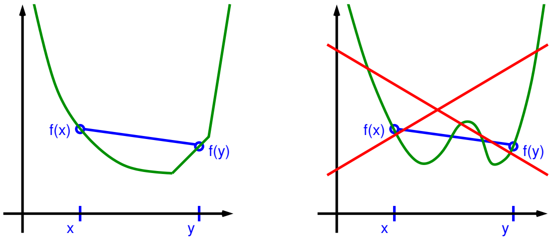

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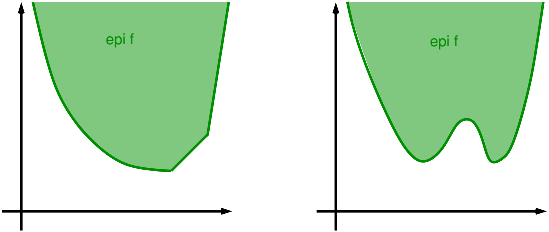

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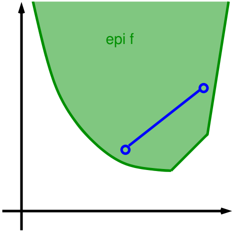

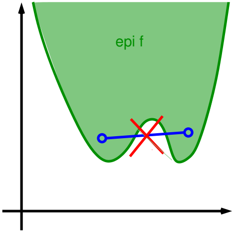

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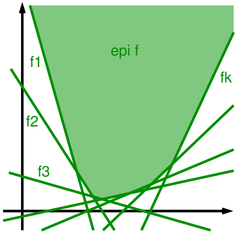

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Theorem

Each local minimum of a convex function is also a global minimum, and for strictly convex functions it is unique (if it exists).

For convex functions there exist rather good optimization methods.

Convex Functions

A function $f : \mathbb{R}^n \to \bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ is convex if $f(\alpha x + (1-\alpha)y) \le \alpha f(x) + (1-\alpha) f(y)$ for $x, y \in \mathbb{R}^n$ and $\alpha \in [0,1]$.

f is strictly convex, if $f(\alpha x + (1-\alpha)y) < \alpha f(x) + (1-\alpha) f(y)$ for $x, y \in \mathbb{R}^n$, $x \ne y$ and $\alpha \in (0,1)$.

The epigraph of a function $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ is the set

$$\operatorname{epi} f := \left\{ \begin{pmatrix} x \\ r \end{pmatrix} : x \in \mathbb{R}^n, r \ge f(x) \right\}$$ [the points "above" $f(x)$]

Theorem

A function is convex if and only if its epigraph is a convex set.

Ex.: for convex $f_i : \mathbb{R}^n \to \bar{\mathbb{R}}$ also $f(x) := \sup_i f_i(x)$, $x \in \mathbb{R}^n$ is convex.

Theorem

Each local minimum of a convex function is also a global minimum, and for strictly convex functions it is unique (if it exists).

For convex functions there exist rather good optimization methods.

A function f is concave, if $-f$ is convex.
(Each local maximum of a concave function is a global one.)

Contents

1.8 Relaxation

Concept applicable to arbitrary optimization problems (here maximize):

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Given two optimization problems with $\mathcal{X}, \mathcal{W} \subseteq \mathbb{R}^n$ *and* $f, f' : \mathbb{R}^n \to \mathbb{R}$

$$(OP) \quad \max\ f(x) \text{ s.t. } x \in \mathcal{X} \qquad \textit{and} \qquad (RP) \quad \max\ f'(x) \text{ s.t. } x \in \mathcal{W},$$

(RP) *is a* ***relaxation*** *of* (OP) *if*

1. $\mathcal{X} \subseteq \mathcal{W}$,
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Let (RP) be a relaxation of (OP).

Observation

1. $v(RP) \geq v(OP)$. *[(RP) yields an upper bound]*
2. *If* (RP) *is infeasible, then so is* (OP),
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Search for a suitable "small" $\mathcal{W} \supseteq \mathcal{X}$ and $f' \geq f$ so that (RP) is efficiently solvable.

A relaxation (RP) of (OP) is called exact if $v(OP) = v(RP)$.

Convex Relaxation

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Ex.: for a combinatorial max problem with finite ground set Ω, feasible solutions $\mathcal{F} \subseteq 2^{\Omega}$ and linear cost function $c \in \mathbb{R}^{\Omega}$

$$\max c^T x \text{ s.t. } x \in \operatorname{conv}\{\chi_{\Omega}(F) : F \in \mathcal{F}\}$$

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In global optimization, nonlinear functions are approximated from below by convex functions on domain subdivisions.

Ex.: Consider $(OP)\ \min f(x) := \frac{1}{2}x^T Q x + q^T x \text{ s.t. } x \in [0,1]^n$
with f not convex, i.e., $\lambda_{\min}(Q) < 0$. [$\lambda_{\min}$... minimal eigenvalue]
$Q - \lambda_{\min}(Q)I$ is positive semidefinite and by $x_i^2 \le x_i$ on $[0,1]^n$ there holds
$f'(x) := \frac{1}{2}x^T(Q - \lambda_{\min}(Q)I)x + (q + \lambda_{\min}(Q)\mathbf{1})^T x \ \le\ f(x) \quad \forall x \in [0,1]^n$.
Thus, $(RP)\ \min\ f'(x) \text{ s.t. } x \in [0,1]^n$ is a convex relaxation of (OP).

LP-Relaxation for Integer Programs

For an integer program $\max\ c^T x$ s.t. $Ax \le b,\ x \in \mathbb{Z}^n$ dropping the integrality constraints yields the LP-relaxation

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It is a relaxation, because

$$\mathcal{X} := \{x \in \mathbb{Z}^n : Ax \le b\} \ \subseteq\ \{x \in \mathbb{R}^n : Ax \le b\} =: \mathcal{W}.$$

[basis of all standard solvers for mixed integer programs]

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Ex.: knapsack problem: $n = 2$, weights $a = (6, 8)^T$, capacity $b = 10$,
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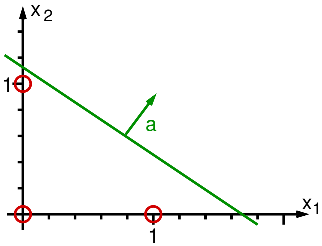

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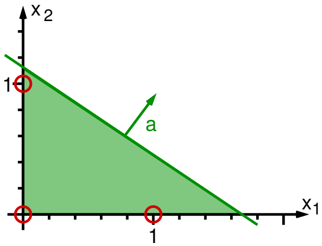

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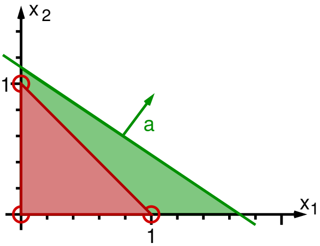

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best relaxation: the convex hull

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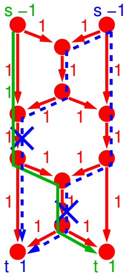
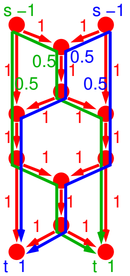
[basis of all standard solvers for mixed integer programs]

Ex: integer multi-commodity flow → fractional multi-commodity flow



infeasible, $\mathcal{X} = \emptyset$



frac. feasible, $\mathcal{W} \neq \emptyset$

too large, would need convex hull

Lagrangian Relaxation

[Appl. to constrained optim. in general, here only for ineq.-constraints]

Inconvenient constraints are lifted into the cost function via a Lagrange multiplier that penalizes violations ($g : \mathbb{R}^n \to \mathbb{R}^k$):

$$(OP) \quad \begin{array}{ll} \max & f(x) \\ \text{s.t.} & g(x) \leq 0 \quad \mid \cdot \lambda \geq 0 \\ & x \in \Omega \end{array} \quad \rightarrow \quad (RP_\lambda) \quad \begin{array}{ll} \max & c^T x - \lambda^T g(x) \\ \text{s.t.} & x \in \Omega \end{array}$$

is a relaxation: $\mathcal{X} := \{x \in \Omega : g(x) \leq 0\} \subseteq \{x \in \Omega\} =: \mathcal{W}$ and for $x \in \mathcal{X}$, $\lambda \geq 0$ there holds $f(x) \leq f(x) - \lambda^T g(x) =: f'(x)$ by $g(x) \leq 0$.

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Define the dual function $\varphi(\lambda) := \sup_{x \in \Omega} \left[f(x) - g(x)^T \lambda \right] = v(RP_\lambda)$

[for each fixed x linear in λ]

- for each $\lambda \ge 0$ there holds $\varphi(\lambda) \ge v(OP)$ [upper bound]
- $\varphi(\lambda)$ easy to compute if (RP_λ) is “easy” to compute
- φ is convex, because sup of linear functions in λ
- best bound is $\inf\{\varphi(\lambda) : \lambda \ge 0\}$ [convex problem!]

well suited for convex optimization methods!

Example: Integer Multi-Commodity Flow

Let A be the node-arc incidence matrix to $D=(V,E)$, 2 goods, relax the coupling capacity constraints by $\lambda \geq 0$:

$$\begin{array}{llcccl}
 & \min\; {c^{(1)}}^T x^{(1)} & + & {c^{(2)}}^T x^{(2)} & & \\
 & \text{s.t.}\quad A x^{(1)} & & & = & b^{(1)} \\
 & & & A x^{(2)} & = & b^{(2)} \\
\lambda \cdot | & x^{(1)} & + & x^{(2)} & \leq & w \\
 & x^{(1)} \leq w, & & x^{(2)} \leq w & & \\
 & x^{(1)} \in \mathbb{Z}^E_+, & & x^{(2)} \in \mathbb{Z}^E_+. & &
\end{array}
\quad \rightarrow \quad
\begin{array}{lllr}
\min & (c^{(1)}+\lambda)^T x^{(1)} + & (c^{(2)}+\lambda)^T x^{(2)} & -\lambda^T w \\
\text{s.t.} & A x^{(1)} & & = b^{(1)} \\
 & & A x^{(2)} & = b^{(2)} \\
 & x^{(1)} \leq w, & x^{(2)} \leq w, & \\
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\end{array}$$

The relaxation consists of two independent min-cost-flow problems

$$(RP^{(i)}_\lambda) \quad \min\,(c^{(i)}+\lambda)^T x^{(i)} \quad \text{s.t.} \quad A x^{(i)} = b^{(i)},\; w \geq x^{(i)} \in \mathbb{Z}^E_+ \quad i \in \{1,2\}$$

These can be solved integrally and efficiently!

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These can be solved integrally and efficiently!

If Lagrangian relaxation splits the problem into independent subproblems, this is sometimes called Lagrangian decomposition. Frequently this allows to solve much bigger problems efficiently.

Does this also yield better bounds?

Comparison of Lagrange- and LP-Relaxation

Given finite $\Omega \subset \mathbb{Z}^n$ and $D \in \mathbb{Q}^{k\times n}$, $d \in \mathbb{Q}^k$,

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In the ex.: $\Omega = \Omega^{(1)} \times \Omega^{(2)}$ with $\Omega^{(i)} = \{x \in \mathbb{Z}_+^E : Ax = b^{(i)}, x \le w\}$, $i \in \{1, 2\}$.

Theorem

$$\inf_{\lambda \ge 0} v(RP_\lambda) = \sup\{c^T x : Dx \le d, x \in \operatorname{conv}\Omega\}.$$

If $\operatorname{conv}\Omega$ is identical to the feasible set of the LP-relaxation of Ω, the values of the best Lagrange relaxation and the LP-relaxation match!

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In general: Let $\{x \in \mathbb{Z}^n : Ax \le b\} = \Omega$ be a formulation of Ω. Only if $\{x \in \mathbb{R}^n : Ax \le b\} \ne \operatorname{conv}\Omega$, Lagrange relaxation may yield a better value Wert than the LP-relaxation.

Contents

1.9 Application: Traveling Salesman Problem (TSP)

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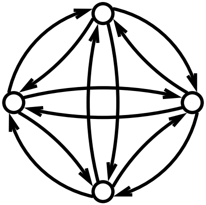
NP-complete problem
Number of tours?



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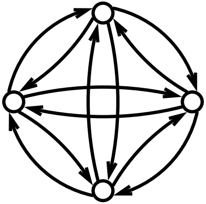
→combinatorial explosion



$n =$ 3:	2
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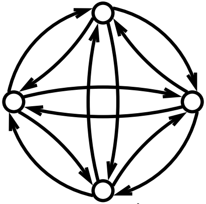

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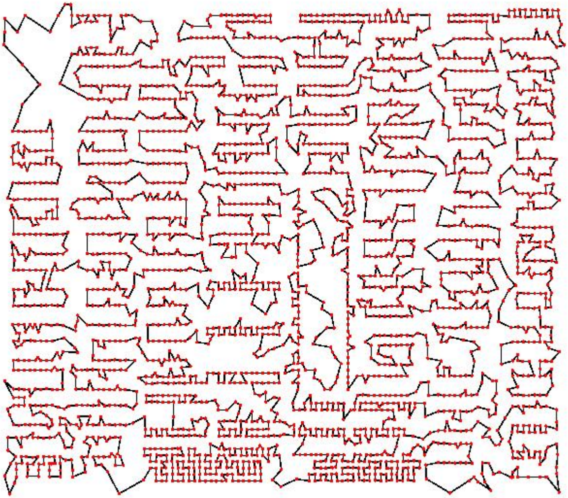
Typical problem for:
- delivery services (parcels, pharmacy-transports)
- taxi services, breakdown services
- scheduling with setup costs [e.g. coloring cars]
- steering robots (mostly nonlinear)

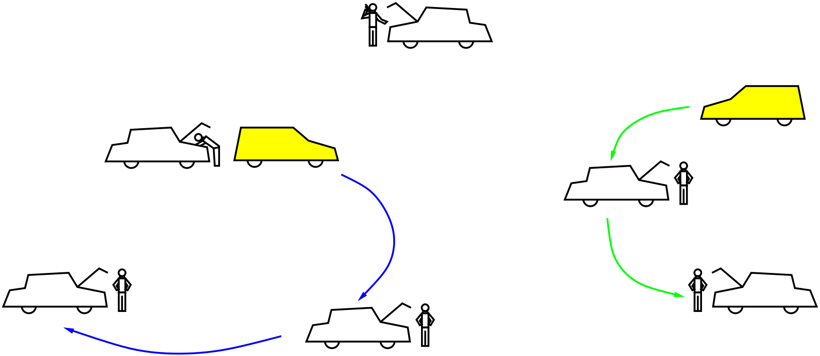
[frequently with several “vehicles” and time windows]

Drilling holes into main boards



[http://www.math.princeton.edu/tsp]

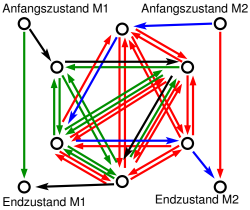
With online aspects: breakdown services

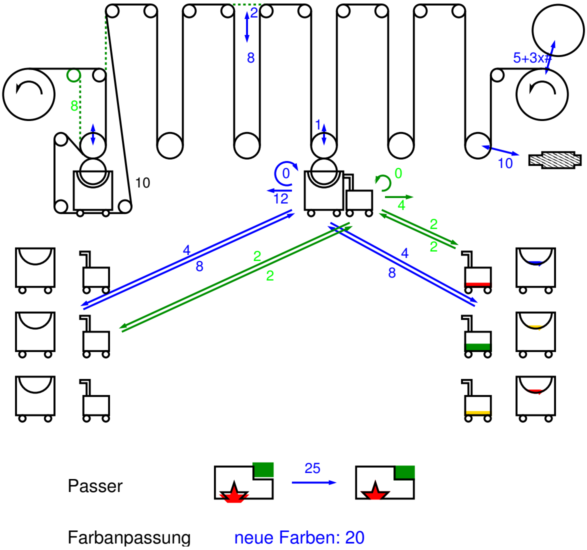


Find an assignment of cars and a sequence for each repair man, so that promised waiting periods are not exceeded.

Scheduling for long setup times

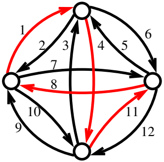
Two rotational printing machines
for printing wrapping papier



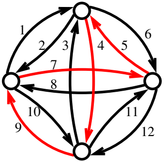


Integer Programming Model

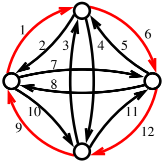
Abstract formulation uses convex hull of incidence vectors:



$R_1 = \{1, 4, 8, 11\}$



$R_2 = \{4, 5, 7, 9\}$



$R_3 = \{1, 6, 9, 12\}$

$$\chi(R_1) = \begin{bmatrix} 1\\0\\0\\1\\0\\0\\0\\1\\0\\0\\1\\0 \end{bmatrix} \qquad \chi(R_2) = \begin{bmatrix} 0\\0\\0\\1\\1\\0\\1\\0\\1\\0\\0\\0 \end{bmatrix} \qquad \chi(R_3) = \begin{bmatrix} 1\\0\\0\\0\\0\\1\\0\\0\\1\\0\\0\\1 \end{bmatrix}$$

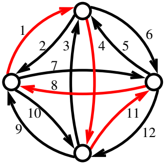
For arc lengths $c \in \mathbb{R}^E$ the length of a tour R is $\sum_{e \in R} c_e = c^T \chi(R)$.

$$(TSP) \quad \min\ c^T x \text{ s.t. } x \in \operatorname{conv}\{\chi(R) : R \text{ tour in } D = (V, E)\} =: P_{TSP}$$

Would be exact, but no linear description $A_I x \leq b_I$ of P_{TSP} is known!

Integer Formulation of (TSP)

Goal: wrap P_{TSP} by a bigger polytope $P = \{x \in \mathbb{R}^n : Ax \leq b\} \supseteq P_{TSP}$ as tightly as possible so that at least $P \cap \mathbb{Z}^E = \{\chi(R) : R \text{ tour}\}$.



An equation/inequality is feasible for P_{TSP}, if it holds for all $x \in \{\chi(R) \colon R \text{ tour}\}$.

Suggestions?

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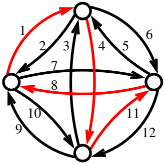

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0-1 cube: $0 \le x \le 1$ is feasible

degree constraints:
exactly one arc exits and enters each node,

$$\text{for } v \in V: \quad \sum_{(v,u)\in E} x_{(v,u)} = 1, \quad \sum_{(u,v)\in E} x_{(u,v)} = 1$$

(exactly one 1 per row/column $\leftrightarrow$ assignment problem)

Is this a formulation? $P \cap \mathbb{Z}^E = \{\chi(R) : R \text{ tour}\}$?

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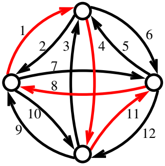

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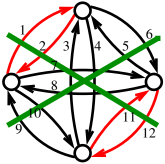


Is this a formulation? $P \cap \mathbb{Z}^E = \{\chi(R) : R \text{ tour}\}$?

Subtour elimination constraints:
At least one arc must exit each proper subset of nodes,

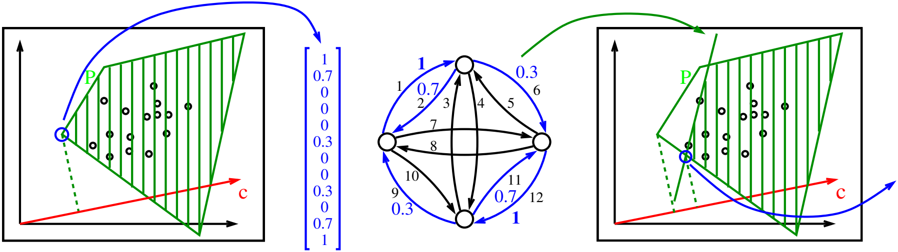
$$\text{for } S \subset V, 2 \le |S| \le n-2: \quad \sum_{e\in\delta^+(S)} x_e \ge 1$$

This is now a formulation, but it needs roughly 2^n inequalities!

Solving the TSP LP-Relaxation

Requires a cutting plane approach:
The first relaxation is the assignment problem (box+degree constr.)
Its solutions is integral and consists of distinct cycles in general.

From now on the bound is improved iteratively by subtour elim. constr.



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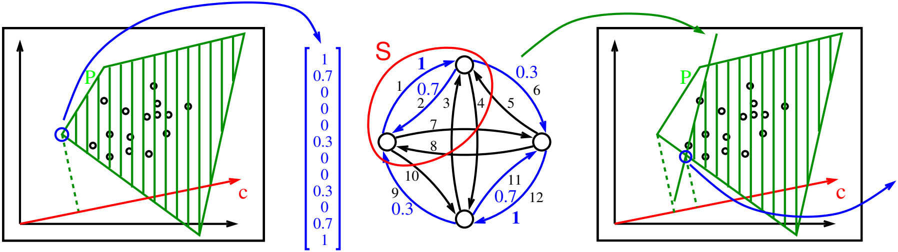

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$\Leftrightarrow$ Find, in network $D = (V, E)$ with capacities x, a cut $x(\delta^+(S)) < 1$.
$\rightarrow$ Maximum s-t-flow/minimum s-t-cut for $s, t \in V$, solvable exactly!

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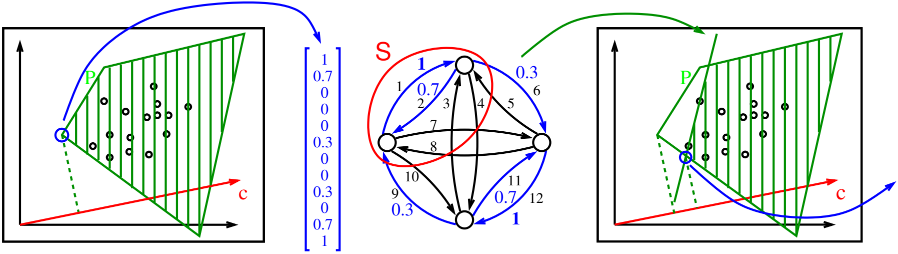

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degree+subtour yield high quality bounds, but are still far from P_{TSP}!
Bound can be improved by further ineqs (comb-, etc.),
but the solution of the relaxation almost never becomes integral!

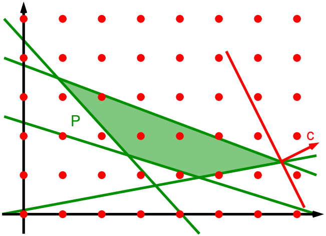
General Cutting Planes

There exist several problem independent general cutting planes, that are used in state-of-the-art solvers for integer programming:

- Gomory-cuts [rounding down coefficients by $\lfloor\cdot\rfloor$]

  if $a^T x \leq \beta$ is feasible for $x \in P \cap \mathbb{Z}^n_+$
  then by $\lfloor a \rfloor^T x \leq a^T x$
  also $\lfloor a \rfloor^T x \leq \lfloor \beta \rfloor$ feasible.

  For non integral OS violated inequs. of this type can be constructed.



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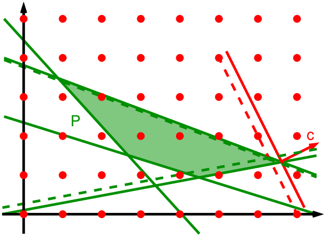

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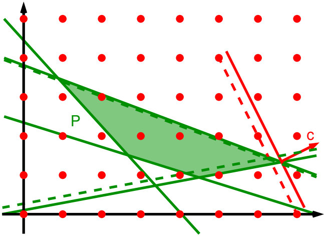

- lift-and-project cuts,
- clique inequalities,

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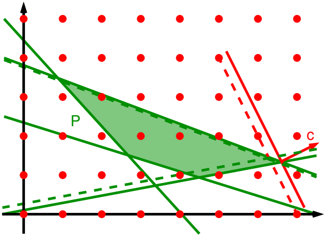

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LP-relaxation with cutting planes gives rise to good bounds (upper for maximization, lower for minimization problems), the solutions of the relaxation are (almost) never integral, but are often close to integer solutions of good quality.

Contents

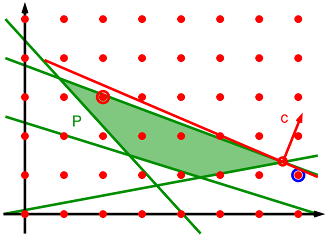
1.10 Finding "Good" Solutions, Heuristics

[Heuristic has greek origin find/invent]

For "small" $x \in \mathbb{Z}^n$ standard rounding of LP solutions typically yields infeasible or bad solutions (even if the bound is good).

It may happen that no feasible point is in the neighborhood of the LP solution!

State-of-the-art solvers employ sophisticated general purpose rounding heuristics (e.g., feasibility pump).



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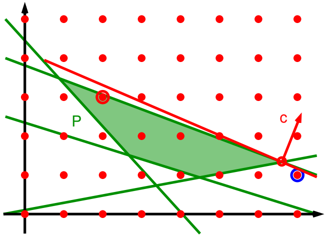

In general: Integer problems are NP-hard, have many “local optima” (no close better solutions) and likely finding the optimum requires (partial) enumeration. It may even be difficult to find any feasible solution!

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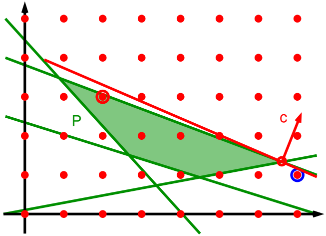

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Rough algorithmic scheme:

- generate a (feasible?) starting solution (often based on LP-sol.)
- iteratively improve the solution by some local search method (locally exact, simulated annealing, tabu search, genetic algorithms, etc.)

Starting Solution based on LP-Relaxation

Typical approaches:

- Often exactly one of several $\{0,1\}$-variables has to be selected:

  LP-relaxation: $\sum_{i \in N} x_i = 1,\ x_i \in [0,1]$

  Interpret value of x_i as the probability that x_i has to be set to 1 and generate several such solutions randomly, select the best.

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For some basic problems there exist rounding methods that generate feasible solutions with quality guarantee from LP solutions (approximation algorithms), these are a valuable source of good ideas for designing new rounding methods.

Improvement Methods: Principles

common basic elements:

- **declare a search neighborhood:** with respect to the current solution it describes which solutions "close by" will or may be investigated (e.g. all obtainable by certain exchange operations or by freeing certain variables with local post optimization etc.)
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  [cmp. merit- and filter-approach in nonlin. opt.]
- **fix an acceptance-scheme:** serves to decide which of the new solutions will be used to continue the search; worse solutions may be accepted sometimes in order to allow leaving local optima.

Locally Exact Methods/Local Enumerartion

Define $\mathcal{N}(\cdot)$ so that $\quad (P_{\hat{x}}) \quad \min f(x)$ s.t. $x \in \mathcal{N}(\hat{x})$
is solvable exactly by a polynomial time algorithm or by complete enumeration for each $\hat{x}$.

0. determine a starting solution $\hat{x}$
1. solve $(P_{\hat{x}}) \to \bar{x}$
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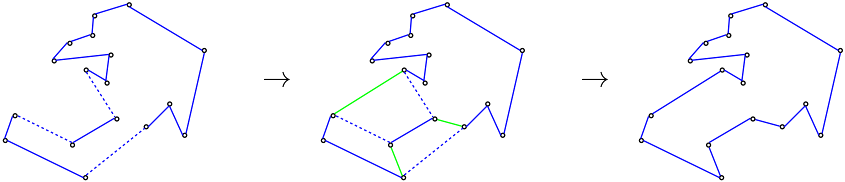
Ex.: 3-opt for TSP: remove 3 edges and concat parts optimally



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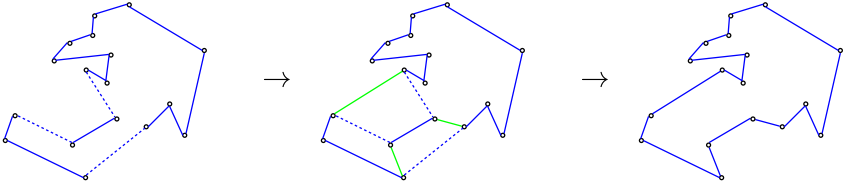

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the art: find a powerful and large neighborhood for which $(P_{\hat{x}})$ is still polynomially solvable.

The number of iterations may be exponential none the less!

Simulated Annealing (simulates a slow cooling process)

Select, in step k, randomly some $\bar{x}$ from $\mathcal{N}(\hat{x})$. Accept it if $f(\bar{x})$ is better than $f(\hat{x})$, otherwise accept it only with probability

$$\exp\left(\frac{-|f(\hat{x}) - f(\bar{x})|}{T_k}\right) \qquad (0 < T_k \to 0 \text{ für } k \to \infty).$$

0. given a starting $\hat{x}$, fix sequence $\{T_k > 0\}_{k\in\mathbb{N}} \searrow 0$, put $k = 0$.
1. choose randomly (uniformly) $\bar{x} \in \mathcal{N}(\hat{x})$, put $k \leftarrow k + 1$
2. if $f(\bar{x})$ is better than $f(\hat{x})$, put $\hat{x} \leftarrow \bar{x}$ and goto 1
3. draw a uniform random number $\zeta \in [0, 1]$;
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Choose $\mathcal{N}(\cdot)$ so that each x is reachable via intermediate steps.

If the (temperature-/cooling-)sequence T_k goes to zero slowly enough, each x is visited with positive probability over time (complete enumeration), therefore also the optimum (but after how long?)

Tabu-Search

Idea: try to generate highly diverse solutions

Describe $\mathcal{N}(\cdot)$ by exchange rules $r \in \mathcal{R}$ and save used rules in a tabu list $\mathcal{L}$. For any new $\bar{x}$ at least one rule $r \in \mathcal{R} \setminus \mathcal{L}$ should be used or its value must improve.

0. determine a starting $\hat{x}$, put $\mathcal{L} = \emptyset$.
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By using rules $\mathcal{R}$ each solution should be reachable.
No general theoretical insights or quality guarantees seem to exist.

Genetic Algorithms

Idea: Let evolution work for you (and wait in the meantime).

From some population generate the next population by selection (choose next parents), recombination (exchange parts of solutions) and mutation (modify some elements randomly).

0. Choose $k \in \mathbb{N}$ and determine a starting population $\mathcal{P}$, $|\mathcal{P}| \geq 2k$.
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Genetic Algorithms

Idea: Let evolution work for you (and wait in the meantime).

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- many experiments use populations of size 1 (!!!)
- theory indicates that simulated annealing is better in locating optima

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- “Convergence” of a method does not imply any quality guarantee. Without some related relaxation the distance to an optimal solution is entirely open (sometimes rather extreme).

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x needs to be integral on some subset of indices $G \subseteq \{1, \ldots, n\}$.

$$\max \; c^T x \text{ s.t. } Ax \geq b, \; x \in \mathbb{R}^n, x_G \in \mathbb{Z}^G$$

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$\min \sum_{k \in K, m \in M} c_{km} x_{km} + \sum_{m \in M} c_m x_m = c^T x$

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- M too big $\rightarrow$ ineq. in LP-relaxation too weak $\rightarrow$ bad bound
- reasonable, if violation gap of ineq. is well controlled (see the example on facility location)
- useful in branch&bound if the decision is used for branching

Modelling Logical Constraints

For $x_i \in \{0,1\}$, $x_i = 1$ often represents "expression i is true".
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$x_1 \Leftrightarrow x_2$	$x_1 = x_2$

Remark: Together with $0 \leq x_i \leq 1$ these constraints describe

$$\operatorname{conv}\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \{0,1\}^3 : \text{die } x_i \text{ satisfy the logical expression} \right\}.$$

Using this technique models of further expressions can be derived.
Exercise: $x_3 = (x_1 \text{ xor } x_2)$

General Cutting Planes for MIP

Like in integer programming, “conv” is the best linear relaxation,

$$P_G := \mathrm{conv}\{x \in \mathbb{R}^n : Ax \leq b, x_G \in \mathbb{Z}^G\}.$$

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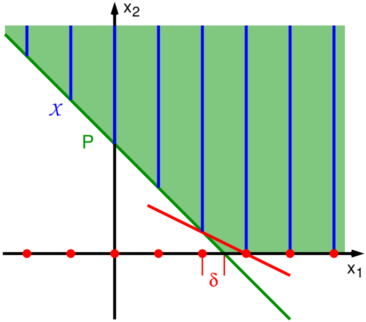
Ex.: Mixed Integer Rounding inequality (MIR)
simplest form: $\mathcal{X} = \{(x_1, x_2) \in \mathbb{Z} \times \mathbb{R}_+ : x_1 + x_2 \ge \beta\}$, $\beta \in \mathbb{R}$



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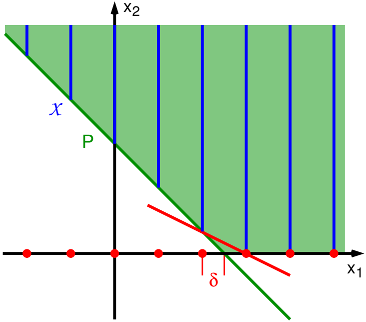

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In state-of-the-art packages many further types are included (flow cover, cliques, etc.)

Branch-and-Cut Frameworks

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There exist packages that provide the entire framework and allow to add further problem specific cutting planes and heuristics.

e.g. SCIP, Cplex, Gurobi, Abacus . . .