Technische Universität Chemnitz
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Chemnitz, 26. November 2019

Graph Theory

Exercise 3

Terms

- Laplacian, (Laplacian) spectrum of a graph,
- path system, (internally) disjoint paths,
- $\kappa_G(v, w)$, $\kappa_G(A, B)$, (vertex-)connectivity $\kappa(G)$, edge-connectivity $\kappa'(G)$,
- (vertex-)separator, edge-separator, k-connected graph, Menger's theorem,
- n-partite graph, bipartite graph, (perfect, maximal, maximum) matching,
- Mader's theorem, Hall's theorem, k-factor, Tutt's 1-factor theorem

Tasks

1. Let G be a graph with minimal degree δ and odd girth $g(G) = 2k + 1$. Show that G has at least
$$1 + \delta + \delta(\delta - 1) + \cdots + \delta(\delta - 1)^{k-1}$$
vertices.

2. Show that a non-empty graph G is connected, if and only if the eigenvalue 0 of the Lapalcian $L(G)$ has multiplicity one.

3. Let G be a graph with two distinct vertices $v, w \in V(G)$. Let G' be the graph which is obtained from G by subdividing an edge of G. Show that $\kappa_G(v, w) = \kappa_{G'}(v, w)$.

4. Prove Whitney's theorem: Let G be a non-complete graph. The following are equivalent:

   (i) each separator of G has size at least k,

   (ii) for any two vertices $v, w \in V(G)$ there are k internally disjoint v-w-paths.

5. Prove that deleting or contracting an edge can change the connectivity by at most one.

6. Prove that every cubic graph without a bridge has a 2-factor.