Fermionic Dyson Expansions and Perturbation Theory in Stochastic Analysis

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This talk is on joint work with Jonas Miehe.

Let X be a Hilbert space and let $H \geqslant 0$ be a self-adjoint operator in X. Moreover, let $P_1, \dots, P_n$ be densely defined closed operators in X.

For $t > 0$ we are interested in bounded operators in X of the form

$$\Phi_t^H(P_1, \dots, P_n)$$
$$:= \int_{t\sigma_n} e^{-s_1 H} P_1 e^{-(s_2 - s_1)H} P_2 \cdots e^{-(s_n - s_{n-1})H} P_n e^{-(t-s_n)H} \, \mathrm{d}s_1 \cdots \mathrm{d}s_n,$$

with the t-scaled n-simplex

$$t\sigma_n := \{0 \leqslant s_1 \leqslant \cdots \leqslant s_n \leqslant t\} \subset \mathbb{R}^n.$$

If $P_1 = \cdots = P_n = P$, then

$$e^{-t(H+P)} = \sum_{l=0}^{\infty} (-1)^l \Phi_t^H(\underbrace{P, \dots, P}_{l \text{ times}}).$$

is the Dyson perturbation expansion (up to questions of convergence: Miyadira, Voigt).

Example:

- In noncommutative geometry, $\Phi_t^H(P_1,\dots,P_n)$ appears naturally, with $H = D^2$ an abstract Dirac operator on a $\mathbb{Z}_2$-graded Hilbert space X carrying a representation of a Banach algebra $\mathscr{A}$. In this case, $P_j = [D, a_j]$, where $a_j \in \mathscr{A}$, and $\Phi_t^{D^2}([D,a_1],\dots,[D,a_n])$ relates the index of D to the K_0-theory of $\mathscr{A}$ **(Connes/Getzler in 1990s)**.
- Let M^{2m} be a closed Riemannian spin manifold with D its Dirac operator. One needs $\Phi_t^{D^2}(P_1,\dots,P_n)$ with P_j a differential operator of order 1 **(unbounded!)**, to develop an integration theory for differential forms on loop space LM. An S^1-localization formula on LM as been conjectured by **Atiyah/Bismut/Witten in 1980s**, and proved by **G./Ludewig in 2022**.

Question: Is $\Phi_t^H(P_1, \ldots, P_n)$ in some sense equivalent to an analytic semigroup ($\rightsquigarrow$ probabilistic representation)?
We make the following **assumption on the P_j's:**

$$P_j(H+1)^{-a_j} \in \mathscr{L}(X) \quad \text{for some } a_j \in (0,1).$$

In particular, $P_j(H+1)^{-1}$ is bounded and $\mathrm{Dom}(H) \subset \mathrm{Dom}(P_j)$, and some straightforward estimates (spectral calculus) show that for $t > 0$ the expression

$$\Phi_t^H(P_1, \ldots, P_n)$$
$$:= \int_{t\sigma_n} e^{-s_1 H} P_1 e^{-(s_2-s_1)H} P_2 \cdots e^{-(s_n-s_{n-1})H} P_n e^{-(t-s_n)H} \,\mathrm{d}s_1 \cdots \mathrm{d}s_n$$

exists as a Bochner integral in $\mathscr{L}(X)$.

The exterior algebra Λ_n of $\mathbb{C}^n$ is multiplicatively generated by $\theta_1, \ldots, \theta_n$ subject to $\theta_j^2 = 0$, $\theta_i \theta_j + \theta_j \theta_i = 0$ for $i \neq j$.
Given a complex linear space V, every $\alpha \in \Lambda_n \otimes V$ can be uniquely written as

$$\alpha = \sum_{k=0}^{n} \sum_{1 \leqslant j_1 < \ldots < j_k \leqslant n} \theta_{j_1} \cdots \theta_{j_k} \otimes \alpha_{j_1,\ldots,j_k}, \quad \alpha_{j_1,\ldots,j_k} \in V.$$

The *Fermionic (or Berezin) integral* is the linear map

$$\oint_V : \Lambda_n \otimes V \to V, \quad \alpha \mapsto \alpha_{1,\ldots,n}.$$

By declaring the products $\theta_{j_1} \cdots \theta_{j_k}$ to be an ONB of Λ_n, we have an inner product on Λ_n, and we can define the Hilbert space

$$X^{(n)} := (\Lambda_n \otimes X)^n = \left\{ f = (f_1, \ldots, f_n)^T : f_j \in \Lambda_n \otimes X \quad \text{for all } j \right\}.$$

Consider the selfadjoint operator $H^{(n)} \geqslant 0$ in the Hilbert space $X^{(n)}$ given by

$$H^{(n)} := \begin{pmatrix} \mathrm{id}_{\Lambda_n} \otimes H & & \\ & \ddots & \\ & & \mathrm{id}_{\Lambda_n} \otimes H \end{pmatrix}$$

on its natural domain of definition (given by all $f \in X^{(n)}$ such that $f_j \in \Lambda_n \otimes \mathrm{Dom}(H)$ for all $j = 1, \ldots, n$).

A densely defined closed operator $P^{(n)}$ in $X^{(n)}$ is given by setting

$$P^{(n)} := \begin{pmatrix} & & & \widehat{\theta}_n \otimes P_n \\ \widehat{\theta}_{n-1} \otimes P_{n-1} & & & \\ & \ddots & & \\ & & \widehat{\theta}_1 \otimes P_1 & \end{pmatrix}.$$

Here, for every $\alpha \in \Lambda_n$, we have defined the multiplication operator

$$\widehat{\alpha} \colon \Lambda_n \longrightarrow \Lambda_n, \quad \beta \longmapsto \alpha\beta.$$

For $a := \max_j a_j \in (0,1)$ we have $P^{(n)}(H^{(n)} + 1)^{-a} \in \mathscr{L}(X)$, and therefore $P^{(n)}(H^{(n)} + 1)^{-1}$ is bounded. In particular, $\mathrm{Dom}(H^{(n)}) \subset \mathrm{Dom}(P^{(n)})$.

Since $X^{(n)} \simeq \Lambda_n \otimes X^n$, we have the canonical isomorphism

$$\mathscr{L}(X^{(n)}) \simeq \mathscr{L}(\Lambda_n) \otimes \mathscr{L}(X^n) \simeq \Lambda_n \otimes \Lambda_n^* \otimes \mathscr{L}(X^n),$$

which induces the Fermionic integral

$$\oint_{\Lambda_n^* \otimes \mathscr{L}(X^n)} : \mathscr{L}(X^{(n)}) \to \Lambda_n^* \otimes \mathscr{L}(X^n).$$

Moreover, we define the projection

$$\pi_n^X \colon \Lambda_n^* \otimes \mathscr{L}(X^n) \to \mathscr{L}(X), \quad \alpha \otimes (a_{ij})_{i,j=1,\dots,n} \mapsto a_{nn}.$$

Theorem (G. / Miehe)

The operator $H^{(n)} + P^{(n)}$ generates an analytic semigroup in X, to be denoted with $e^{-t(H^{(n)}+P^{(n)})}$, and for all $t > 0$ one has

$$e^{-t(H^{(n)}+P^{(n)})} = \sum_{l=0}^{n} (-1)^l \Phi_t^{H^{(n)}}\big(\underbrace{P^{(n)}, \dots, P^{(n)}}_{l \text{ times}}\big),$$

$$\pi_n^X \oint_{\Lambda_n^* \otimes \mathscr{L}(X^n)} e^{-t(H^{(n)}+P^{(n)})} = (-1)^n \Phi_t^H(P_1, \dots, P_n).$$

Idea of proof for $n = 1$ (no combinatorics involved):

$$H^{(1)} + P^{(1)} = \mathrm{id}_{\Lambda_1} \otimes H + \widehat{\theta}_1 \otimes P_1 \quad \text{in } X^{(1)} = \Lambda_1 \otimes X.$$

Dyson expansion of $e^{-t(H^{(1)}+P^{(1)})}$ gives

$$e^{-t(H^{(1)}+P^{(1)})} = \mathrm{id}_{\Lambda_1} \otimes e^{-tH} - \widehat{\theta}_1 \otimes \int_0^t \left(e^{-sH} P_1 e^{-(t-s)H}\right) \mathrm{d}s$$
$$+ \text{ terms of order} \geqslant 2 \text{ in } \widehat{\theta}_1,$$

and so

$$\pi_1^X \oint_{\Lambda_1^* \otimes \mathscr{L}(X^{(1)})} e^{-t(H^{(1)}+P^{(1)})} = -\int_0^t e^{-sH} P_1 e^{-(t-s)H} \, \mathrm{d}s = -\Phi_t^H(P_1).$$

It is a simple consequence of the above theorem and the theory of analytic semigroups that

$$[0,\infty) \ni t \longmapsto \Phi^H_t(P_1,\dots,P_n) \in \mathscr{L}(X)$$

is smooth on $(0,\infty)$ and extends continuously 0 at $t=0$, and for all $t>0$ one has

$$\mathrm{Ran}\big(\Phi^H_t(P_1,\dots,P_n)\big) \subset \mathrm{Dom}(H), \tag{1}$$

while, somewhat surprisingly, in general $\Phi^H_t(P_1,\dots,P_n)$ does **not** even map

$$\bigcap_{l\in\mathbb{N}} \mathrm{Dom}(H^l) \to \mathrm{Dom}(H^2).$$

To see the latter, assume $n = 1$, let P be bounded and let $f \in \bigcap_l \mathrm{Dom}(H^l)$. Then

$$H\Phi_t^H(P)f = -\mathrm{e}^{-tH}Pf + \Phi_t^H(PH)f + P\mathrm{e}^{-tH}f.$$

The first and second summand are in $\mathrm{Dom}(H)$. Thus $\Phi_t^H(P)f$ is not in $\mathrm{Dom}(H^2)$, if and only if $P\mathrm{e}^{-tH}f$ is not in $\mathrm{Dom}(H)$.

This can be constructed easily: $H = -\Delta/2$ on a closed, connected Riemannian manifold M and $P = 1_U$, where $U \subset M$ is open with $M \setminus \overline{U}$ nonempty. Then for

$$f := 1 \in \bigcap_l \mathrm{Dom}(H^l) = C^\infty(M)$$

and $t > 0$ one has $\mathrm{e}^{-tH}f = 1$, thus

$$P\mathrm{e}^{-tH}f = 1_U \notin \mathrm{Dom}(H) = W^2(M).$$

Feynman-Kac on manifolds.....

- M: closed connected Riemannian manifold with volume measure μ
- $H = \nabla^{\dagger}\nabla/2 + V$, where ∇ is a metric connection on a metric vector bundle $E \to M$ and where V is a potential on $E \to M$; locally $V : M \to \mathrm{Mat}(l \times l; \mathbb{C})$
- $P_1, \dots, P_n$ are differential operators on $E \to M$ of order $\leqslant 1$; locally $P_j = \sum_i P_{ji}\partial_i$, with $P_{ji} : M \to \mathrm{Mat}(l \times l; \mathbb{C})$

H is essentially self-adjoint in the Hilbert space $\Gamma_{L^2}(M, E)$ (locally: L^2-functions $M \to \mathbb{C}^l$), and $P_j(H+1)^{-1/2}$ is a pseudo-differential operator of order 0 and thus bounded.

$H^{(n)}$ is again a Schrödinger type operator on the metric vector bundle $\Lambda_n \otimes E \otimes \mathbb{C}^n \to M$, and $P^{(n)}$ is a differential operator of order $\leqslant 1$ on this bundle.

How can we get a Feynman-Kac formula for the integral kernel $\Phi_t^H(P_1,\dots,P_n)(x,y) \in \mathrm{Hom}(E_y,E_x)$?

Decompose $P_j = \alpha_j \circ \nabla + V_j^\nabla$ with $\alpha_j = \mathrm{Sym}(P_j)$ and V_j^∇ an endomorphism.

Define for fixed $t > 0$, $x, y \in M$ and $\mathrm{B}^{x,y;t}$ a Brownian bridge on M, a semimartingale in $\mathrm{End}(E_x)$ for times $s \in [0,t]$ by

$$\Psi^{x,y;t}_{\nabla,P_j}(s) := \int_0^s //^{x,y;t}_{\nabla}(r)^{-1}\Big(\alpha_j^\flat(\delta \mathrm{B}^{x,y;t}_r) + V_j^\nabla(\mathrm{B}^{x,y;t}_r)\Big)//^{x,y;t}_{\nabla}(r)\mathrm{d}r.$$

Inductively, we can then define an iterated Ito integral as a semimartingale

$$\int_{s\sigma_n} \delta\Psi^{x,y;t}_{\nabla,P_1}(s_1)\cdots\delta\Psi^{x,y;t}_{\nabla,P_n}(s_n) \in \mathrm{End}(E_x), \quad s \in [0,t].$$

Let the process $\mathscr{V}_{\nabla}^{x,y;t}(s) \in \mathrm{End}(E_x)$, $s \in [0,t]$, be given as the solution of the pathwise ordinary differential equation

$$(\mathrm{d}/\mathrm{d}s)\mathscr{V}_{\nabla}^{x,y;t}(s) = -\mathscr{V}_{\nabla}^{x,y;t}(s)\Big(//_{\nabla}^{x,y;t}(s)^{-1} V(\mathrm{B}_s^{x,y;t}) //_{\nabla}^{x,y;t}(s) \Big),$$

$$\mathscr{V}_{\nabla}^{x,y;t}(0) = \mathrm{id}_{E_x}.$$

Corollary (Feynman-Kac formula)

For $H = \nabla^{\dagger}\nabla/2 + V$ *and* $x, y \in M$, $t > 0$,

$$\Phi_t^H(P_1, \dots, P_n)(x,y)$$
$$= p(t,x,y)\mathbb{E}\left[\mathscr{V}_{\nabla}^{x,y;t}(t)\left(\int_{t\sigma_n} \delta\Psi_{\nabla,P_1}^{x,y;t}(s_1)\cdots\delta\Psi_{\nabla,P_n}^{x,y;t}(s_n)\right) //_{\nabla}^{x,y;t}(t)^{-1}\right].$$

Proof: The integral kernel of $e^{-t(H^{(n)}+P^{(n)})}(x,y)$ is given by a Feynman-Kac formula (Boldt/G.), and the Fermionic integral and π_n of this kernel can be calculated explicitly with some efforts.

- If M is even dimensional and spin and D is the Dirac operator, then this result applies to $D^2 = \nabla^\dagger \nabla + (1/4)\mathrm{scal}$. With particular choices of P_j one can obtain a Gaussian integration theory for Chen iterated differential forms on LM.
- Using the above Feynman-Kac formula, one can calculate the asymptotics of the supertrace of $t^l \Phi_t^{D^2}(P_1, \ldots, P_n)(x, x)$ as $t \to 0+$ in order to obtain the localization formula on LM.
- Connection to Bismut derivative formulae?

Thank you very much for your attention!