COMPARING THE ILL-POSEDNESS FOR LINEAR OPERATORS IN HILBERT SPACES

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ABSTRACT. The difficulty for solving ill-posed linear operator equations in Hilbert space is reflected by the strength of ill-posedness of the governing operator, and the inherent solution smoothness. In this study we focus on the ill-posedness of the operator, and we propose a partial ordering for the class of all bounded linear operators which lead to ill-posed operator equations. For compact linear operators, there is a simple characterization in terms of the decay rates of the singular values. In the context of the validity of the spectral theorem the partial ordering can also be understood. We highlight that range inclusions yield partial ordering, and we discuss cases when compositions of compact and non-compact operators occur. Several examples complement the theoretical results.

1. Introduction

The goal of this study is to *compare the strength (degree) of ill-posedness* between *two* linear operator equations

$$Ax = y \qquad (x \in X,\, y \in Y) \tag{1.1}$$

and

$$A'x' = y' \qquad (x' \in X',\, y' \in Y'). \tag{1.2}$$

Both equations represent mathematical models for inverse problems, and they are characterized by the *injective* and *bounded* linear forward operators $A : X \to Y$ and $A' : X' \to Y'$ mapping between the *infinite dimensional and separable real Hilbert spaces* X, Y as well as X', Y'. It is supposed that the operators A and A' possess non-closed ranges $\mathcal{R}(A)$ and $\mathcal{R}(A')$. This has the consequence that both operator equations (1.1) and (1.2) are ill-posed. To avoid additional technical notation we shall assume that the operators have dense ranges in Y and Y', respectively. To simplify the formulation, we speak in the following of comparing the strength of the ill-posedness of the operators A and A', but actually mean the comparison of the strength of ill-posedness of the operator equations (1.1) and (1.2).

The comparison of the ill-posedness of linear operator equations was raised earlier, especially when comparing equations with compact and non-compact operators. In [20, p.55], M. Z. Nashed states that "…an equation involving a bounded non-compact operator with non-closed range is 'less' ill-posed than an equation with a compact operator with infinite dimensional range". Often the comparison

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of compact operators is based on the *degree of ill-posedness* or the *interval of ill-posedness*, and we refer to [13] for formal definitions. Such comparison seems rough, as there are very different operators, sharing the same degree of ill-posedness. It may even happen that the degree of ill-posedness equals zero, although the operator equation is ill-posed. Other authors consider the growth rate of distribution functions, as in [23], or the decay rate of the decreasing rearrangements, see [19] to measure the ill-posedness.

Here we introduce a direct comparison by means of a partial ordering. Such partial ordering must take into account that the governing operators A and A' may have different domain and target spaces. The formal definition is given in Section 2. Then we show in Section 3 that for compact operators the present definition coincides with comparing the decay rates of the singular values. In a side step, we touch the cases when the compact operator A' is a composition of another compact operator A with a non-compact one possessing a non-closed range, see Section 4. Comparison of two operators may also be viewed as comparison of the ranges, both dense in their target spaces. We highlight in Section 5 that this is indeed covered by the present approach. Section 6 indicates how range inclusions have impact on regularization theory; and this was actually the starting point for our investigations. The comparison of non-compact operators may be reduced to comparing the self-adjoint non-negative companions via the polar decomposition. In Section 7 we use the spectral theorem to indicate when and how such non-compact operators can be compared. Throughout the study, we present examples that explain our attitude, and the final Section 8 complements the analysis with two further examples.

2. Partial ordering of operators: Definition and general results

For the bounded linear operators A and A' with non-closed ranges, we introduce a partial ordering as follows:

Definition 1 (partial ordering)**.** The operator $A'\colon X' \to Y'$ is said to be *more ill-posed than* $A\colon X \to Y$ whenever there exist a bounded linear operator $S\colon X' \to X$ and an orthogonal operator $R\colon Y \to Y'$ such that $A' = R\,A\,S$. In this case we shall write $A' \prec_{R,S} A$.

Remark 1. Not every pair of operators may be comparable in the sense of the partial ordering introduced by Definition 1. If a pair S and R of operators with $A' \prec_{R,S} A$ does not exist, we shall write $A' \not\prec A$. If we have that $A' \prec_{R,S} A$ but $A \not\prec A'$, then the operator A' is said to be *strictly more ill-posed than* A. If, however, $A' \not\prec A$ as well as $A \not\prec A'$, then A and A' are said to be *incomparable*. Finally we shall write $A' \asymp A$ if both $A' \prec A$ and $A \prec A'$. With respect to Definition 1, the relation '$\asymp$' represents the specific kind of 'equality' of two operators such that the partial ordering is not only reflexive and transitive, but also antisymmetric.

Factorization as shown in Figure 1 can hold true in many ways. There are two notable situations, namely the one-sided compositions, i.e., when either $A' = TA$, or $A' = AS$. These are called *factorization from the left* or *factorization from the right*, respectively. Let us mention that factorization from the left with $T = R$ being orthogonal is not interesting, because then we have that also $A = R^*A'$, and hence that $A' \asymp A$. However, the situation for $A' = TA$, and non-orthogonal operator T is less trivial, and we refer to examples in Section 4.

$$\begin{array}{ccc} X' & \xrightarrow{A'} & Y' \\ {\scriptstyle S}\downarrow & & \uparrow{\scriptstyle R} \\ X & \xrightarrow{A} & Y \end{array}$$

Figure 1. Comparison of operators A and A' via factorization, where using the orthogonal mapping R.

A natural example for factorization from the right $A' = AS$, and hence ordering, occurs when $X' \subset X$, i.e., when X' is continuously embedding into X via the mapping $S = J^X_{X'}$, and the mapping A' is the restriction of $A\colon X \to Y$. In this case we can write $A' = A\, J^X_{X'}$, thus $A' \prec_{I_Y, J^X_{X'}} A$.

Finally we mention that the partial ordering is compatible with the absolute value. Indeed, given a bounded operator $A\colon X \to Y$ we assign the absolute value operator $|A| := (A^*A)^{1/2} : X \to X$. By virtue of the polar decomposition, see [2, Chapt. 1, 3.9] the following result holds true.

Proposition 1. *Let A be a bounded operator and $|A|$ its absolute value. Then we have that $A \asymp |A|$.*

Proof. By assumption we have that $A = U\,|A|$ for an orthogonal operator $U\colon X \to Y$, as well as $|A| = U^*A$. Therefore $A \prec_{U,I_X} |A|$, and $|A| \prec_{U^*,I_X} A$, which yields the conclusion. □

3. Partial ordering of compact operators

This definition, depicted in the Figure 1, is well-motivated for *compact* operators A and A'. As the following two propositions show, Definition 1 compares in that case the behavior of the decay rates of singular values s_n to zero as $n \to \infty$ of the operators A and A'.

Proposition 2. *Suppose that the compact operator $A'\colon X' \to Y'$ is more ill-posed than $A\colon X \to Y$, i.e. $A' \prec_{R,S} A$. Then the decay rate to zero of the singular values of A' is not slower than the corresponding decay rate of A, which means that*

$$s_n(A') = \mathcal{O}(s_n(A)) \quad as \quad n \to \infty. \tag{3.1}$$

Proof. Indeed, $A' \prec_{R,S} A$ means that there are a bounded linear operator $S\colon X' \to X$ and an orthogonal operator $R\colon Y \to Y'$ such that $A' = R\,A\,S$. Then, by the definition of singular values, we find that

$$s_n(A') \le \|R\|_{Y \to Y'}\, s_n(A)\, \|S\|_{X' \to X} \quad (n = 1, 2, \dots),$$

which implies that (3.1) is valid. □

The converse assertion holds also true.

Proposition 3. *Suppose that the compact operators $A\colon X \to Y$ and $A'\colon X' \to Y'$ are such that $s_n(A') = \mathcal{O}(s_n(A))$ as $n \to \infty$. Then there exist a bounded linear operator $S\colon X' \to X$ and an orthogonal operator $R\colon Y \to Y'$ such that $A' \prec_{R,S} A$.*

Proof. By the Schmidt Representation Theorem (SVD) we have orthonormal systems u_j, v_j $(j \in \mathbb{N})$ and u'_j, v'_j $(j \in \mathbb{N})$ such that $A = \sum_{i=1}^{\infty} s_i(A) u_i \otimes v_i$, and $A' = \sum_{i=1}^{\infty} s_i(A') u'_i \otimes v'_i$. By assumption the sequence $\sigma_i := \frac{s_i(A')}{s_i(A)}$, $(i \in \mathbb{N})$, is bounded. Now we can set $S_\sigma := \sum_{i=1}^{\infty} \sigma_i u'_i \otimes u_i$ and $R := \sum_{i=1}^{\infty} v_i \otimes v'_i$. By construction, the operator S_σ is bounded, and R constitutes an orthogonal mapping. It is straightforward to see that $A' = R\,A\,S_\sigma$. □

Both the above propositions yield

Corollary 1. *Let A and A' be compact operators. The following assertions are equivalent.*

1. *We have that $A' \asymp A$.*
2. *It holds true that $s_n(A') \asymp s_n(A)$ as $n \to \infty$.*

Remark 2. By Definition 1 the operator A' is strictly more ill-posed than A, if $A' \prec_{R,S} A$, but there are no corresponding operators S' (bounded linear) and R' (orthogonal) such that $A \prec_{R',S'} A'$. For compact operators, this is the case under the stronger assumption

$$s_n(A') = o\left(s_n(A)\right) \quad \text{as} \quad n \to \infty. \tag{3.2}$$

4. Composition with non-compact operators

We extend the previous discussion with the following observation. Suppose that we have a factorization from the left in the form $A' = TA$ or from the right in the form $A' = AT$ for a compact operator A and a compact or a bounded non-compact linear operator T. Then we find that A' is also compact and obeys the condition $s_n(A') = \mathcal{O}\left(s_n(A)\right)$ as $n \to \infty$ for the singular values of A and A'. Now, Proposition 3 implies that $A' \prec A$, and there will thus be a factorization $A' = RAS$ with an orthogonal operator R and a bounded operator S.

A typical situation for the factorization from the left is met when the operators are connected by some bounded but non-compact multiplication operator T as discussed in the following example.

Example 1. Let $X = X' = Y = Y' = L_2(0,1)$ and consider compositions $A' = TA$ with non-compact multiplication operators $T := M$ defined as

$$(Mx)(t) := f(t)\,x(t) \quad (0 \le t \le 1)$$

and mapping in $L_2(0,1)$. The occurring non-negative multiplier functions $f \in L_\infty(0,1)$ are supposed to possess essential zeros. Such compositions with focus on the simple integration operator $A := J$ defined as

$$(Jx)(s) := \int_0^s x(t)dt \quad (0 \le s \le 1)$$

were considered in [9, 14, 15]. It was shown there that for wide classes of functions f, including the monomials $f(t) = t^\kappa$ for all $\kappa > 0$, the decay rates of the singular values of J and MJ coincide, which implies that $J \asymp MJ$ for such multiplier functions f.

Another aspect for factorizations from the left is highlighted by the following general assertions, Proposition 4, and its Corollary 2, here with with focus on ill-posed situations. The subsequent Example 2 below illustrates the situation of the corollary.

First the following result, privately communicated by A. Pietsch, seems interesting.

Proposition 4. *The following assertions are equivalent for an arbitrary bounded linear operator $T\colon Y \to Z$.*

((a)) *There is a constant $c > 0$ such that $s_n(A) \le c s_n(TA)$ for all compact linear operators $A\colon X \to Y$, and all $n = 1, 2, \dots$.*

((b)) *The operator T admits a bounded left inverse $B\colon Z \to Y$, i.e., $BT = I_Y$.*

Proof. Clearly, item (b) yields (a), because then

$$s_n(A) = s_n(BTA) \le \|B\|_{Z\to Y}\, s_n(TA), \quad n = 1, 2, \dots.$$

Next, if item (a) holds true, then this must hold for arbitrary rank one operators $A = x_0 \otimes y_0$, mapping $x \mapsto \langle x, x_0\rangle y_0,\ x \in X$, and for $n = 1$. In this case the assumption with $n = 1$ translates to

$$\|x_0\|_X\, \|y_0\|_Y = s_1(x_0 \otimes y_0) \le c s_1(x_0 \otimes Ty_0) = c\, \|Ty_0\|_Z\, \|x_0\|_X$$

Since $x_0 \neq 0$ and $y_0 \neq 0$ may be chosen arbitrary we find that the operator $T\colon Y \to \mathcal{R}(T)$ is continuously invertible, and the inverse T^{-1} can be continuously extended to $\overline{\mathcal{R}(T)}$. Denoting $P\colon Z \to \overline{\mathcal{R}(T)}$ the projection, the mapping $B := T^{-1}P$ constitutes the left inverse to T. The proof is complete. □

Within the present context of ill-posed operator equations this yields the following.

Corollary 2. *Let $T\colon Y \to Z$ be an injective bounded linear operator with non-closed and dense range. Then for every (arbitrarily large) constant $C < \infty$ there are a compact operator $A\colon X \to Y$ and an index n such that $s_n(A)/s_n(TA) \ge C$.*

Proof. Since we assume that $\overline{\mathcal{R}(T)} = Z$, the existence of a bounded left inverse $B\colon Z \to Y$ to T actually requires the existence of a bounded inverse T^{-1}. This contradicts the ill-posedness of T coming from the non-closedness of the range $\mathcal{R}(T)$. Hence, item (a) of Proposition 4 is violated. The assertion of the corollary is a reformulation of the violation of item (a). □

Example 2. Here we consider the non-compact ill-posed Hausdorff moment operator $T := B^{(H)}$. This operator $B^{(H)}\colon Y = L_2(0,1) \to Z = \ell^2$ is given as

$$[B^{(H)}z]_j := \int_0^1 t^{j-1} z(t)dt \qquad (j = 1, 2, \dots). \tag{4.1}$$

In the composition $A' := B^H \circ J$ with the simple integration operator $A := J$ acting in $X = Y = L_2(0,1)$ the situation of Corollary 2 is met. Indeed, it was shown in [12] that we actually have that $s_n(A') = \mathcal{o}\,(s_n(J))$ as $n \to \infty$, and hence $B^H \circ J$ is strictly more ill-posed than J.

Also for the composition of the Hausdorff moment operator $T := B^{(H)}$ with the compact embedding operator $A := E^k\colon H^k(0,1) \to L_2(0,1)$, mapping from the Hilbertian Sobolev spaces $X = H^k(0,1)$ $(k = 1, 2, ...)$ to $Y = L_2(0,1)$, the situation of Corollary 2 occurs. In both cases, $B^{(H)}J$ and $B^{(H)}E^k$, it is still an open problem whether the composition operator has power type or exponential decay of the singular values.

Subsequently, a series of studies turned to such questions, see e.g. [3], and references therein.

As we have seen, Definition 1 also applies in combination with non-compact bounded linear operators. A direct comparison is in the focus of the following proposition, which highlights that bounded non-compact linear operators with non-closed ranges can never be more ill-posed in the sense of Definition 1 than compact linear operators.

Proposition 5. *Suppose that A is a* non-compact *bounded linear operator with non-closed range, and that A' a* compact *linear operator with infinite dimensional range. Then we have that $A \not\prec A'$. If, however, $A' \prec A$ holds, then A' is even* strictly more ill-posed *than A.*

Proof. Suppose to the contrary that $A \prec A'$ holds true, and hence a factorization $A = R'A'S'$ exists. The family of compact operators constitutes an operator ideal, and hence the composition $R'A'S'$ will be a compact operator, which contradicts the assumption. The second assertion holds true because under the made assumptions we cannot have that $A \asymp A'$. □

The partial ordering of Definition 1 with respect to the strength of ill-posedness of compact and ill-posed non-compact operators has various facets and was discussed contradictorily in the literature, see also Remark 3, below. As mentioned in the introduction, Nashed's opposite claim in [20, p.55] is rather problematic due to occurring incomparability phenomena. We refer in this context also to the study [11], where it has been shown that compact and non-compact operators are fundamentally different with respect to projections into finite dimensional spaces. The reason for this seems to be that sequences of compact operators can never converge in norm to a non-compact operator.

5. Range inclusions yield ordering

A further strong motivation for Definition 1 is due to range inclusions as a tool for comparing the ill-posedness of two operators A and A'. It can be expected that *smaller ranges of operators A'* compared to A *indicate a higher degree of ill-posedness*. A stringent justification of this comes from Douglas' Range Inclusion Theorem, and we refer to the original study in [4]. In light of this theorem, and using the recent formulation in [18], we can formulate the following result.

Theorem 1. *The following assertions are equivalent:*

((a)) *There exists an orthogonal mapping $R\colon Y \to Y'$, for which the range inclusion*
$$R^* \mathcal{R}(A') = \mathcal{R}(R^* A') \subset \mathcal{R}(A)$$
is satisfied.

((b)) *There is a constant $0 \le C < \infty$ such that, for all $y \in Y$,*
$$\| (A')^* Ry\| \le C\|A^* y\|.$$

((c)) *There exist a bounded linear operator $S\colon X' \to X$ and an orthogonal operator $R\colon Y \to Y'$ such that A' obeys the factorization $A' = R\,A\,S$, which means $A' \prec_{R,S} A$ in the sense of Definition 1.*

Proof. We apply the original theorem to the operators R^*A', and A, respectively. Since R was orthogonal, the adjoint mapping is $(A')^* R$, which shows the equivalence of item (a) and (b). Again, from the original formulation we find a factor, say S, such that $R^*A' = AS$. Using the orthogonality once more this yields $A' = RAS$, which is the assertion from item (c), and the proof is completed. □

The following corollary characterizes the special case, when the image spaces of A and A' coincide, i.e. when $Y = Y'$.

Corollary 3. *Let $A\colon X \to Y$ and $A'\colon X' \to Y$ be both bounded linear operators with non-closed ranges. The following assertions are equivalent.*

((i)) *There is a range inclusion $\mathcal{R}(A') \subset \mathcal{R}(A)$.*
((ii)) *There is a bounded linear operator $S : X \to X'$ such that the factorization $A' = A\,S$ holds true.*

In either case we have that $A' \prec_{I_Y,S} A$.

In particular we see that factorizations from the right always yield comparison.

Remark 3. Proposition 5 yields that the range inclusion $R(A') \subset \mathcal{R}(A)$ cannot hold when A' is non-compact, but A is compact. The interplay of compact and non-compact operators possessing $\mathcal{R}(A') \subset \mathcal{R}(A)$ has been discussed in [1, Example 10.2], where also a concrete example of a compact operator A' is presented that is strictly more ill-posed than a concrete non-compact operator A. Situations of comparable compact and non-compact operators are typical for factorizations from the right when the compact operator A' is factorized as $A' = A \circ T$ with an ill-posed non-compact bounded operator A. Here, T is mostly a compact operator (e.g. $A' = B^{(H)} \circ J$ in Example 2), but T can also be a non-compact operator as comprehensively discussed in [16].

If the ranges coincide, i.e. $R(A') = \mathcal{R}(A)$, then by virtue of Theorem 1 we have that $A' \asymp A$, and hence $s_n(A') \asymp s_n(A)$ for $n \to \infty$, by Corollary 1. Note that the pre-image spaces X and X' in this context can be very different.

Unfortunately, if $R(A')$ is a *proper subset* of $\mathcal{R}(A)$, then one cannot conclude that A' is strictly more ill-posed than A, and we refer to Lemma 1 below for counterexamples.

Lemma 1. Let for the compact operators with infinite dimensional range $A\colon X \to Y$ and $A'\colon X' \to Y$ hold that the range $\mathcal{R}(A')$ is a m-codimensional subspace of $\mathcal{R}(A)$ with $m \in \mathbb{N}$. Moreover assume that the singular values of A satisfy the condition

$$s_{2n}(A)/s_n(A) \geq \underline{C} \quad \text{for all } n \in \mathbb{N} \text{ and some constant } \underline{C} > 0. \tag{5.1}$$

Then the decay rates to zero of the singular values of A and A' coincide, i.e. $s_n(A') \asymp s_n(A)$ as $n \to \infty$, and hence $A' \asymp A$.

Proof. Let Q denote the orthogonal projection from $\mathcal{R}(A)$ onto the m-codimensional subspace $\mathcal{R}(A')$. We see that $A = (I - Q)A + Q\,A$, and the calculus with singular values provides us with the estimate

$$s_{m+n}(A) \leq s_n(Q\,A) + s_{m+1}((I - Q)A) = s_n(Q\,A),$$

because $\mathrm{rk}(I - Q) = m < m + 1$. Thus, we have

$$s_{2n}(A) \leq s_n(Q\,A) \leq \hat{C}\, s_n(A')$$

for some constant $0 < \hat{C} < \infty$ and $n \geq m$. The right inequality is a consequence of the construction $\mathcal{R}(Q\,A) = \mathcal{R}(A')$, which implies $\mathcal{R}(Q\,A) \subset \mathcal{R}(A')$ and yields with Corollary 3 in combination with Proposition 2 the existence of the constant

$\hat{C}$ satisfying the inequality $s_n(Q\,A) \le \hat{C}\, s_n(A')$. Moreover, there is a constant $0 < \overline{C} < \infty$ from $\mathcal{R}(A') \subset \mathcal{R}(A)$ such that we have $s_n(A') \le \overline{C}\, s_n(A)$. Together we find, by recalling the condition (5.1) with the constant $\underline{C} > 0$, the estimate

$$\underline{C}\, s_n(A) \le s_{2n}(A) \le \hat{C}\, s_n(A') \le \hat{C}\,\overline{C}\, s_n(A) \quad (n \ge m),$$

which proves the lemma. □

This lemma will be applied in Section 8 for comparing the ill-posedness of pairs of compact operators, where the range of one is a proper subset of the range of the other. We note that the required assumption (5.1) of the lemma is satisfied if the decay rate of the singular values of A is polynomial, i.e. if there are constants $0 < c_1, c_2 < \infty$ and exponents $0 < \theta_1 \le \theta_2 < \infty$ such that $c_1\, n^{\theta_1} \le s_n(A) \le c_2\, n^{\theta_2}$ $(n \in \mathbb{N})$.

6. Impact on regularization

The comparison of the smoothing properties of the operators A and A', say $A' \prec A$, my loosely be understood in the sense that the application of the inverse of the injective mapping A may be compensated by the application of the operator A'. This reasoning would assume that the range $\mathcal{R}(A') \subset \mathcal{D}(A)$, because then $A^{-1} \circ A'$ could be defined. Therefore, the orthogonal mapping R^* may be used to put the range of A' into the target space Y of A. Then it makes sense to consider the potentially unbounded mapping $A^{-1}R^*A'$.

The solution theory of ill-posed operator equations deals with replacing the unbounded mapping A^{-1} by bounded mappings

$$g_\alpha(A^*A)A^* \colon Y \to X \qquad (\alpha > 0),$$

which are constructed by a family of generator functions g_α for regularization. We will not dwell into details here, rather we refer to the monograph [5]. In this focus we can formulate the following result.

Proposition 6. *Let g_α be a family of generator functions for regularization. We have the following dichotomy.*

((i)) *Either $\mathcal{R}\left(R^*A'\right) \subset \mathcal{R}(A)$, and then*

$$\left\|g_\alpha(A^*A)A^*\ R^*\ A'\right\|_{X' \to X} \text{ is uniformly bounded as } \alpha \to 0,$$

((ii)) *or $\mathcal{R}\left(R^*A'\right) \not\subset \mathcal{R}(A)$, and then the family*

$$\left\|g_\alpha(A^*A)A^*\ R^*\ A'\right\|_{X' \to X},\ \alpha > 0$$

is unbounded.

Proof. This is a consequence of the well-known dichotomy, see e.g. [22, Chapt. 2, Thm. 5.2]. In ibid., this is formulated for iterative regularization schemes. However, such philosophy is also valid for arbitrary types of regularization. For an operator A with dense range, it says the following: Either $y \in \mathcal{R}(A)$ and then $g_\alpha(A^*A)A^*y$ converges to $A^{-1}y$, or $y \notin \mathcal{R}(A)$ in which case $\left\|g_\alpha(A^*A)A^*y\right\|_X$ is unbounded as $\alpha \searrow 0$. The assertion of the proposition is a direct consequence of this. □

7. Application of spectral theorem to partial ordering

Definition 1 is also applicable for comparing two non-compact operators via Halmos' version (cf. [8]) of the spectral theorem, applicable to self-adjoint linear operators. Proposition 2, for example, which has been formulated for compact operators via decay rates of singular values, can be extended by exploiting a measure space (Ω, μ) to multiplication operators $M_f\colon L_2(\Omega,\mu) \to L_2(\Omega,\mu)$ defined as

$$[M_f \xi](\omega) := f(\omega)\, \xi(\omega) \quad (\mu - a.e. \text{ on } \Omega),$$

with multiplier functions $f \in L^\infty(\omega, \mu)$.

We know from Proposition 1 that it is enough to consider non-negative self-adjoint operators. Hence we consider a pair $H\colon X \to X$ and $H'\colon X' \to X'$ of non-negative self-adjoint semi-definite operators. In this case the occuring multiplier functions f and f' are real-valued and non-negative.

Proposition 7. *Suppose that the two self-adjoint positive semi-definite bounded linear operators $H\colon X \to X$ and $H'\colon X' \to X'$, both with non-closed ranges $\mathcal{R}(H)$ and $\mathcal{R}(H')$, admit a spectral representation with respect to a* common semi-finite measure space (Ω, μ). *Thus we have representations*

$$H = U M_f U^* \quad \text{and} \quad H' = U' M_{f'} U'^*$$

with orthogonal operators $U\colon L_2(\Omega,\mu) \to X$, $U'\colon L_2(\Omega,\mu) \to X'$, and with the two multiplication operators

$$M_f\colon L_2(\Omega,\mu) \to L_2(\Omega,\mu) \quad \text{and} \quad M_{f'}\colon L_2(\Omega,\mu) \to L_2(\Omega,\mu),$$

respectively, characterized by the multiplier functions f and f', which are both non-negative a.e. on Ω and belong to $L^\infty(\Omega,\mu)$. If also the quotient function $\frac{f'}{f}$ belongs to $L^\infty(\Omega,\mu)$, and hence the multiplication operator $M_{\frac{f'}{f}}$ is bounded, then H' is more ill-posed than H, *i.e. $H' \prec_{R,S} H$, with the orthogonal mapping $R := U'U^*\colon X \to X'$ and the bounded linear operator $S := U M_{f'/f} U'^*\colon X' \to X$.*

Proof. It suffices to draw the following diagram, given in Figure 2. □

$$\begin{array}{ccc}
X' & \xrightarrow{H'} & X' \\
{\scriptstyle U'^*}\big\downarrow & & \big\uparrow{\scriptstyle U'} \\
L_2(\Omega,\mu) & \xrightarrow{M_{f'}} & L_2(\Omega,\mu) \\
{\scriptstyle M_{\frac{f'}{f}}}\big\downarrow & & \big\uparrow{\scriptstyle I_{L_2(\Omega,\mu)}} \\
L_2(\Omega,\mu) & \xrightarrow{M_f} & L_2(\Omega,\mu) \\
{\scriptstyle U^*}\big\uparrow & & \big\downarrow{\scriptstyle U} \\
X & \xrightarrow{H} & X
\end{array}$$

Figure 2. Comparison of self-adjoint operators H and H'

Remark 4. For the factorization $H = UM_fU^*\colon X \to X$ of the self-adjoint positive semi-definite operator H with non-closed range $\mathcal{R}(H)$ we have that the spectrum of H and the essential range of the multiplier function f coincide (cf. [7, Theorem 2.1(g)]). They represent the same closed subset of the bounded interval $[0, \|H\|_{X\to X}] \subset \mathbb{R}$, and zero belongs to that subset, because of the ill-posedness. It is important to distinguish the case of a finite measure space (Ω, μ) with $\mu(\Omega) < \infty$, where *increasing rearrangements* (see, e.g., [6]) of the multiplier function f characterize the situation, and the case of an infinite measure $\mu(\Omega) = \infty$, where *decreasing rearrangements* of f play this role. We refer for detailed studies in this context to the recent papers [19] and [23].

Example 3. As a typical example for comparing non-compact operators in the case $\Omega = [0, 1]$ with Lebesgue measure μ on $\mathbb{R}$ and $\mu(\Omega) < \infty$ serves the family of pure multiplication operators in $X = X' = L_2(0, 1)$ with $U = U' = I : X \to X$. We compare the two bounded, non-compact, self-adjoint and positive semi-definite operators $Hx = M_f x$ and $H'x = M_{f'}x$ mapping on $L_2(0, 1)$ with continuous and strictly increasing multiplier functions $f(t)$ and $f'(t)$ for $0 < t \le 1$ satisfying the conditions $\lim_{t\to+0} f(t) = \lim_{t\to+0} f'(t) = 0$. If there is a finite constant $C > 0$ such that $\frac{f'(t)}{f(t)} \le C$ for all $0 < t \le 1$, then H' is *more ill-posed than* H, where precisely $H' \prec_{I,M_{f'/f}} H$. This case occurs when $f(t) = c_1t$ and $f'(t) = c_2t$ $(0 < t \le 1)$ with positive constants c_1 and c_2. But then also $H \prec_{I,M_{f/f'}} H'$ takes place. If, however, the decay rate $f'(t) \searrow 0$ is higher than the rate of $f(t) \searrow 0$ as $t \to +0$, we have $H' \prec_{I,M_{f'/f}} H$ but $H \not\prec H'$ and H' is even *strictly more ill-posed than* H like in the example $f(t) = t^{\kappa_1}$ and $f'(t) = \exp\left(-\frac{1}{t^{\kappa_2}}\right)$ $(0 < t \le 1)$ with positive constants κ_1 and κ_2.

Example 4. An example for non-compact operators analog to Example 3, but in the infinite measure case $\Omega = [0, \infty)$ with Lebesgue measure μ on $\mathbb{R}$ and pure multiplication operators in $X = X' = L_2(0, \infty)$ can be found by comparing continuous multiplier functions $f(t)$ and $f'(t)$, which are strictly decreasing on $[0, \infty)$ and tend to zero as $t \to \infty$. Here, the decay rate of the multiplier functions at infinity determines the partial ordering. If the decay rate of f' at infinity is higher than those of f, then H' is strictly more ill-posed than H, i.e. $H' \prec_{I,M_{f'/f}} H$ but $H \not\prec H'$. This if for example the case for $f(t) = (1 + t)^{-1}$ and $f'(t) = (1 + t)^{-2}$ $(0 \le t < \infty)$.

8. Further examples

In this section, we are going to study by means of examples the partial ordering of compact operators with common image spaces $Y = Y'$. This case has already been discussed in Corollary 3. The goal is to check whether the previous results allow us to draw conclusions about the comparability for specific operators.

Example 5. First let us compare

((i)) the Riemann–Liouville fractional integration operators $J^m : L_2(0, 1) \to L_2(0, 1)$ of order $m = 1, 2, \ldots$, defined as

$$[J^m x](s) := \int_0^s \frac{(s-t)^{m-1}}{\Gamma(m)} x(t)\, dt \quad (0 \le s \le 1),$$

for which solving the associated operator equation (1.1) requires to find the m-th derivative of the function y, and ((ii)) the embedding operators $E^k\colon H^k(0,1) \to L_2(0,1)$ mapping from the Hilbertian Sobolev spaces H^k to L_2 for $k = 1, 2, ...$

Of course, the decay of the singular values of the embeddings are known, specifically we have that

$$s_n(E^k) \asymp n^{-k} \quad \text{as} \quad n \to \infty \tag{8.1}$$

(see, e.g., [17]).

Also, the ranges $\mathcal{R}(J^m)$ of the mapping J^m can be characterized. These are known to be subsets of $H^m(0,1)$, with a finite set of linear constraints, thus the range is a finite-codimensional subspace of $H^m(0,1)$. From this we conclude by virtue of Corollary 3 that $J^m \prec E^m$. Moreover, since the decay rate given in (8.1) is polynomial, Lemma 1 applies and actually yields that $J^m \asymp E^m$. Consequently, by Corollary 1 we find that $s_n(J^m) \asymp n^{-m}$ as $n \to \infty$. Of course, this is known, and we refer to [21], for instance. Next, having these decay rates for both operators E^m and J^m we can apply Proposition 3, see Remark 2, to immediately conclude that for $k > m$ the operator E^k is strictly more ill-posed than J^m, whereas for $m > k$ the opposite assertion holds true.

The following example deals with functions of two real variables defined on the unit square $[0,1]^2$.

Example 6. Here we compare for $m = 1, 2, ...$

((i)) the compact embedding operators $\mathbf{E}_2^m : H^m([0,1]^2) \to L_2([0,1]^2)$, with

((ii)) the compact *mixed integration operator* $\mathbf{J}_2 : L_2([0,1]^2) \to L_2([0,1]^2)$ defined as

$$[\mathbf{J}_2\, x(t_1,t_2)](s_1,s_2) := \int_0^{s_1} \int_0^{s_2} x(t_1,t_2)\, dt_1\, dt_2 \quad ((s_1,s_2) \in [0,1]^2).$$

The associated operator equation (1.1) aims at finding the second mixed derivative $\frac{\partial}{\partial s_1 \partial s_2} y(s_1,s_2)$ of the right-hand side function y of (1.1).

We first discuss smoothness $m = 1$. In this case we know that $\mathcal{R}(\mathbf{J}_2) \subset H^1(0,1)^2 = \mathcal{R}(\mathbf{E}_2^1)$, and hence $\mathbf{J}_2 \prec \mathbf{E}_2^m$.

However, for $m = 2$ we have that the range $\mathcal{R}(\mathbf{E}_2^1) = H^2([0,1]^2)$, but we have $\mathcal{R}(\mathbf{E}_2^2) \not\subset \mathcal{R}(\mathbf{J}_2)$ as outlined in [10, §4] by a counterexample. Thus range inclusions do not apply here. However, see e.g., [17, §3c], we know that

$$s_n(\mathbf{E}_2^2) \asymp n^{-1} \quad \text{as} \quad n \to \infty. \tag{8.2}$$

Thus

$$s_n(\mathbf{E}_2^2) = \mathcal{O}\left(\frac{\log(n)}{n}\right) \quad \text{as} \quad n \to \infty,$$

where the right hand side rate corresponds to the decay rate of the singular values, i.e., $s_n(\mathbf{J}_2) \asymp \frac{\log(n)}{n}$, see, e.g., [10, Prop. 3.1]. By virtue of Proposition 3 there must hold true that $\mathbf{E}_2^2 \prec_{R,S} \mathbf{J}_2$, for some orthogonal mapping R and bounded S. This factorization is not known to us.

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