

REGULARIZABILITY OF ILL-POSED PROBLEMS AND THE MODULUS OF CONTINUITY

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ABSTRACT. The regularization of linear ill-posed problems is based on their conditional well-posedness when restricting the problem to certain classes of solutions. Given such class one may consider several related real-valued functions, which measure the well-posedness of the problem on such class. Among those functions the modulus of continuity is best studied. For solution classes which enjoy the additional feature of being star-shaped at zero, the authors develop a series of results with focus on continuity properties of the modulus of continuity. In particular it is highlighted that the problem is conditionally well-posed if and only if the modulus of continuity is right-continuous at zero. Those results are then applied to smoothness classes in Hilbert space. This study concludes with a new perspective on a concavity problem for the modulus of continuity, recently addressed by two of the authors in *A note on the modulus of continuity for ill-posed problems in Hilbert space*, 2012.

1. INTRODUCTION

We shall consider linear ill-posed problems, given in the form

$$(1) \quad y^\delta = Ax + \delta\xi,$$

where $A: X \rightarrow Y$ denotes an injective and bounded linear operator acting between Banach spaces X and Y . If the range $\mathcal{R}(A) \subset Y$ is non-closed, then the problems (1) are ill-posed, meaning that the inverse mapping $A^{-1}: \mathcal{R}(A) \subset Y \rightarrow X$ is not bounded and hence not continuous. However, one may restrict a problem as posed in (1) by introducing the *a priori information* that $x \in M \subseteq X$, for a non-empty subset M . By doing so we may study the following questions.

Question 1. Does the mapping $A: M \rightarrow Y$ have a bounded inverse?

If this is the case, then the mapping $\Omega_M: [0, +\infty) \rightarrow [0, +\infty]$, given as

$$(2) \quad \Omega_M(\delta) := \sup \{ \|x - x'\| : x, x' \in M, \|Ax - Ax'\| \leq \delta \},$$

2010 *Mathematics Subject Classification.* 47A52, secondary: 65J20.

The research of Radu Ioan Boț and Bernd Hofmann was partially supported by Deutsche Forschungsgemeinschaft (DFG) under grants BO 2516/4-1 and HO 1454/8-1, respectively.

is of interest. The above function Ω_M is analyzed in various studies, and we mention [7, § 2.3] for a monograph.

Question 2. Is the problem (1) *regularizable* on M in the sense of Tikhonov?

This question is related to the reconstruction of x from noisy data, and we introduce the error criterion. If S is any mapping of the form

$$y^\delta \in Y \rightarrow S(y^\delta) \in X,$$

then we let

$$e(S, x, \delta) := \sup_{y^\delta: \|Ax - y^\delta\| \leq \delta} \|S(y^\delta) - x\|, \quad \delta > 0,$$

and the corresponding *uniform error* of any reconstruction S on the set M will be given by

$$e(S, M, \delta) := \sup_{x \in M} e(S, x, \delta), \quad \delta > 0.$$

By *regularizability* on M in the sense of Tikhonov we mean that there is a family R_δ of reconstructions such that for each $x \in M$ we have that

$$(3) \quad e(R_\delta, x, \delta) \rightarrow 0 \quad \text{as } \delta \rightarrow 0.$$

The question whether a problem is regularizable received attention very early in the analysis of ill-posed problems (for an early study we refer to [12], and we also mention the monograph [1, Chapt. 1]). If the above is the case then we call the restriction of A^{-1} on

$$A(M) := \{z \in Y : z = Ax, x \in M\}$$

regularizable on $A(M)$. In addition we may ask for *uniform regularizability* of the problem (1) on the set M , thus asking whether a family R_δ of reconstructions exists for which $e(R_\delta, M, \delta) \rightarrow 0$ as $\delta \rightarrow 0$.

While the first question studies intrinsic continuity features of the problem, the latter question asks for the ability to find reconstruction methods, which are capable to recover x based on noisy data y^δ as δ is getting smaller. In this study we will not address the regularizability problem in the sense of Tikhonov. Instead we analyze uniform regularizability for classes M .

2. MODULUS OF CONTINUITY

For a variety of classes of subsets M of X that contain the zero element, instead of the function Ω_M from (2) one can consider the function $\omega_M : [0, +\infty) \rightarrow [0, +\infty]$, defined as

$$(4) \quad \omega_M(\delta) := \sup \{\|x\| : x \in M, \|Ax\| \leq \delta\},$$

usually called *modulus of continuity*. Clearly, for $0 \in M$ we have that $\omega_M(\delta) \leq \Omega_M(\delta)$ for all $\delta \geq 0$. The modulus of continuity is obviously a non-decreasing function of δ .

2.1. Elementary properties. The following elementary properties are easily verified.

Lemma 1. *Let $M, N \subseteq X$ be two subsets. Then for all $\delta \geq 0$ it holds*

- (i) $\omega_{M \cup N}(\delta) = \max\{\omega_M(\delta), \omega_N(\delta)\}$.
- (ii) $\omega_M(\delta) = \omega_{-M}(\delta)$.

In particular, we have that

$$\omega_{M \cup -M}(\delta) = \omega_M(\delta).$$

Below we shall confine ourselves to sets M from the following class of sets.

Definition 1. A subset $M \subseteq X$ is said to be *star-shaped at zero* if $0 \in M$ and if $x \in M$ implies $\alpha x \in M$ for all $0 \leq \alpha \leq 1$. Equivalently, for each $C \geq 1$ we have that $\frac{1}{C}M \subseteq M$.

Lemma 2. *Suppose that M is star-shaped at zero. Then for all $\delta \geq 0$ and all $C \geq 1$ it holds*

$$\omega_M(C\delta) \leq C\omega_M(\delta) \text{ and } \Omega_M(C\delta) \leq C\Omega_M(\delta).$$

In particular, we have that $\omega_M(t)/t \leq \omega_M(s)/s$ as well as $\Omega_M(t)/t \leq \Omega_M(s)/s$ whenever $0 < s \leq t$.

Proof. Let $C \geq 1$ be arbitrary. Then for all $\delta \geq 0$ it holds

$$\begin{aligned} \omega_M(C\delta) &= \sup \{ \|x\| : x \in M, \|Ax\| \leq C\delta \} \\ &= C \sup \left\{ \left\| \frac{1}{C}x \right\| : \frac{1}{C}x \in \frac{1}{C}M, \left\| A \frac{1}{C}x \right\| \leq \delta \right\} \\ &= C \sup \left\{ \|z\| : z \in \frac{1}{C}M, \|Az\| \leq \delta \right\} \\ &= C\omega_{\frac{1}{C}M}(\delta) \leq C\omega_M(\delta). \end{aligned}$$

Taking into account that $\Omega_M = \omega_{M-M}$, where we let $M - M := \{x - y, x, y \in M\}$, and the fact that the difference of two star-shaped at zero sets is again star-shaped at zero, the assertion for Ω_M follows. \square

We summarize the above elementary findings.

Proposition 1. *Suppose that the set $M \subseteq X$ is star-shaped at zero. Then*

- (i) *the mapping ω_M is non-decreasing, $\omega_M(0) = 0$, and the mapping $\delta \rightarrow \omega_M(\delta)/\delta$ is non-increasing on $(0, +\infty)$.*

- (ii) Likewise the mapping Ω_M is non-decreasing, $\Omega_M(0) = 0$, and the mapping $\delta \rightarrow \Omega_M(\delta)/\delta$ is non-increasing on $(0, +\infty)$.

These elementary properties have immediate consequences for the continuity properties of both ω_M and Ω_M .

2.2. Modulus functions.

Definition 2. We agree to call a mapping $f: [0, +\infty) \rightarrow [0, +\infty]$ a *modulus function* if it is non-decreasing, $f(0) = 0$, and $t \rightarrow f(t)/t$ is non-increasing on $(0, +\infty)$. It is called *proper* if for some $t > 0$ it has a finite value $f(t) < +\infty$.

With this notion both the functions ω_M and Ω_M are modulus functions provided that the set M is star-shaped at zero.

Proposition 2. *Let f be any proper modulus function. Then*

- (i) *the values $f(t)$ are finite for every $t > 0$.*
- (ii) *If $f(t_0) = 0$ for some $t_0 > 0$ then $f(t) = 0$ for all $t \geq 0$.*
- (iii) *The mapping f is continuous on $(0, +\infty)$, and*
- (iv) *for every pair $t_1, t_2 > 0$ we have that*

$$f(t_1 + t_2) \leq f(t_1) + f(t_2).$$

Proof. Suppose that f is finite at $t_0 > 0$. Then it is finite for every $0 < t \leq t_0$ by monotonicity. Also, if $t > t_0$ then $f(t)/t \leq f(t_0)/t_0 < +\infty$, which proves the assertion (i).

For the second assertion (ii), suppose that $f(t_0) = 0$, and $t_0 > 0$. Then, due to monotonicity we have that $f(t) = 0$, $0 \leq t \leq t_0$. For any $t_1 > t_0$ we see that $0 \leq f(t_1) = f(\frac{t_1}{t_0}t_0) \leq \frac{t_1}{t_0}f(t_0) = 0$.

For proving (iii) let $t > 0$ be any real number. If $f(t) = 0$ then by item (ii) f is identically zero, and the assertion is obvious. Otherwise, we first prove right continuity of f at t . Let $t_n \searrow t$. Then $1 \leq f(t_n)/f(t) \leq t_n/t \rightarrow 1$. Similarly, if $t_n \nearrow t$ then $1 \leq f(t)/f(t_n) \leq t/t_n \rightarrow 1$, which proves the continuity of f at $t > 0$.

The proof of the sub-additivity in item (iv) is well-known, and we recall this here for convenience. Plainly, $t_1, t_2 \leq t_1 + t_2$, and therefore

$$\begin{aligned} f(t_1 + t_2) &= t_1 \frac{f(t_1 + t_2)}{t_1 + t_2} + t_2 \frac{f(t_1 + t_2)}{t_1 + t_2} \\ &\leq t_1 \frac{f(t_1)}{t_1} + t_2 \frac{f(t_2)}{t_2} = f(t_1) + f(t_2), \end{aligned}$$

and this completes the proof of the proposition. \square

Remark 1. Proposition 2(iv) yields that $|f(t_1) - f(t_2)| \leq f(|t_1 - t_2|)$, $t_1, t_2 \geq 0$. Therefore, proper modulus functions are sub-additive, and continuous at every $t > 0$.

Within the classical context, the modulus of continuity, say ω , of a real function on a bounded interval is non-decreasing, $\omega(0) = 0$, and sub-additive, see e.g. [8, § 6.1]. Therefore any such function is called *modulus of continuity* if, in addition, it is continuous at zero.

Corollary 1. *A proper modulus function is continuous on $[0, +\infty)$ if and only if it is right-continuous at zero.*

We conclude this subsection with gathering more, and important, properties of modulus functions, we refer to [8, Lemma 6.1.4].

Proposition 3. *For every proper modulus function f which is right-continuous at zero there is a concave right-continuous proper modulus function f_* with*

$$f(t) \leq f_*(t) \leq 2f(t), \quad t > 0.$$

The constant 2 cannot be improved, in general.

We finally mention the following result.

Corollary 2. *If a proper modulus function f does not vanish identically then it tends to zero at most linearly, i.e., $t = \mathcal{O}(f(t))$ as $t \searrow 0$.*

Proof. Suppose that f does not vanish identically and that it is finite for $t_0 > 0$. According to Proposition 2(i), f is finite on $[0, +\infty)$. Then the assertion is immediate from the fact that $f(t)/t$ is non-increasing, which implies that for all $0 < t \leq t_0$ we have that $f(t) \geq \frac{f(t_0)}{t_0}t$. \square

2.3. Conditional well-posedness. The right-continuity at zero of the functions ω_M and Ω_M is intimately related to the continuity of the inverse A^{-1} , when considered as acting from $A(M) \subset Y$ to X . Indeed, we have the following proposition.

Proposition 4. *Let $M \subseteq X$ be star-shaped at zero. Then the modulus ω_M is right-continuous at zero if and only if the mapping A^{-1} restricted to $A(M)$ is continuous at zero.*

Proof. One has that ω_M is right-continuous at zero if and only if

$$\forall \varepsilon > 0 \exists \bar{\delta} > 0 \text{ such that } \forall \delta < \bar{\delta} \text{ it holds } \omega_M(\delta) < \varepsilon \Leftrightarrow$$

$$\forall \varepsilon > 0 \exists \bar{\delta} > 0 \text{ such that } \forall \delta < \bar{\delta} \forall y \in A(M) \text{ with } \|y\| \leq \delta$$

$$\text{it holds } \|A^{-1}y\| < \varepsilon \Leftrightarrow$$

$$\forall \varepsilon > 0 \exists \bar{\delta} > 0 \text{ such that } \forall y \in A(M) \text{ with } \|y\| < \bar{\delta} \text{ it holds } \|A^{-1}y\| < \varepsilon,$$

which is nothing else than $A^{-1}: A(M) \rightarrow X$ is continuous at zero. \square

We summarize the preceding discussion.

Definition 3. We call the problem (1) *conditionally well-posed* on M if the mapping A^{-1} restricted to $A(M)$ is continuous.

A famous theorem by A. N. Tikhonov, see [11], asserts that the problem is conditionally well-posed whenever the set $M \subset X$ is compact. For some further discussion and examples concerning conditional well-posedness we also refer to [4, 5, 14].

Theorem 1. *Suppose that the set $M \subseteq X$ is star-shaped at zero. Then the following assertions are equivalent:*

- (i) *The problem (1) is conditionally well-posed on M .*
- (ii) *The modulus of continuity ω_M is right-continuous at zero.*
- (iii) *The modulus of continuity ω_M is continuous on $[0, \infty)$.*

Proof. The equivalence of the first two assertions is a consequence of Proposition 4. The equivalence of the last two assertions follows from Corollary 1. \square

Tikhonov's result translates to the following statement.

Proposition 5. *If the set $M \subset X$ is compact and star-shaped at zero, then we have $\omega_M(\delta) \rightarrow 0$ as $\delta \rightarrow 0$, and hence the modulus of continuity ω_M is continuous on $[0, \infty)$.*

We present the following examples.

Example 1. Suppose that the operator A is injective and has a non-closed range. By denoting with $B_X := \{x \in X : \|x\| \leq 1\}$ the closed unit ball of X , we show that for all $\delta > 0$ it holds

$$\omega_{B_X}(\delta) = \sup\{\|x\| : \|x\| \leq 1, \|Ax\| \leq \delta\} = 1.$$

Indeed, consider a fixed $\delta > 0$. Obviously, $\omega_{B_X}(\delta) \leq 1$. Since the range of A is non-closed, there is no $K > 0$ such that $\|x\| \leq K\|Ax\|$ for all $x \in X$. Thus there exists a sequence $\{\tilde{x}_n\}_{n \geq 1} \subset X \setminus \{0\}$ such that $\frac{\|\tilde{x}_n\|}{\|A\tilde{x}_n\|} \rightarrow +\infty$ as $n \rightarrow +\infty$. By defining for all $n \geq 1$ $x_n := \frac{1}{\|\tilde{x}_n\|}\tilde{x}_n$, one has that $\|x_n\| = 1$ and $\|Ax_n\| \rightarrow 0$ as $n \rightarrow +\infty$. Thus there exists $n(\delta) \geq 1$ such that $\|Ax_n\| \leq \delta$ for all $n \geq n(\delta)$ and this provides the desired assertion. Consequently, the modulus of continuity ω_M is not right-continuous at zero for $M = B_X$.

Example 2. If $\{0\} \neq L \subset X$ is a finite-dimensional linear subspace then we have $\omega_M(\delta) \asymp \delta$ as $\delta \rightarrow 0$ for $M = L$. Indeed, the image space $A(L)$ is a closed linear subspace in Y , and therefore $A^{-1}: A(L) \rightarrow L$ is a bounded linear operator. Thus there is a constant $C < +\infty$ for which $\|x\| \leq C\|Ax\|$, $x \in L$. But this yields that $\omega_L(\delta) \leq C\delta$ for all $\delta > 0$. Next, if ω_L would vanish, say at $\delta_0 > 0$, then $\{x \in L : \|Ax\| \leq \delta_0\} = \{0\}$, which is not true. In the light of Corollary 2 the above assertion is proved.

The above extremal decay rate, together with Lemma 1 gives rise to the following

Conjecture 1. *Let $M \subset X$ be any set which is star-shaped at zero, and let L be a finite-dimensional subspace of X . If ω_M does not vanish identically then $\omega(M + L, \delta) \asymp \omega(M, \delta)$ as $\delta \rightarrow 0$.*

2.4. Regularizability. We now highlight the problem of the continuity of the modulus of continuity to the regularizability problem. To this end we introduce the local companion to the function Ω_M from (2), and we introduce, given any $x \in M$, the function

$$(5) \quad \Omega_M^{loc}(x, \delta) := \sup \{ \|x - x'\| : x' \in M, \|Ax - Ax'\| \leq \delta \}.$$

Plainly, $\sup_{x \in M} \Omega_M^{loc}(x, \delta) = \Omega_M(\delta)$.

In agreement with the usual nomenclature we call a reconstruction S *interpolatory at level δ* if for each $x \in M$, and data y^δ with $\|Ax - y^\delta\| \leq \delta$, we have that $S(y^\delta) \in M$ and $\|AS(y^\delta) - y^\delta\| \leq \delta$.

Lemma 3. *Let S be any reconstruction.*

(i) *If S is interpolatory, then we have for each $x \in M$ that*

$$e(S, x, \delta) \leq \Omega_M^{loc}(x, 2\delta), \quad \delta > 0.$$

(ii) *If the set M is centrally-symmetric, then*

$$\omega_M(\delta) \leq e(S, M, \delta), \quad \delta > 0.$$

(iii) *If the set M is convex and centrally-symmetric, then for each $x \in M$ it holds*

$$\Omega_M^{loc}(x, \delta) \leq \Omega_M(\delta) = 2\omega_M(\delta/2), \quad \delta > 0$$

Thus, for convex centrally-symmetric sets M we have

$$(6) \quad \omega_M(\delta) \leq e(S, M, \delta) \leq 2\omega_M(\delta), \quad \delta > 0,$$

for any reconstruction S which is interpolatory at the level δ .

Proof. Given data y^δ and interpolatory reconstruction S we let $x' := S(y^\delta) \in M$. Then $\|x - S(y^\delta)\| \leq \Omega_M^{loc}(x, 2\delta)$. Since this is true for any data y^δ with $\|Ax - y^\delta\| \leq \delta$ the proof of the first assertion (i) can be completed.

For the second assertion (ii) we argue as follows. For every $x \in M$, and due to symmetry also $-x \in M$ with $\|Ax\| = \|A(-x)\| \leq \delta$ the data $y^\delta := 0$ are possible data, and we bound, using the triangle inequality

$$e(S, M, \delta) \geq \max \{ \|S(0) - x\|, \|S(0) + x\| \} \geq \|x\|.$$

Therefore we conclude that

$$e(S, M, \delta) \geq \sup \{ \|x\| : x \in M, \|Ax\| \leq \delta \} = \omega_M(\delta).$$

To prove item (iii) we fix any $x \in M$, and let $x' \in M$ be arbitrary. Then $\hat{x} := (x - x')/2 \in M$, and we have that $\|A\hat{x}\| \leq \delta/2$, provided that $\|Ax - Ax'\| \leq \delta$. Thus, $\|x - x'\| = 2\|\hat{x}\| \leq 2\omega_M(\delta/2)$. Next we prove that $2\omega_M(\delta/2) \leq \Omega_M(\delta)$. To this end, given $0 < \varepsilon \leq \omega_M(\delta/2)$, let $z \in M$, $\|Az\| \leq \delta/2$, be such that $\|z\| \geq \omega_M(\delta/2) - \varepsilon$. This yields that $z, -z \in M$, $\|A(z - (-z))\| \leq \delta$, and

$$\|z - (-z)\| = 2\|z\| \geq 2\omega_M(\delta/2) - 2\varepsilon,$$

hence that $\Omega_M(\delta) \geq 2\omega_M(\delta/2)$. The bounds in (6) are now easy consequences. \square

Remark 2. Indeed, interpolatory reconstructions always exist. To check this, fix $x \in M$ and data y^δ with $\|Ax - y^\delta\| \leq \delta$ and consider the set

$$D(y^\delta) := \{z \in M : \|Az - y^\delta\| \leq \delta\}.$$

Since $x \in D(y^\delta)$ this set is non-empty, and any selection $S(y^\delta) \in D(y^\delta)$ will yield an interpolatory reconstruction. If the set $M \subset X$ is convex and compact, then there is even a continuous selection by *Michael's continuous selection theorem*, we refer to [2, § 7] for details and extensions. For compact sets M the construction of interpolatory reconstructions may be achieved by solving the optimization problem

$$(7) \quad x_{qu} := \arg \min_{z \in M} \|Az - y^\delta\|,$$

which exists due to the compactness of M . It is readily checked that $x_{qu} \in D(y^\delta)$. This construction goes back to Ivanov [6] and it is called *method of quasi-solutions*, there.

Corollary 3. *For convex centrally-symmetric sets M the problem (1) is uniformly regularizable on M if and only if the modulus of continuity ω_M is right-continuous at zero.*

3. SMOOTHNESS CLASSES IN HILBERT SPACE

Here and in the subsequent section let X and Y be separable Hilbert spaces. We recall that the linear operator $A : X \rightarrow Y$ is assumed to be bounded and injective. For that case one can consider its self-adjoint companion $H := A^*A$, where we set $a := \|H\| = \|A\|^2$.

The typical smoothness classes as considered in inverse problems, and we mention *source sets* expressing general smoothness assumptions, and more recently, *level sets*, are based on the distribution function

$$F_x^2(t) := \|\chi_{(0,t]}(H)x\|^2 = \int_0^t d\|E_\tau x\|^2, \quad t \geq 0,$$

which is well-defined and finite for each $x \in X$. Above, we let $\chi_{(0,t]}$ be the characteristic function of the interval $(0, t]$, and $E_t = E_t(H)$, $0 \leq t \leq a$, be the spectral resolution of the operator H . The following elementary properties are easily seen, for a further discussion and consequences cf. [3].

Lemma 4. *Let $x \in X$ be arbitrary.*

- (i) *The function $t \mapsto F_x(t)$ is right-continuous and non-decreasing.*
- (ii) *If the operator H is injective then $F_x(0) = 0$.*
- (iii) *For all $0 < t \leq a$ we have that*

$$F_{\chi_{(0,t]}(H)x}(\tau) = F_{\chi_{(0,t]}(H)x}(t), \quad \tau \geq t.$$

For a class $M \subset X$ we consider the associated function \bar{F}_M , given as

$$(8) \quad \bar{F}_M(t) := \sup_{x \in M} F_x(t), \quad t \geq 0.$$

This function is finite whenever M is bounded, and we thus will assume boundedness of M , throughout. It is also non-decreasing, and we have $\bar{F}_M(0) = 0$. As it will turn out, the right-continuity of the function $\bar{F}_M(t)$ at zero and the right-continuity of the function ω_M at zero are closely related, and we will dwell into this, now. Best results are obtained for smoothness which is expressed through sets M , which are determined in the vicinity of zero of the distribution function $F_x(t)$, $t > 0$, only.

Definition 4 (spectral smoothness). We call a smoothness class M *spectral*, if for each $t > 0$ we have that $x \in M$ yields that $\chi_{(0,t]}(H)x \in M$.

Definition 5 (index function). We call a function $\varphi : (0, a] \rightarrow (0, \infty)$ *index function* if it is continuous and increasing with $\lim_{t \searrow 0} \varphi(t) = 0$.

Example 3. For an index function φ we assign the smoothness class \mathcal{M}_φ as

$$\mathcal{M}_\varphi := \{x = \varphi(H)v, \quad \|v\| \leq 1\},$$

i.e., the image of the unit ball under the mapping $\varphi(H)$. Such classes are spectral since with $x = \varphi(H)v \in \mathcal{M}_\varphi$ we also have that

$$\chi_{(0,t]}(H)x = \chi_{(0,t]}(H)\varphi(H)v = \varphi(H)\chi_{(0,t]}(H)v,$$

and $\|\chi_{(0,t]}(H)v\| \leq \|v\| \leq 1$. We mention that

$$\bar{F}_{\mathcal{M}_\varphi}(t) = \|\chi_{(0,t]}(H)\varphi(H)\| \leq \sup_{0 < s \leq t} \varphi(s) = \varphi(t), \quad 0 < t \leq a,$$

such that $\bar{F}_{\mathcal{M}_\varphi}$ is right-continuous at zero exactly if φ was an index function.

Example 4. For an index function φ we assign the level set \mathcal{E}_φ as

$$\mathcal{E}_\varphi := \{x \in X : F_x(t) \leq \varphi(t), 0 < t \leq a\}.$$

In view of Lemma 4(iii) such classes also constitute spectral smoothness classes, and $\bar{F}_{\mathcal{E}_\varphi}$ is right-continuous at zero for index functions φ .

Proposition 6. *Suppose that the set $M \subset X$ is bounded. If the function \bar{F}_M is right-continuous at zero then ω_M is also right-continuous at zero. Moreover, for spectral smoothness classes M the converse also holds true.*

Proof. To prove the first assertion we observe that for $0 < t < a$ we can estimate the norm square as

$$\|x\|^2 = \int_0^t dF_x^2(\tau) + \int_t^a dF_x^2(\tau) \leq F_x^2(t) + \frac{1}{t} \int_t^a \tau dF_x^2(\tau).$$

From this it follows $\|x\|^2 \leq F_x^2(t) + \frac{1}{t} \|Ax\|^2$ and taking the supremum over the set $\{x \in M : \|Ax\| \leq \delta\}$ we obtain for $\delta > 0$ and all sufficiently small $t > 0$

$$(9) \quad \omega_M^2(\delta) \leq \bar{F}_M^2(t) + \frac{\delta^2}{t}.$$

By setting $t := \delta$ and under the condition $\bar{F}_M(t) \rightarrow 0$ as $t \searrow 0$ the upper bound of $\omega_M^2(\delta)$ in (9) tends to zero as $\delta \searrow 0$. Hence $\lim_{\delta \searrow 0} \omega_M(\delta) = 0$. The second assertion is proved by contraposition. Without loss of generality we assume that M belongs to the unit ball in X . Suppose that there is some $\varepsilon > 0$ such that for all $t > 0$ we have that $\bar{F}_M(t) = \sup_{x \in M} \|\chi_{(0,t]}(H)x\| > \varepsilon$. Thus we can find $\tilde{x}_t \in M$ with $\|\chi_{(0,t]}(H)\tilde{x}_t\| \geq \varepsilon$.

We assign $x_t := \chi_{(0,t]}(H)\tilde{x}_t$, $t > 0$, and $x_t \in M$ since the set M was assumed to be spectral. We thus have that for this $\varepsilon > 0$ we can find a family $x_t \in M$, $t > 0$ with $\|x_t\| \geq \varepsilon$. We claim that $\|Ax_t\| \leq t$. Indeed, taking into account item (iii) of Lemma 4 we bound

$$\|Ax_t\|^2 = \int_0^a \tau dF_{x_t}^2(\tau) = \int_0^t \tau dF_{x_t}^2(\tau) \leq t \int_0^t dF_{x_t}^2(\tau) \leq t \|x_t\|^2 \leq t.$$

Consequently we see that $\omega_M(\sqrt{t}) \geq \varepsilon$, $t > 0$, which is a contradiction. The proof is complete. \square

Remark 3. Tight bounds for the modulus of continuity $\omega_{\mathcal{M}_\varphi}$ can be obtained under additional geometric (convexity) assumptions by means of general interpolation results within the framework of *variable Hilbert scales*, and we mention [13] for an early work on this. More recently this is pursued within the framework of *conditionally stability estimates*, see the recent study [14].

Evidently, Proposition 6 provides us with a characterization for the convex and centrally-symmetric smoothness classes \mathcal{M}_φ and \mathcal{E}_φ .

In many cases the set M is an ellipsoid in Hilbert space, i.e., there is an operator $G: Z \rightarrow X$, for a Hilbert space Z such that

$$(10) \quad M(G) := \{Gv : \|v\| \leq 1\},$$

the image of the unit ball in Z under the mapping G . Such ellipsoids $M(G)$ need not be spectral smoothness classes, in general. An obvious exception occurs for commuting operators G and H .

Example 5. In case that $G = \varphi(H): X \rightarrow X$ is a function of H , where φ is any index function we have $M(G) = \mathcal{M}_\varphi$. Hence the concept of ellipsoidal sets generalizes general smoothness classes from Example 3.

For ellipsoids $M(G)$ we can rewrite

$$\bar{F}_{M(G)}(t) = \sup_{\|v\| \leq 1} \|\chi_{(0,t]}(H)Gv\| = \|\chi_{(0,t]}(H)G\|, \quad 0 < t \leq a.$$

where the latter is the operator norm of $\chi_{(0,t]}(H)G: Z \rightarrow X$.

The following was proved in [5, Thm. 4.4], and this shows that the bound in Lemma 3(iii) can be attained for ellipsoidal sets.

Theorem 2. *For each ellipsoid $M(G)$ and each $\delta > 0$ there is a linear reconstruction method S_δ such that*

$$e(S_\delta, M(G), \delta) = \omega_{M(G)}(\delta).$$

For compact operators A we have the following characterization.

Proposition 7. *Suppose that A is a compact operator and $H := A^*A$. The function $\bar{F}_{M(G)}$ is right-continuous at zero exactly if the operator G is compact.*

Proof. Since the operator H is self-adjoint and compact it has a monotone Schmidt representation (svd) in the form $Hx = \sum_{j=1}^{\infty} s_j \langle x, u_j \rangle u_j$, $x \in X$ with $\|H\| = s_1 \geq s_2 \geq \dots \geq 0$, and orthonormal system u_j , $j = 1, 2, \dots$. Then the operator $\chi_{(0,t]}(H)$ is a finite co-dimensional orthogonal projection, and hence it can be written as $\chi_{(0,t]}(H) = I - P_N$, where P_N is the projection onto the finite dimensional space $X_N := \text{span}(\{u_j, s_j > t\})$. Thus, we see that $\sup_{x \in M(G)} F_x(t) = \|(I - P_N)G\|$. As $t \searrow 0$ the dimension of the spaces X_N will increase to $+\infty$. From this we conclude that $\sup_{x \in M(G)} F_x(t) \rightarrow 0$ if and only if the operator G is approximable by finite rank operators, and thus compact. In Hilbert space, as a consequence of its (metric) approximation property, the notions of compactness and approximability coincide, see e.g. [9, Chapt. 10]. \square

Conjecture 2. *Suppose that A is compact and that the set $M(G)$ is an ellipsoid generated by the operator G , see (10). The modulus of continuity $\omega_{M(G)}$ is right-continuous at zero if and only if G is compact.*

Remark 4. For commuting operators G and H this holds true, in view of Proposition 6.

4. ON THE CONCAVITY OF THE MODULUS OF CONTINUITY

Now, under the setting and notation of the previous section we (re)prove (see [4, Theorem]) the concavity of the function $\omega_M^2(\sqrt{\delta})$, $0 \leq \delta < \infty$, for $M = \mathcal{M}_\varphi$, and $M = \mathcal{E}_\varphi$ (for an index function φ , see Examples 3 and 4), by using some tools of convex analysis.

For the beginning we notice that, according to the spectral theorem for bounded self-adjoint linear operators in Hilbert spaces (see [10, Chapt. VII]), there exist a measurable space $(\Omega, \mathcal{A}, \mu)$, a unitary transformation $U : X \rightarrow L^2(\Omega, \mathcal{A}, \mu)$ and a measurable function

$$f : \Omega \rightarrow \sigma(H) \setminus \{0\} \subseteq (0, \|H\|) \subset \mathbb{R}$$

such that $T_f := UHU^* : L^2(\Omega, \mathcal{A}, \mu) \rightarrow L^2(\Omega, \mathcal{A}, \mu)$ is a multiplication operator defined as

$$[T_f h](\omega) := f(\omega) h(\omega) \text{ for all } \omega \in \Omega.$$

By [4, Proposition 3] we get for an arbitrary set $M \subseteq X$ that

$$(11) \quad \omega_M(\delta) := \omega_{M,A}(\delta) = \omega_{M,H^{\frac{1}{2}}}(\delta) = \omega_{UM,T_f}(\delta) \text{ for all } \delta \geq 0.$$

In the formula above the second lower index in the modulus of the continuity denotes the bounded linear operator to which this is associated, while $UM := \{Ux : x \in M\}$ stands for the image of the set M through the operator U .

For the images of the sets \mathcal{M}_φ and \mathcal{E}_φ through the unitary transformation U we have, according to [4, Lemma 1], the representations

$$UM_\varphi = \{g \in L^2(\Omega, \mathcal{A}, \mu) : g = \varphi(f)h, \|h\|_{L^2(\Omega, \mathcal{A}, \mu)} \leq 1\},$$

and, respectively,

$$UE_\varphi = \left\{ g \in L^2(\Omega, \mathcal{A}, \mu) : \int_{0 < f(\omega) \leq t} g^2(\omega) d\mu(\omega) \leq \varphi^2(t) \quad \forall t \in (0, a] \right\}.$$

The main result of this section follows.

Theorem 3. *Let $A : X \rightarrow Y$ be an injective and bounded linear operator with non-closed range $\mathcal{R}(A)$ mapping between separable Hilbert spaces X and Y and let $\varphi : (0, \|A\|^2] \rightarrow (0, +\infty)$ be an arbitrary index function. Then the functions*

$$\delta \mapsto \omega_{\mathcal{M}_\varphi}^2(\sqrt{\delta})$$

and

$$\delta \mapsto \omega_{\mathcal{E}_\varphi}^2(\sqrt{\delta})$$

are concave on the interval $[0, +\infty)$.

Proof. To prove that $\delta \mapsto \omega_{\mathcal{M}_\varphi}^2(\sqrt{\delta})$ is concave on $[0, +\infty)$, we let be $\Theta(t) := \sqrt{t} \varphi(t)$ for all $0 < t \leq a$. According to (11) and by making use of the representation given for $U\mathcal{M}_\varphi$ above, it holds, for all $\delta \geq 0$,

$$(12) \quad \begin{aligned} -\omega_{\mathcal{M}_\varphi}^2(\sqrt{\delta}) &= \inf \left\{ -\|g\|_{L^2(\Omega, \mathcal{A}, \mu)}^2 : g = \varphi(f)h, \|h\|_{L^2(\Omega, \mathcal{A}, \mu)} \leq 1, \|\sqrt{f}g\|_{L^2(\Omega, \mathcal{A}, \mu)}^2 \leq \delta \right\} \\ &= \inf \left\{ -\|\varphi(f)h\|_{L^2(\Omega, \mathcal{A}, \mu)}^2 : \|h\|_{L^2(\Omega, \mathcal{A}, \mu)}^2 \leq 1, \|\Theta(f)h\|_{L^2(\Omega, \mathcal{A}, \mu)}^2 \leq \delta \right\}. \end{aligned}$$

It will be convenient to consider the duality pairing $\langle \cdot, \cdot \rangle$ between $L^\infty(\Omega, \mathcal{A}, \mu)$ and $L^1(\Omega, \mathcal{A}, \mu)$. Because both the functions $\varphi(f)$ and $\Theta(f)$ are uniformly bounded, and the function $k := h^2 \geq 0$ belongs to $L^1(\Omega, \mathcal{A}, \mu)$, the representation (12) rewrites as

$$-\omega_{\mathcal{M}_\varphi}^2(\sqrt{\delta}) = \inf \left\{ \langle -\varphi^2(f), k \rangle : \langle 1, k \rangle \leq 1, \langle \Theta^2(f), k \rangle \leq \delta, k \geq 0 \right\},$$

which results in

$$-\omega_{\mathcal{M}_\varphi}^2(\sqrt{\delta}) = \inf_{k \in L^1(\Omega, \mathcal{A}, \mu)} \Psi(k, \delta),$$

where the function $\Psi : L^1(\Omega, \mathcal{A}, \mu) \times \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$ is given as

$$\Psi(k, \delta) = \begin{cases} \langle -\varphi^2(f), k \rangle, & \langle 1, k \rangle \leq 1, \langle \Theta^2(f), k \rangle \leq \delta, k \geq 0, \delta \geq 0, \\ \infty, & \text{otherwise.} \end{cases}$$

The function Ψ is a convex function in both variables, and this fact implies the convexity of marginal function $\delta \mapsto -\omega_{\mathcal{M}_\varphi}^2(\sqrt{\delta})$ (cf. [15, Theorem 2.1.3 (v)]), which further yields the concavity of $\omega_{\mathcal{M}_\varphi}^2(\sqrt{\cdot})$ on $[0, +\infty)$, and which completes the proof.

The proof of the the concavity of $\delta \mapsto \omega_{\mathcal{E}_\varphi}^2(\sqrt{\delta})$ is similar and hence omitted. \square

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