A note on the degree of ill-posedness for mixed differentiation on the d-dimensional unit cube

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Abstract

Numerical differentiation of a function over the unit interval of the real axis, which is contaminated with noise, by inverting the simple integration operator J mapping in L^2 is discussed extensively in the literature. The complete singular system of the compact operator J is explicitly given with singular values $\sigma_n(J)$ asymptotically proportional to $1/n$. This indicates a degree *one* of ill-posedness for the associated inverse problem of differentiation. We recall the concept of the degree of ill-posedness for linear operator equations with compact forward operators in Hilbert spaces. In contrast to the one-dimensional case, there is little specific material available about the inverse problem of mixed differentiation, where the d-dimensional analog J_d to J, defined over unit d-cube, is to be inverted. In this note, we show for that problem that the degree of ill-posedness stays at one for all dimensions $d \in \mathbb{N}$. Some more discussion refers to the two-dimensional case in oder to characterize the range of the operator J_2.

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1 Introduction and main results

For measuring the strength of ill-posedness of an operator equation

$$A\,x = y \qquad (x \in X,\ y \in Y)\,, \tag{1}$$

with an injective and *compact* linear operator $A \in \mathcal{L}(X,Y)$ mapping between the infinite dimensional and separable real Hilbert spaces X and Y, the concept the *degree of ill-posedness* $\mu = \mu(A) \in (0,\infty]$ has been developed. Since G. Wahba in 1980 distinguished

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in her paper [32] between *mildly, moderately and severely* linear ill-posed equation (1), this degree was used as one ingredient for characterizing the ill-posedness of linear inverse problems. For example, in [8, p. 31] $\mu \le 1$ designates mildly, $1 < \mu < \infty$ moderately, and $\mu = \infty$ severely ill-posed problems. With growing μ, the instability of the approximate solutions with respect to noisy data increases, because the condition numbers of $n \times n$-matrices arising from A in (1) by discretization are in the best case of order $\mathcal{O}(n^\mu)$ (cf., e.g., [34]). In combination with the smoothness of the solution x to (1), the value μ is a factor influencing error estimates and convergence rates in regularization (cf., e.g., [19, 21]). For more details in this context, example situations and alternative concepts, we also refer to [2, 4, 5, 9, 11, 14, 16, 22, 31].

In this note, we use a definition of the degree of ill-posedness along the lines of [10]:

Definition 1.1 *Let $\{\sigma_n(A)\}_{n=1}^\infty$ the non-increasing sequence of singular values of the injective and compact linear operator $A \in \mathcal{L}(X,Y)$, tending to zero as $n \to \infty$. Based on the well-defined* interval of ill-posedness *introduced as*

$$[\underline{\mu}(A), \overline{\mu}(A)] = \left[\liminf_{n\to\infty} \frac{-\log \sigma_n(A)}{\log n}\, ,\, \limsup_{n\to\infty} \frac{-\log \sigma_n(A)}{\log n}\right] \subset [0,\infty],$$

we say that the operator A, and respectively the associated operator equation (1)*, is* ill-posed of degree *$\mu = \mu(A) \in (0,\infty)$ if $\mu = \underline{\mu}(A) = \overline{\mu}(A)$, i.e., if the interval of ill-posedness degenerates into a single point.*

Evidently, if the singular value asymptotics of A is of the form $\sigma_n(A) \asymp n^{-\mu}$ [1], then this operator is ill-posed of degree $\mu > 0$. An important example of this type is the *simple integration operator* over the unit interval $J: X \to Y$ with $X = Y = L^2([0,1])$, frequently discussed in the literature, defined as

$$[Jx](s) := \int_0^s x(t)\,dt \qquad (0 \le s \le 1)\,, \tag{2}$$

with range $\mathcal{R}(J) = \{y \in H^1([0,1]) : y(0) = 0\}$, where solving the operator equation $Jx = y$ for $y \in \mathcal{R}(J)$ corresponds with the differentiation in the weak sense $x(t) = y'(t)$ a.e. for $t \in (0,1)$. For this operator J with adjoint J^*, one explicitly knows the *singular system* $\{\sigma_n(J); u_n(J); v_n(J)\}_{n=1}^\infty$ with complete orthonormal systems $\{u_n(J)\}_{n=1}^\infty$ in X and $\{v_n(J)\}_{n=1}^\infty$ in Y, satisfying $Ju_n(J) = \sigma_n(J)v_n(J)$ and $J^*v_n(J) = \sigma_n(J)u_n(J)$ for all $n \in \mathbb{N}$, as

$$\left\{\frac{2}{(2n-1)\pi}, \sqrt{2}\cos\left(n - \frac{1}{2}\right)\pi t\ (0 \le t \le 1); \sqrt{2}\sin\left(n - \frac{1}{2}\right)\pi t\ (0 \le t \le 1)\right\}_{n=1}^\infty . \tag{3}$$

Thus we have a strictly decreasing sequence of singular values of J with $\sigma_n(J) \sim \frac{1}{n\pi}$, hence with $\sigma_n(J) \asymp n^{-1}$, and the ill-posedness degree of J is one.

[1] We use the notation $a_n \asymp b_n$ for sequences of positive numbers a_n and b_n satisfying inequalities $\underline{c}\, b_n \le a_n \le \overline{c}\, b_n$ for constants $0 < \underline{c} \le \overline{c} < \infty$ for sufficiently large $n \in \mathbb{N}$. If moreover $\lim_{n\to\infty} a_n/b_n = 1$, we write $a_n \sim b_n$. If the quotients a_n/b_n are only bounded from above by a constant, then we write $a_n = \mathcal{O}(b_n)$.

The injective and compact operator $J_d : X \to Y$ mapping in the infinite dimensional real Hilbert space $X = Y = L^2([0,1]^d)$ and restricted to functions defined over the d-dimensional unit cube as

$$[J_d\, x](s_1, \ldots, s_d) := \int\limits_0^{s_1}\int\limits_0^{s_2} \ldots \int\limits_0^{s_d} x(t_1, t_2, \ldots, t_d)\, dt_d \ldots dt_2 dt_1 \tag{4}$$

is a d-dimensional analog to J, where solving the operator equation $J_d\, x = y$ for $y \in \mathcal{R}(J_d)$ corresponds with the *mixed differentiation* in the weak sense as

$$x(s_1, \ldots, s_d) = \frac{\partial^d}{\partial s_1 \ldots \partial s_d} y(s_1, \ldots, s_d) \quad \text{a.e. for} \quad (s_1, \ldots, s_d) \in [0,1]^d.$$

Solving an operator equation (1) with $A := J_d$ mapping in $L^2([0,1]^d)$ occurs in statistics when y represents a d-dimensional copula, from which an associated d-dimensional *copula density* x is to be reconstructed, and we refer for details to [20] and [23, Chap. 2.4], but with respect to numerical challenges for growing dimensions d in the context of this inverse problem to the report [29].

The following theorem represents the main result of this note.

Theorem 1.2 *For all $d \in \mathbb{N}$, the operator $J_d : L^2([0,1]^d) \to L^2([0,1]^d)$ defined by formula* (4) *is ill-posed of degree one.*

The proof of Theorem 1.2 will be given in Section 3, based on a proposition that characterizes with some more discussion the shape $\frac{(\log n)^{d-1}}{n}$ for the asymptotics of the non-increasingly ordered singular values $\sigma_n(J_d)$ of the compact operator J_d from (4) mapping in $L^2([0,1]^d)$. Required auxiliary results for that reason are presented in Section 2 and proven in the final Section 5. The particular case of dimension $d = 2$ will be outlined with respect to range properties in Section 4.

2 Preliminaries

In this section, we present in form of Proposition 2.1 and of the three Lemmas 2.2, 2.3 and 2.4 auxiliary results, which are substantial for the proof of Theorem 1.2 in Section 3. The proofs of those auxiliary results are given in Section 5.

Proposition 2.1 *Based on the strictly decreasing sequence $\{\sigma_n(J)\}_{n=1}^\infty$ taken from* (3) *of singular values of the operator J defined in* (2)*, we can fully characterize the set of singular values of the operator J_d defined in* (4) *as*

$$\{\tilde{\sigma}_{i_1 i_2 \ldots i_d}(J_d)\}_{i_1, i_2, \ldots, i_d = 1}^\infty = \{\sigma_{i_1}(J)\sigma_{i_2}(J) \ldots \sigma_{i_d}(J)\}_{i_1, i_2, \ldots, i_d = 1}^\infty . \tag{5}$$

Note that in the proof of Proposition 2.1 in Section 5 also the singular functions of J_d are explicitly given.

In coincidence with the reasons stated in the introduction, our focus is on the degree of ill-posedness of the operator J_d, which requires determining the decay rate of its singular values. For this purpose, the singular values in (5) must be arranged in decreasing order as $\|J_d\|_{\mathcal{L}(L^2([0,1]^d))} = \sigma_1(J_d) \geq \sigma_2(J_d) \geq \ldots \geq \sigma_n(J_d) \geq \sigma_{n+1}(J_d) \geq \ldots \to 0$ as $n \to \infty$. To verify the singular value asymptotics expressed by Proposition 3.1 as basis for the main result of this note, the following three lemmas are of importance.

Lemma 2.2 *For $d \in \mathbb{N} := \{1, 2, \ldots\}$ and sufficiently large x we have the formulas*

$$\sum_{i \leq x} \frac{1}{i} (\log i)^{d-1} = \frac{1}{d} (\log x)^d + \mathcal{O}(1) \tag{6}$$

and

$$\sum_{i \leq x} \frac{1}{i} \left(\log\left(\frac{x}{i}\right)\right)^{d-1} = \frac{1}{d} (\log x)^d + \mathcal{O}\left((\log x)^{d-1}\right), \tag{7}$$

where the running index i can take values in $\mathbb{N}$.

Lemma 2.3 *Let $\{\sigma_i\}_{i=1}^{\infty}$ be a non-increasing sequence of positive numbers tending to zero with the asymptotics $\sigma_i \asymp i^{-1}$ and let us define the function*

$$n_d(x) := \#\{(i_1, \ldots, i_d) : x\, \sigma_{i_1} \ldots \sigma_{i_d} \geq 1\}. \tag{8}$$

Then we have for all $d \in \mathbb{N}$ the asymptotic property

$$n_d(x) \asymp x\, (\log x)^{d-1} \quad \text{as} \quad x \to \infty\,. \tag{9}$$

Note that for the model case $\sigma_i = \frac{1}{i}$, the function $n_d(x)$ counts the integer d-tuples $(i_1, \ldots, i_d)$ satisfying $i_1 i_2 \cdots i_d \leq x$. In the two-dimensional case $d = 2$, the value of $n_d(x)$ then coincides with the sum of numbers of divisors of all positive integers $\leq x$. The study of the asymptotical expansion of such divisor sums as $x \to \infty$ has a long history and is known as Dirichlet's division approximation, see, e.g. [7, Theorem 318].

Lemma 2.4 *If, for $d \in \mathbb{N}$,*

$$n = n(x) \asymp x\, (\log x)^d \quad \text{as} \quad x \to \infty, \tag{10}$$

then we have

$$x \asymp \frac{n}{(\log n)^d} \quad \text{as} \quad x \to \infty. \tag{11}$$

3 Proof of the theorem and some discussion

We start with a proposition, which acts a basic ingredient for the proof of the theorem.

Proposition 3.1 *For $d \in \mathbb{N}$ and the operator $J_d : L^2([0,1]^d) \to L^2([0,1]^d)$ defined by formula (4), we have the singular value asymptotics*

$$\sigma_n(J_d) \asymp \frac{(\log n)^{d-1}}{n} \quad \text{as} \quad n \to \infty\,. \tag{12}$$

Proof: For $d = 1$ all is clear. Therefore, we let $d \geq 2$ and set $\sigma_i := \sigma_i(J)$ for all $i \in \mathbb{N}$ in the sense of Lemma 2.3. Then we can use, due to the formula (5) from Proposition 2.1, the number function $n_d(x)$ from formula (8) of Lemma 2.3 with the asymptotic property $n_d(x) \asymp x\,(\log x)^{d-1}$ as $x \to \infty$ of formula (9). Now $n_d(x)$ counts, for the non-increasingly ordered singular values $\sigma_n(J_d)$ of the operator J_d, the number of singular values obeying the condition $1/\sigma_n(J_d) \leq x$. Thus, we have the obvious inequality

$$n_d(1/\sigma_n(J_d)) = \max\{m \in \mathbb{N} : \sigma_m(J_d) = \sigma_n(J_d)\} \geq n. \tag{13}$$

Since, for sufficiently large $x > 0$,

$$n_d(x) \leq \overline{c}\, x\,(\log x)^{d-1}$$

for some positive constant $\overline{c}$, we see by Lemma 2.4 that

$$1/\sigma_n(J_d) \leq c_1\, \frac{n}{(\log n)^{d-1}}\,,$$

with some positive constant c_1 and for sufficiently large $n \in \mathbb{N}$. If we approach the possible jump at $1/\sigma_n(J_d)$ from the other side, we see that

$$\lim_{x \to 1/\sigma_n(J_d)-} n_d(x) < n. \tag{14}$$

Consequently, since

$$n_d(x) \geq \underline{c}\, x\,(\log x)^{d-1},$$

with some $\underline{c} > 0$ and for sufficiently large $x > 0$, this provides us with

$$1/\sigma_n(J_d) \geq c_2\, \frac{n}{(\log n)^{d-1}}$$

for some constant $c_2 > 0$ in the same way. Together, we then have $\sigma_n(J_d) \asymp \frac{(\log n)^{d-1}}{n}$ as $n \to \infty$, which yields the singular value asymptotics (12) of the operator J_d and completes the proof of Proposition 3.1. □

Proof of Theorem 1.2: From formula (12) of Proposition 3.1 we simply derive the limit condition

$$\lim_{n\to\infty} \left(\frac{-\log\left(\frac{(\log n)^{d-1}}{n}\right)}{\log n} \right) = 1\,.$$

This, however, indicates in the sense of Definition 1.1 the degree of ill-posedness one for J_d and confirms the assertion of the theorem. □

Remark 3.2 We note that the above given rather simple proof our asymptotics formula (12) is based on the fact that for J and J_d the singular systems are explicitly available using the multipliability of singular values in the sense of Proposition 2.1. Indeed, this asymptotics result is not new and can be found, but without fully comprehensible proof, as a special case of Proposition 5.4 in [17] for $\alpha_i = 1$ $(i = 1, 2, ..., d)$ when considering Riemann-Liouville operators of fractional integration. An asymptotics of the shape $\frac{(\log n)^{d-1}}{n}$ as $n \to \infty$ also occurs for related s-numbers of compact operators in similar contexts. So we have this asymptotics (cf. [27, p. 197]) for the n-the entropy number e_n of an embedding operator from the Hilbert-type Sobolev space $S_2^1 W([0,1]^d)$ of dominating mixed smoothness up to derivatives of order one into $L^2([0,1]^d)$ as well as (cf. [30, p. 192]) for the Kolmogorov n-width d_n for periodic functions over the d-dimensional torus $\mathbb{T}^d$.

Remark 3.3 In many multivariate inverse problems, the degree of ill-posedness strongly depends on the spatial dimension d. For example, this applies for the compact embedding operator $A = \mathcal{E}_d : H^p([0,1]^d) \to L^2([0,1]^d)$ $(d = 1, 2, \dots)$ with the singular value asymptotics $\sigma_n(\mathcal{E}_d) \asymp n^{-p/d}$ (cf. [15, § 3c]). In contrast, Theorem 1.2 shows that the dimension d does not influence the ill-posedness degree of J_d. One reason for this effect may be that in the d-dimensional case the element $y \in \mathcal{R}(J_d)$ is (weakly) d-times differentiable and consequently the smoothness of y grows in coincidence with d. However, if we try to find a constant $K(d)$ such that

$$\sigma_n(J_d) \sim K(d)\,\frac{(\log n)^{d-1}}{n}$$

holds, then we could verify $K(d) = (d-1)!$ under the stronger assumption $\sigma_n(J) \sim n^{-1}$. This is a consequence of equation (21) in the proof of Lemma 2.3 below. Hence, for moderate size of n but large dimension d the constant $K(d)$ dominates the reconstruction error, and the expected asymptotics cannot be seen. Therefore, the problem of mixed differentiation can be *intractable* (cf., e.g., [24, § 4.4]), even if the degree of ill-posedness is small.

□

4 The special case of mixed differentiation in the two-dimensional case

For the two-dimensional case of mixed differentiation, we have to solve the operator equation (1) with the injective and compact forward operator $J_2 : X \to Y$ defined as

$$[J_2\, x](s_1, s_2) := \int_0^{s_1}\int_0^{s_2} x(t_1, t_2)\, dt_2 dt_1\,, \tag{15}$$

mapping in the real Hilbert space $X = Y = L^2([0,1]^2)$, where solving (1) with $A := J_2$ requires to verify from $y \in \mathcal{R}(J_2)$ the *mixed second derivative* $x(s_1, s_2) = \frac{\partial^2}{\partial s_1 \partial s_2} y(s_1, s_2)$ a.e. for $(s_1, s_2) \in [0,1]^2$. Some essential properties of the range $\mathcal{R}(J_2)$ are given by the following proposition.

Proposition 4.1 *The range $\mathcal{R}(J_2)$ of the operator $J_2 : L^2([0,1]^2) \to L^2([0,1]^2)$ defined in (15) is a subset S of both the Sobolev space $H^1([0,1]^2)$ and the Hölder space $C^{0,\beta}([0,1]^2)$ with Hölder exponent $\beta = 0.5$, additionally satisfying boundary conditions in the sense of*

$$S = \{y \in H^1([0,1]^2) \cap C^{0,0.5}([0,1]^2) :\ y(s_1, 0) = 0\ (0 \le s_1 \le 1),\ y(0, s_2) = 0\ (0 \le s_2 \le 1).\} \tag{16}$$

Proof: The partial derivatives of $y(s_1, s_2) = [J_2(x)](s_1, s_2)$ with respect to s_1 and s_2 belong to $L^2([0,1]^2)$, because $\|x\|^2_{L^2([0,1]^2)} < \infty$ is a majorant of the L^2-norm squares of $\frac{\partial}{\partial s_1} y$ and $\frac{\partial}{\partial s_2} y$. Moreover, the boundary conditions are evident. Then it remains to show the Hölder continuity of y with Hölder exponent $1/2$. Now we can estimate $|y(s_1, s_2) - y(\tilde{s}_1, \tilde{s}_2)|$ from above, for all $0 \le s_1, s_2, \tilde{s}_1, \tilde{s}_2 \le 1$, by the sum

$$\left| \int_0^{s_1} \left(\int_0^{s_2} x(t_1, t_2) dt_2 - \int_0^{\tilde{s}_2} x(t_1, t_2) dt_2 \right) dt_1 \right| + \left| \int_0^{\tilde{s}_2} \left(\int_0^{s_1} x(t_1, t_2) dt_1 - \int_0^{\tilde{s}_1} x(t_1, t_2) dt_1 \right) dt_2 \right|.$$

This yields the inequalities

$$|y(s_1,s_2)-y(\tilde{s}_1,\tilde{s}_2)| \le \int\limits_0^{s_1}\int\limits_{\min(s_2,\tilde{s}_2)}^{\max(s_2,\tilde{s}_2)} |x(t_1,t_2)|\,dt_2dt_1 + \int\limits_0^{\tilde{s}_2}\int\limits_{\min(s_1,\tilde{s}_1)}^{\max(s_1,\tilde{s}_1)} |x(t_1,t_2)|\,dt_1dt_2$$

and

$$|y(s_1,s_2)-y(\tilde{s}_1,\tilde{s}_2)| \le \int\limits_0^{1}\int\limits_{\min(s_2,\tilde{s}_2)}^{\max(s_2,\tilde{s}_2)} |x(t_1,t_2)|\,dt_2dt_1 + \int\limits_0^{1}\int\limits_{\min(s_1,\tilde{s}_1)}^{\max(s_1,\tilde{s}_1)} |x(t_1,t_2)|\,dt_1dt_2\,, \qquad (17)$$

which allows us to use the Cauchy–Schwarz inequality in the form

$$\begin{aligned}|y(s_1,s_2)-y(\tilde{s}_1,\tilde{s}_2)| &\le \left(|s_1-\tilde{s}_1|^{1/2}+|s_2-\tilde{s}_2|^{1/2}\right)\|x\|_{L^2([0,1]^2)}\\ &\le \sqrt{2}\|x\|_{L^2([0,1]^2)}\left\|\begin{pmatrix}s_1\\ s_2\end{pmatrix}-\begin{pmatrix}\tilde{s}_1\\ \tilde{s}_2\end{pmatrix}\right\|_1^{1/2}.\end{aligned}$$

This completes the proof. □

Remark 4.2 In the context of Proposition 4.1 let us mention that $\mathcal{R}(J_2)$ is a subspace of the *Sobolev space* $S_2^1W([0,1]^2)$ *of dominating mixed smoothness* up to derivatives of order one, but not a subspace of $H^2([0,1]^2)$, because for factored functions $x(t_1,t_2)=x_1(t_1)x_2(t_2)$ with $x_i \notin H^1([0,1])$, the second derivatives $\frac{\partial^2}{\partial {s_i}^2}J_2(x)$ do not belong to $L^2([0,1]^2)$. On the other hand, the set S from (16), including a specific Hölder continuity property, does not coincide with $\mathcal{R}(J_2)$, as the even Lipschitz continuous function $y(s_1,s_2)=\sqrt{s_1s_2}$ shows, for which the mixed derivative $\frac{\partial^2}{\partial s_1\partial s_2}y(s_1,s_2)=\frac{1}{4\sqrt{s_1s_2}}$ is not in $L^2([0,1]^2)$. This contrasts the situation of an extension of J_2 to the larger domain $L^1([0,1]^2)$, where one can conclude from [3, Theorem 4] that its range can be fully characterized by the subspace

$$\tilde{S}=\{y\in AC([0,1]^2):\, y(s_1,0)=0\ (0\le s_1\le 1),\ y(0,s_2)=0\ (0\le s_2\le 1)\}$$

of the space $AC([0,1]^2)$ of absolutely continuous functions over $[0,1]^2$.

On the other hand, a restriction of the domain of J_2 may lead to higher smoothness of the range elements $J_2\,x$, since stronger assumptions are imposed there on the function x. From formula (17) above it becomes evident that $J_2\,x$ is a Lipschitz continuous function if $x\in L^\infty([0,1]^2)$. Also if x is a copula density, we have $J_2\,x\in \mathrm{Lip}([0,1]^2)$, even with the uniform Lipschitz constant one, see [20, Lemma 1.2]. □

From Proposition 3.1, we immediately obtain the singular value asymptotics

$$\sigma_n(J_2)\asymp\frac{\log n}{n}\quad \text{as}\quad n\to\infty\,, \qquad (18)$$

implying a degree of ill-posedness one for the inverse problem of mixed differentiation in the two-dimensional case. Even if the degree of ill-posedness is only one, some kind of regularization is required in order to find stable approximate solutions to the ill-posed linear operator equation (1) with $A := J_2$ mapping in $X = Y = L^2([0,1]^2)$. For the one-dimensional case with $A := J$ in $X = Y = L^2([0,1])$, the forward operator is monotone and Lavrentiev's regularization method (cf., e.g., [18]) applies, because we have for the inner product $\langle Jx, x\rangle_{L^2([0,1])} \ge 0$ for all $x \in L^2([0,1])$. This, however, is not the case for $A = J_2$. Indeed, there exist functions $x \in L^2([0,1]^2)$ with negative values $\langle J_2 x, x\rangle_{L^2([0,1]^2)}$ and variational or iterative regularization methods are the means of choice (cf., e.g., [1, 5, 6, 12, 13, 26, 28]).

5 Proofs of auxiliary results

Proof of Proposition 2.1: We start by recalling the notation $\sigma_n = \sigma_n(J), u_n = u_n(J)$ and $v_n = v_n(J)$, $n = 1, 2, \ldots$, considered in Section 1. From those two complete orthonormal systems (u_n) and (v_n) of $L^2([0,1])$, we construct two orthonormal bases $(U_{i_1,\ldots,i_d})$ and $(V_{i_1,\ldots,i_d})$ of $L^2([0,1]^d)$ by considering tensor products:

$$\begin{aligned} U_{i_1,\ldots,i_d}(t_1,\ldots,t_d) &= u_{i_1}(t_1)u_{i_2}(t_2)\cdots u_{i_d}(t_d), \\ V_{i_1,\ldots,i_d}(t_1,\ldots,t_d) &= v_{i_1}(t_1)v_{i_2}(t_2)\cdots v_{i_d}(t_d), \qquad t_1,\ldots,t_d \in [0,1], \end{aligned}$$

for $i_1, i_2, \ldots, i_d = 1, 2, \ldots$. It may be concluded, e.g., from [25, example following Proposition 2, Section II.4] or [33, Theorem 3.8], in combination with induction over d, that this in fact yields complete orthonormal systems of $L^2([0,1]^d)$, respectively. Now, the statement of the proposition follows from the following computations:

$$\begin{aligned} [J_d U_{i_1,\ldots,i_d}](s_1,\ldots,s_d) &= \int_0^{s_1}\int_0^{s_2}\ldots\int_0^{s_d} U_{i_1,\ldots,i_d}(t_1,t_2,\ldots,t_d)dt_d\ldots dt_2 dt_1 \\ &= \int_0^{s_1} u_{i_1}(t_1)dt_1 \int_0^{s_2} u_{i_2}(t_2)dt_2 \cdots \int_0^{s_d} u_{i_d}(t_d)dt_d = [Ju_{i_1}](s_1)[Ju_{i_2}](s_2)\cdots[Ju_{i_d}](s_d) \\ &= \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_d} V_{i_1,\ldots,i_d}(s_1,\ldots,s_d), \qquad s_1,\ldots,s_d \in [0,1]. \end{aligned}$$

This shows that $\{\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_d}; U_{i_1,\ldots,i_d}; V_{i_1,\ldots,i_d}\}_{i_1,\ldots,i_d=1}^{\infty}$ defines a singular value decomposition of the tensor product J_d, with unsorted singular values in fact. $\square$

Proof of Lemma 2.2: We first note that

$$(\log i)^d - (\log(i-1))^d = \frac{d}{i}(\log i)^{d-1} + \mathcal{O}\left(\frac{1}{i^2}(\log i)^{d-1}\right)$$

holds, which follows by Taylor expansion. Since the series $\sum_{i=1}^{\infty} \frac{1}{i^2}(\log i)^{d-1}$ is convergent, by summing up, we get

$$(\log n)^d = \sum_{i=1}^{n} \frac{d}{i}(\log i)^{d-1} + \mathcal{O}(1).$$

For not necessarily integer $x = n$, (6) is still valid, because of

$$(\log x)^d - (\log\lfloor x\rfloor)^d = \mathcal{O}(1) \quad \text{as} \quad x \to \infty.$$

To prove the formula (7), we consider the equation

$$\log\left(\frac{x}{i}\right)^{d-1} = (\log x - \log i)^{d-1} = \sum_{j=0}^{d-1} \binom{d-1}{j} (-1)^j (\log x)^{d-1-j} (\log i)^j,$$

stating that the leading term will be due to (6) as

$$(\log x)^d \sum_{j=0}^{d-1} \binom{d-1}{j} \frac{(-1)^j}{j+1} = \frac{1}{d} (\log x)^d,$$

by taking into account that

$$\sum_{j=0}^{d-1} \binom{d-1}{j} \frac{(-1)^j}{j+1} = \int_0^1 (1-u)^{d-1}\, du = \frac{1}{d}.$$

This completes the proof of the lemma. □

Proof of Lemma 2.3: First we note that $n_d(x)$ introduced in (8) is an increasing function for $x > 0$, and we have $n_d(x) = 0$ for $x < 1/\sigma_1^d$ and all $d \in \mathbb{N}$. From the definition of n_d, we immediately get the recursion

$$n_{d+1}(x) = \sum_{i\in\mathbb{N}} n_d(x\sigma_i) \quad (d = 1, 2, \ldots). \tag{19}$$

The sum in (19) is a finite one, because $n_d(x)$ vanishes for $x > 0$ small enough. By definition we have $n_1(x) = n$ for each n, x satisfying $\sigma_{n+1} < 1/x \le \sigma_n$. Thus $\sigma_n \asymp n^{-1}$ implies $n_1(x) \asymp x$ as $x \to \infty$. Consequently, the maximal index in the sum of (19) possesses also the asymptotics $\asymp x$.

Now for $d = 1$, equation (19) attains the form

$$n_2(x) \asymp \sum_{i\le x} \frac{x}{i} \asymp x \log x \quad \text{as} \quad x \to \infty\,.$$

For general dimensions $d \ge 2$ we obtain

$$n_d(x) \asymp x\,(\log x)^{d-1} \quad \text{as} \quad x \to \infty\,, \tag{20}$$

and we can prove this by induction noting that we already have that this asymptotics is valid for $d = 1$ and $d = 2$. Then assuming that the formula (20) is valid for level d, we make the jump to level $d + 1$ as

$$n_{d+1}(x) \asymp \sum_{i\le x} \frac{x}{i} \left(\log\left(\frac{x}{i}\right)\right)^{d-1} \asymp x\,(\log x)^d \quad \text{as} \quad x \to \infty\,,$$

by using the recursion formula (19) in combination with formula (7) from Lemma 2.2. We remark here that under the stronger assumption $\sigma_i \sim i^{-1}$ we even obtain

$$n_d(x) \sim \frac{1}{(d-1)!}\, x\,(\log x)^{d-1} \tag{21}$$

with the leading constant $\frac{1}{(d-1)!}$, since the induction step from n_d to n_{d+1} introduces a factor $\frac{1}{d}\log x$ due to (7). $\square$

Proof of Lemma 2.4: The asymptotic property

$$n = n(x) \asymp x\,(\log x)^d \quad \text{as} \quad x \to \infty$$

means

$$\log n = \log x + d\,\log\log x + \mathcal{O}(1),$$

i.e.

$$\log n \sim \log x.$$

This implies

$$x \asymp \frac{n}{(\log x)^d} \sim \frac{n}{(\log n)^d} \quad \text{as} \quad n \to \infty\,.$$

$\square$

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