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ERRORS OF REGULARISATION UNDER RANGE INCLUSIONS USING VARIABLE HILBERT SCALES

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Abstract. Based on the variable Hilbert scale interpolation inequality, bounds for the error of regularisation methods are derived under range inclusions. In this context, new formulae for the modulus of continuity of the inverse of bounded operators with non-closed range are given. Even if one can show the equivalence of this approach to the version used previously in the literature, the new formulae and corresponding conditions are simpler than the former ones. Several examples from image processing and spectral enhancement illustrate how the new error bounds can be applied.

1. Introduction.

Let X and Y be infinite dimensional separable Hilbert spaces with norms $\|\cdot\|$ and scalar products $(\cdot,\cdot)$. We study linear inverse problems which take the form of ill-posed operator equations

$$Af = g, \qquad f \in X,\ g \in Y, \tag{1}$$

characterised by an injective bounded linear forward operator $A : X \to Y$ for which the range, denoted by range(A), is a non-closed subset of Y. Then equation (1) is unstable in the sense that the inverse operator $A^{-1} : \text{range}(A) \subseteq Y \to X$ is unbounded. If one replaces the exact right-hand side g of equation (1) by perturbed data g^δ, satisfying

$$\|g - g^\delta\| \le \delta \tag{2}$$

for some (small) noise level $\delta > 0$, one may get arbitrarily large errors in the solution or no solution at all. As a consequence of this ill-posedness phenomenon, regularisation methods are required for the stable approximate solution of inverse problems. Their basic idea consists in finding approximations to the exact solution f in form of solutions $f_\alpha = f_\alpha(g^\delta)$ to stable auxiliary problems neighbouring (1). Those solutions are obtained by using the noisy data g^δ. The degree of neighbourhood of the exploited auxiliary problems is controlled by a regularisation parameter $\alpha > 0$. In this context, small α express closeness to (1) in combination with a low level of stability, whereas larger α ensure better stability, however combined

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with a low level of approximation. For the success of any regularisation method an appropriate trade-off between stability and approximation has to be found when choosing the regularisation parameter.

A successful way for doing regularisation for linear ill-posed problems in Hilbert spaces, which guarantees convergence and convergence rates of constructed methods, requires some knowledge about the impact of smoothness on the regularised solutions. Here smoothness is understood to mean both solution smoothness and smoothing properties of the operator A. For expressing solution smoothness the variable Hilbert scales suggested by HEGLAND in [13, 14] are a powerful tool and the consequences formulated there have introduced a new facet of regularisation theory. We are going to present a couple of new results on convergence rates derived from the interpolation inequality in variable Hilbert scales. In particular, we consider the chances of incorporating range inclusions in that context. Another goal of this paper is to compare Hegland's approach with an alternative approach developed and published by MATHÉ and PEREVERZEV (see [28, 29, 30]) and extended by HOFMANN and other co-workers (see, e.g., [10, 18, 19, 20]) also exploiting variable Hilbert scales.

The paper is organised as follows: In the second section we review the definition and some properties of index functions and variable Hilbert scales. The fundamental interpolation inequality is given together with an application to a general regularisation method. We then show how variable Hilbert scales provide natural source conditions. In the third section bounds for the modulus of continuity are given in a variable Hilbert scale setting. An important part of this section compares the new bounds on the modulus of continuity with some obtained earlier and shows how the new results have a substantially simpler structure. The fourth section analyses linear regularisation methods and parameter choices using the variable Hilbert scale approach. In order to illustrate the abstract theory we conclude the paper in Section 5 with several examples from image processing and spectral enhancement.

2. **Interpolation inequalities and consequences.** The main tool used here to derive error bounds for regularised solutions is an extension of *interpolation inequalities* to variable Hilbert scales. For classical Hilbert scales $\{X_r\}_{r\in\mathbb{R}}$ – with real numbers as scale index r – interpolation inequalities are well-established. These interpolation inequalities were initially applied to the treatment of linear ill-posed problems (1) by NATTERER in [34]. A detailed discussion can be found in the monograph [9, §8.4]). For variable Hilbert scales, new interpolation inequalities have to be formulated. A *variable Hilbert scale* as introduced in [13, 14] is a family of Hilbert spaces $\{X_\theta\}_{\theta\in\mathcal{I}}$ indexed by continuous functions $\theta : (0,\infty) \to (0,\infty)$. The index set $\mathcal{I}$ of the variable Hilbert scale is thus the set of all continuous positive functions defined for positive real numbers. This set of *index functions* $\mathcal{I}$ includes the positive constant functions and all power functions but not the zero function. For index functions ϕ and ψ, the functions $\phi+\psi$, $\phi\psi$ and ϕ/ψ are defined pointwise and are again index functions.

The original Hilbert scales only allowed the derivation of convergence rates of the form $O(\delta^\kappa)$ where δ is the norm of the data error. This is too limited, especially for mildly ill-posed problems like the computation of the derivative of a smooth function, and for very severely ill-posed problems arising when integral equations with smooth kernels are solved if the data is not sufficiently smooth. In both cases variable Hilbert scale theory overcomes the limitation and provides the framework for the derivation of convergence rates of the form $O(\overline{\theta}(\delta))$ where the *rate function* $\overline{\theta}(t)$ – defined and positive for all $0 < t < \infty$ and tending to zero as $t \to +0$ – is assumed to be continuous and monotonically increasing. Note that any monotonically increasing continuous function $\theta(t)$ defined on a finite interval $(0, t_0]$ satisfying $\lim_{t\to+0} \theta(t) = 0$ can be extended to a rate function $\bar{\theta}$ such that $\bar{\theta}(t) = \theta(t)$ for $t \in (0, t_0]$. Furthermore, the index functions corresponding to the classical Hilbert scales X_r can be seen to be power functions $\theta(\lambda) = \lambda^r$ for real r. Rate functions are obtained for this case if $r > 0$. We will in the following denote the index functions by lower case Greek letters and rate functions by overlined lower case Greek letters.

Variable Hilbert scales as introduced in [13, 14] are generated by an injective positive definite linear operator T. They are families $\{X_\theta\}_{\theta\in\mathcal{I}}$ of Hilbert spaces X_θ which themselves are defined as the completion of the domain $\mathcal{D}$ of the operator $\theta(T)$ with respect to the topology defined by the quadratic form

$$\|f\|_\theta^2 = (f, \theta(T)\, f). \tag{3}$$

In general, different operators T give rise to different variable Hilbert scales. In this paper, in the context of ill-posed problems (1) we often assume that T *is unbounded but has a bounded inverse*, i.e. the spectrum of T is contained in the interval $[\|T^{-1}\|^{-1}, \infty)$ and has $+\infty$ as an accumulation point. As the function $1/\lambda$ is an index function and the set of index functions is closed under composition, the inverse T^{-1} generates the same Hilbert scale as T. It is thus not necessary to consider variable Hilbert scales generated by invertible T and bounded T^{-1} separately. The more general case where both T and the inverse T^{-1} are unbounded is only considered for the negative Laplacian $T = -\Delta$ and in particular $T = -d^2/dt^2$. For the more general case where also A is unbounded we refer to the recent paper [20]. To get a link with (1), a particular T is suggested with respect to the forward operator A. A common choice is $T = (A^*A)^{-1}$ for injective operators A with a non-closed range. It follows that $A^*A = \theta(T)$ if $\theta(\lambda) = 1/\lambda$. For classes of problems connected with deconvolution, however, $T = -d^2/dx^2$ on $L_2(\mathbb{R})$ is the canonical choice as T is the generator of symmetric convolutions. An index function θ such that $A^*A = \theta(T)$ is then found using Fourier transforms. More generally, for problems where the source conditions relate to smoothness, $T = -\Delta$ can be chosen.

The most important connection between the norms of different spaces X_θ is the interpolation inequality for variable Hilbert scales. This inequality is stated in the following lemma. A proof can essentially be found in [14] and is also given in an extended version [16] of the current paper (which also reviews other variable Hilbert scale properties).

Lemma 2.1 (Interpolation inequality). *Let T be an unbounded injective self-adjoint positive definite linear operator densely defined on the Hilbert space X with bounded inverse $T^{-1} : X \to X$. Moreover let ϕ, ψ, θ and Ψ be index functions such that Ψ is concave and*

$$\phi(\lambda) \leq \Psi(\psi(\lambda)), \quad \textit{for } \|T^{-1}\|^{-1} \leq \lambda < \infty. \tag{4}$$

Then for any element $0 \neq f \in X_\theta \cap X_{\psi\theta}$ one gets $f \in X_{\phi\theta}$ and

$$\frac{\|f\|_{\phi\theta}^2}{\|f\|_\theta^2} \leq \Psi\left(\frac{\|f\|_{\psi\theta}^2}{\|f\|_\theta^2}\right). \tag{5}$$

The concavity of Ψ is the key property which enables us to use Jensen's inequality. We will see below that concavity only needs to hold for large arguments. We can focus on large arguments, if the spectrum of T under consideration contains only sufficiently large values and has $+\infty$ as an accumulation point. We need some auxiliary result:

Lemma 2.2. *If $\theta : [t_0, \infty) \to (0, \infty)$ is concave for some $t_0 > 0$ then θ is monotonically increasing. If moreover $\lim_{t\to\infty} \theta(t) = \infty$, then θ is strictly increasing.*

Proof. Let $0 < t_1 < t_2$ and let $t > t_2$. Then $t_2 = (1-\lambda)t_1 + \lambda t$ where $\lambda = (t_2 - t_1)/(t - t_1)$. Using the concavity of θ, we have $\theta(t_2) \geq (1-\lambda)\theta(t_1) + \lambda\theta(t) \geq (1-\lambda)\theta(t_1)$. Taking the limit $t \to \infty$, we have $\lambda \to 0$ so that $\theta(t_2) \geq \theta(t_1)$. If $\theta(t_1) = \theta(t_2)$ then concavity yields $\theta(t_2) \geq \theta(t)$. Using the monotonicity one has $\theta(t) = \theta(t_2)$. Consequently $\theta(t) \to \theta(t_2) < \infty$ for $t \to \infty$. The strict increase follows by contraposition. □

We observe that for any index function θ which is concave on $[\lambda_0, \infty)$ for some $\lambda_0 > 0$ the index function Ψ defined by

$$\Psi(\lambda) = \begin{cases} \theta(\lambda), & \lambda_1 \leq \lambda < \infty, \\ \lambda\theta(\lambda_1)/\lambda_1, & 0 < \lambda \leq \lambda_1, \end{cases}$$

is concave on $(0, \infty)$.

The interpolation inequality is the main tool to obtain error bounds for solvers of linear ill-posed problems. However, by inspection it becomes clear that rate results derived from Lemma 2.1 are only based on the behaviour of $\Psi(\lambda)$ for large $\lambda \geq \lambda_1$. Without loss of generality Ψ can be amended for $0 < \lambda \leq \lambda_1$ by the linear function $\Psi(\lambda) = \Psi(\lambda_1)\lambda/\lambda_1$ for $0 < \lambda < \lambda_1$.

Three typical choices for $\Psi(\lambda)$ being concave at least for sufficiently large λ are $\Psi(\lambda) = \lambda^\kappa$ where $\kappa \in (0,1)$, $\Psi(\lambda) = \lambda/\log(\lambda)$ and $\Psi(\lambda) = \log(\lambda)$. For all three choices we have the limit condition

$$\lim_{\lambda\to\infty} \frac{\Psi(\lambda)}{\lambda} = 0 \tag{6}$$

and one gets the following versions of interpolation inequalities from Lemma 2.1:

- For $\Psi(\lambda) = \lambda^\kappa$ one gets
$$\|f\|_{\phi\theta} \leq \|f\|_\theta^{1-\kappa} \, \|f\|_{\psi\theta}^\kappa \,,$$
- for $\Psi(\lambda) = \lambda/\log(\lambda)$ one gets
$$\|f\|_{\phi\theta} \leq \frac{\|f\|_{\psi\theta}}{\sqrt{2\log(\|f\|_{\psi\theta}/\|f\|_\theta)}} \,,$$
- and for $\Psi(\lambda) = \log(\lambda)$ one has
$$\|f\|_{\phi\theta} \leq \|f\|_\theta \sqrt{2\log(\|f\|_{\psi\theta}/\|f\|_\theta}.$$

Asymptotically, i.e. for $\|f\|_\theta \to 0$, the interpolation inequality allows us to find error bounds in the application to the error estimation for the solution of equation (1). One aims to get bounds for the norm $\|f\|$ in X using values of the image norm $\|Af\|$ in Y and values of the norm $\|f\|_{\psi\theta}$ which expresses the specific additional smoothness of f. The terms in the interpolation inequality (5) are then

$$\|f\|_{\phi\theta} = \|f\| \ \text{ for } \ f \in X_{\phi\theta} \quad \text{and} \quad \|f\|_\theta = \|Af\| \ \text{ for } \ f \in X_\theta.$$

The first condition leads to $\phi(\lambda)\theta(\lambda) = 1$ for all λ and the second condition gives $\theta(T) = A^*A$ and with $\theta(\lambda) := 1/\lambda$ the relations $T = (A^*A)^{-1}$ and $\phi(\lambda) = \lambda$. We are still free to choose the index functions ψ and do it in the form $\psi(\lambda) := \chi(\lambda)\,\lambda$ with an appropriate index function χ.

For later use we add here some observations about concave functions which are stated as a lemma:

Lemma 2.3. *Let $\Psi : (0,\infty) \to (0,\infty)$ be a concave function. Then we have the following properties:*

(a) The function $\Xi : (0,\infty) \to (0,\infty)$ defined by

$$\Xi(\lambda) := \frac{\Psi(\lambda)}{\lambda}, \qquad 0 < \lambda < \infty \tag{7}$$

is monotonically decreasing.

(b) The function $\Phi : (0,\infty) \to (0,\infty)$ defined by

$$\Phi(\mu) := \mu\,\Psi\left(\frac{1}{\mu}\right), \qquad 0 < \mu < \infty \tag{8}$$

is concave and monotonically increasing.

Proof. (a) Let $0 < \lambda_0 < \lambda_1 < \lambda_2$. As Ψ is concave and positive one has

$$\begin{aligned}\Psi(\lambda_1) &\geq \frac{\lambda_1 - \lambda_0}{\lambda_2 - \lambda_0}\,\Psi(\lambda_2) + \frac{\lambda_2 - \lambda_1}{\lambda_2 - \lambda_0}\,\Psi(\lambda_0)\\ &\geq \frac{\lambda_1 - \lambda_0}{\lambda_2 - \lambda_0}\,\Psi(\lambda_2).\end{aligned}$$

As this holds for arbitrarily small $\lambda_0 > 0$ one has

$$\Psi(\lambda_1) \geq \frac{\lambda_1}{\lambda_2}\,\Psi(\lambda_2)$$

and consequently $\Xi(\lambda_1) \geq \Xi(\lambda_2)$. This proves assertion (a) of the lemma.

(b) Let $0 < \mu_0 < \mu_1 < \mu_2$ and $\lambda_i = 1/\mu_i$. Then one has $0 < \lambda_2 < \lambda_1 < \lambda_0$ and by the concavity of Ψ and some simple algebraic manipulations one gets

$$\begin{aligned}\frac{\mu_1 - \mu_0}{\mu_2 - \mu_0}\Phi(\mu_2) + \frac{\mu_2 - \mu_1}{\mu_2 - \mu_0}\Phi(\mu_0) &= \frac{\frac{1}{\lambda_1} - \frac{1}{\lambda_0}}{\frac{1}{\lambda_2} - \frac{1}{\lambda_0}}\,\frac{\Psi(\lambda_2)}{\lambda_2} + \frac{\frac{1}{\lambda_2} - \frac{1}{\lambda_1}}{\frac{1}{\lambda_2} - \frac{1}{\lambda_0}}\,\frac{\Psi(\lambda_0)}{\lambda_0}\\ &= \frac{1}{\lambda_1}\left(\frac{\lambda_0 - \lambda_1}{\lambda_0 - \lambda_2}\Psi(\lambda_2) + \frac{\lambda_1 - \lambda_2}{\lambda_0 - \lambda_2}\Psi(\lambda_0)\right)\\ &\leq \frac{1}{\lambda_1}\,\Psi(\lambda_1) = \Phi(\mu_1).\end{aligned}$$

It follows that Φ is concave and hence by Lemma 2.2 also increasing. This completes the proof of the lemma. ☐

Remark 1. We note here that the transformation $\mathcal{S} :\ \Psi \in \mathcal{I} \mapsto \Phi \in \mathcal{I}$ according to formula (8), applicable to every index function and *preserving concavity*, is an *involution*, that is, $\mathcal{S}^2$ is the identity map on $\mathcal{I}$, and hence $\mathcal{S}$ is bijective. If the concave index function Ψ satisfies $\lim\limits_{\lambda\to\infty} \Psi(\lambda) = \infty$, then by Lemma 2.2 the function is even strictly increasing and if, in addition, Ψ is a rate function, i.e., it satisfies the additional limit condition $\lim\limits_{\lambda\to+0} \Psi(\lambda) = 0$, the inverse function Ψ^{-1} is a well-defined and convex index function. If, on the other hand, the limit condition (6) holds, then we have

$$\lim_{\mu\to+0} \Phi(\mu) = \lim_{\mu\to+0} \mu\,\Psi(1/\mu) = \lim_{\lambda\to\infty} \Psi(\lambda)/\lambda = 0$$

and taking into account Lemma 2.3 (a) and (b) one sees that $\Phi = \mathcal{S}(\Psi)$ is a concave *rate function*. Vice versa we have that $\Psi = \mathcal{S}(\Phi)$ satisfies (6) whenever Φ is a rate function.

By inspection of the proof of Lemma 2.3 one can also see the following facts: If $\Psi(\lambda)$ is only concave for $\lambda \in [\lambda_0, \infty)$, then $\Phi(\mu) = [\mathcal{S}(\Psi)](\mu)$ is concave for $\mu \in (0, \mu]$ with $\mu_0 = 1/\lambda_0$. The involution $\mathcal{S}$ preserves also the convexity of an index function and if the concavity or convexity is strict, then the strictness carries over to the transformed function.

The following corollaries provide direct applications of the interpolation inequality to obtain error bounds for abstract regularisation methods. The proofs are essentially in [14], see also [16].

Corollary 2.4. *Let $A : X \to Y$ be an injective bounded linear operator with non-closed range mapping between the two Hilbert spaces X and Y. Furthermore let the variable Hilbert scale $\{X_\nu\}_{\nu\in I}$ be generated by $T = (A^*A)^{-1}$. Moreover let χ and Ψ be index functions and Ψ be concave such that*

$$\Psi\left(\chi(\lambda)\,\lambda\right) \ge \lambda \quad \text{for all} \;\; \lambda \in [\|T^{-1}\|^{-1}, \infty)\,. \tag{9}$$

If the solution f to (1) in addition satisfies the condition $f \in X_\chi$ and if $f_\alpha \in X_\chi$ is such that $f \neq f_\alpha$ then

$$\|f - f_\alpha\| \le \epsilon\,\sqrt{\Psi(\zeta^2/\epsilon^2)} \tag{10}$$

where $\epsilon = \|Af_\alpha - Af\|$ and $\zeta = \|f_\alpha - f\|_\chi$.

Results similar to those of Corollary 2.4 can be found for other choices of T, see for example Corollary 2.5 where $T = -d^2/dt^2$. The bound in the corollary guarantees that any method which defines some family of f_α which

- is stable in the sense that $\|f_\alpha\|_\chi$ is bounded uniformly in α and
- is consistent in the sense that the residuum $Af_\alpha \to Af$ for $\alpha \to 0$

is also convergent in the sense that $f_\alpha \to f$ for $\alpha \to 0$.

Corollary 2.5. *Let X_ν be the Hilbert scales generated by $T = -d^2/dt^2$ from $L^2(\mathbb{R})$. Furthermore, let $A : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be a (convolution) operator satisfying*

$$A^*A = \theta(T)$$

for some bounded index function θ. Moreover, let ϕ, ψ and Ψ be index functions and Ψ be concave such that

$$\phi(\lambda) \le \Psi(\psi(\lambda)), \quad \phi(\lambda)\theta(\lambda) = 1, \quad \text{for} \quad \lambda > 0.$$

If $Af = g \in X_\psi$ and if f_α is such that $Af_\alpha \in X_\psi$ and

$$\|Af_\alpha - g\|_\psi = \zeta$$
$$\|Af_\alpha - g\| = \epsilon$$

then

$$\|f - f_\alpha\| \le \epsilon\sqrt{\Psi(\zeta^2/\epsilon^2)}.$$

In comparison with Corollary 2.4 this corollary uses an operator T which is not necessarily equal to $(A^*A)^{-1}$ but more importantly, the source condition is here not given as a property of the solution f but of the data g.

Note that the error estimate (10) of Corollary 2.4 requires the essential conditions $f \in X_\chi$ and $f_\alpha \in X_\chi$, i.e. the approximate solutions f_α are constructed such that they obtain the same smoothness level with respect to T as the exact solution f. A next step for drawing conclusions of Lemma 2.1 will be formulated in Corollary 2.6 by assuming that f belongs to the ball

$$B_\chi(R) := \{h \in X_\chi : \ \|h\|_\chi \le R\} \tag{11}$$

in X_χ with radius $R = R_1$ and that the approximate solutions f_α for all $\alpha > 0$ under consideration belong to another such ball with radius $R = R_2$. Moreover, we consider for data g^δ satisfying (2) the limit process $\delta \to +0$ in correspondence with associated regularised solutions $f_\alpha = f_\alpha^\delta$, where the regularisation parameter $\alpha > 0$ is chosen either a priori as $\alpha = \alpha(\delta)$ or a posteriori as $\alpha = \alpha(\delta, g^\delta)$.

Corollary 2.6. *Under the setting of Corollary 2.4 let the limit condition (6) be satisfied and let $f \in B_\chi(R_1)$, $R_1 > 0$ and let $Af = g$. Let $\alpha = \alpha(\delta)$ be such that for some $\delta_{\max} > 0$ one has $f^\delta_{\alpha(\delta)} \in B_\chi(R_2)$ and*

$$0 < \|Af^\delta_{\alpha(\delta)} - g\| \le \bar{C}\,\bar{\xi}(\delta), \tag{12}$$

for all $\delta \in (0, \delta_{\max}]$ and for some rate function $\bar{\xi}$ and some constant $\bar{C} > 0$.

Then we have

$$\|f - f^\delta_{\alpha(\delta)}\| \le \bar{C}\bar{\xi}(\delta)\sqrt{\Psi\left(\left[\frac{R_1 + R_2}{\bar{C}\bar{\xi}(\delta)}\right]^2\right)}, \qquad 0 < \delta \le \delta_{\max}\,, \tag{13}$$

where the upper bound in (13) is a rate function, i.e., it tends to zero as $\delta \to 0$.

Proof. As $\|Af^\delta_{\alpha(\delta)} - g\| > 0$ one has $f^\delta_{\alpha(\delta)} \ne f$ and by Corollary 2.4 one has $\|f^\delta_{\alpha(\delta)} - f\| \le \epsilon\sqrt{\Psi(\zeta^2/\epsilon^2)}$ where $\epsilon = \|Af^\delta_{\alpha(\delta)} - g\|$ and $\zeta = \|f^\delta_{\alpha(\delta)} - f\|_\chi$. From the conditions and the triangle inequality one obtains $\|f^\delta_{\alpha(\delta)} - f\|_\chi \le R_1 + R_2$.

By Lemma 2.2 the function Ψ is monotonically increasing and consequently $\|f^\delta_{\alpha(\delta)} - f\| \le \epsilon\sqrt{\Psi((R_1+R_2)^2/\epsilon^2)}$. Substituting $\lambda = (R_1 + R_2)^2/\epsilon^2$ gives $\|f^\delta_{\alpha(\delta)} - f\| \le (R_1 + R_2)\sqrt{\Psi(\lambda)/\lambda}$. By Lemma 2.3 the expression $\Psi(\lambda)/\lambda$ is monotonically decreasing in λ.

By assumption $\epsilon = \|Af^\delta_{\alpha(\delta)} - g\| \le \bar{C}\bar{\xi}(\delta)$ and thus $\lambda \ge (R_1 + R_2)^2/(\bar{C}^2\bar{\xi}(\delta)^2)$. Using the monotonicity of $\Psi(\lambda)/\lambda$ leads to

$$\|f^\delta_{\alpha(\delta)} - f\| \le (R_1 + R_2)\sqrt{\frac{\Psi\left(\left(\frac{R_1+R_2}{\bar{C}\,\bar{\xi}(\delta)}\right)^2\right)}{\left(\frac{R_1+R_2}{\bar{C}\,\bar{\xi}(\delta)}\right)^2}}.$$

The claimed bound then follows directly. The upper bound in (13) is a function of δ which decreases to zero as $\bar{\xi}(\delta)$ decreases to zero. This is a direct consequence of the limit condition (6) which Ψ satisfies. □

Remark 2. We can extend the situation of Corollary 2.6 to aposteriori choices $\alpha_{\text{dis}} = \alpha_{\text{dis}}(\delta, g^\delta)$ for the regularisation parameter realised by a *discrepancy principle*

$$\|Af^\delta_{\alpha_{\text{dis}}} - g^\delta\| = C_{\text{dis}}\delta \tag{14}$$

with some prescribed $C_{\text{dis}} > 0$. Then by using the triangle inequality we obtain with (2) as noise model

$$\|Af^\delta_{\alpha_{\text{dis}}} - g\| \le \|Af^\delta_{\alpha_{\text{dis}}} - g^\delta\| + \|g^\delta - g\| \le (C_{\text{dis}} + 1)\,\delta = \bar{C}\,\delta\,.$$

Then for such $\alpha = \alpha_{\rm dis}$ under (6) the regularisation method converges strongly in X with the convergence rate

$$\|f - f_\alpha^\delta\| \,=\, \mathcal{O}\left(\delta\,\sqrt{\Psi(\bar{K}/\delta^2)}\right) \quad \text{as} \quad \delta \to 0 \tag{15}$$

for some constant $\bar{K} > 0$. Note that, in addition to the assumption $f \in B_\chi(R_1)$ on the solution smoothness, for that result the strong condition $f^\delta_{\alpha(\delta,g^\delta)} \in B_\chi(R_2)$, for all $\delta \in (0, \delta_{\max}]$ and all associated g^δ satisfying (2), is required.

The convergence rate in (15) depends only on the asymptotic behaviour of $\Psi(\lambda)$ as $\lambda \to \infty$. Thus the alteration of $\Psi(\lambda)$ for small λ has no influence on that rate. For the class of functions $\Psi(\lambda) = \lambda^\kappa$ with $0 < \kappa < 1$ rate functions proportional to $\delta^{1-\kappa}$ occur in (15). All those error rates are lower than the rate δ typically occurring for well-posed problems. It should be mentioned that $\Psi(\lambda) = \lambda$ fails to satisfy the condition (6) and in Corollary 2.4 does not yield a convergence rate.

To get a feeling for the role of the solution smoothness $f \in X_\chi$ we can study consequences of the inequality (9) as a hypothesis of Corollary 2.4 taking into account Lemma 2.2. One consequence of (9) is the limit condition $\lim_{\lambda\to\infty} \Psi(\lambda) = \infty$ for the function Ψ which is, because of its concavity, strictly increasing and invertible with convex $\Psi^{-1}(\lambda)$ also tending to infinity as $\lambda \to \infty$. Then (9) implies $\chi(\lambda) \geq \frac{\Psi^{-1}(\lambda)}{\lambda}$ for large λ. Under that condition (6) is equivalent to $\lim_{\lambda\to\infty} \Psi^{-1}(\lambda)/\lambda = \infty$. Hence, the index function $\chi(\lambda)$ tends to infinity for $\lambda \to \infty$ provided that (6) holds true.

For a non-closed range of A regularised solutions can only converge with a convergence rate that is expressed by a rate function $\bar{\theta}$ if some source condition is fulfilled, see [38]. *General source conditions* are in the standard case of the form

$$f = \bar{\psi}(A^*A)v \tag{16}$$

with some source element $v \in X$ and with some rate function $\bar{\psi}$. The standard case with monomials $\bar{\psi}(t) = t^\kappa,\ \kappa > 0$, was comprehensively discussed in the monograph by ENGL, HANKE and NEUBAUER [9], the logarithmic case $\bar{\psi}(t) = (\log(1/t))^{-p}$ with $p > 0$ was motivated and worked out in detail by HOHAGE, see [22, 23]. However, there exist numerous examples of inverse problems like parameter identification in PDEs or problems analysed in frequency space, see also Example 4 below, where $\bar{\psi}$ cannot be taken from the two classes mentioned above. Therefore the concept of general source conditions using index functions $\bar{\psi}$ other than monomials and logarithmic functions was early applied to regularisation theory by TAUTENHAHN in [40, 41]. Further progress including error estimates and convergence rates was, for example, obtained by NAIR and coauthors, see [31, 32].

When setting $\phi(\lambda) := \lambda,\ \theta(\lambda) := 1/\lambda$ and $\psi(\lambda) := \Psi^{-1}(\lambda)$ in the interpolation inequality (5) then the corresponding *regularity condition* $f \in X_{\psi\theta}$ is equivalent to a *source condition* (16) which expresses the specific smoothness of the solution f with respect to the forward operator A of equation (1).

Proposition 2.7. *Let $\Psi(\lambda)$, for $0 < \lambda < \infty$, be a concave and strictly increasing index function satisfying the limit conditions $\lim\limits_{\lambda\to+0} \Psi(\lambda) = 0, \quad \lim\limits_{\lambda\to\infty} \Psi(\lambda) = \infty$ and (6). Moreover let $T = (A^*A)^{-1}$ and set $\phi(\lambda) := \lambda,\ \theta(\lambda) := 1/\lambda$ as well as $\psi(\lambda) := \Psi^{-1}(\lambda)$ for $0 < \lambda < \infty$. Then we have $f \in X_{\psi\theta}$ if and only if f satisfies a* *source condition (16) with the function*

$$\bar{\psi}(t) = \frac{1}{\sqrt{t\Psi^{-1}(1/t)}}, \qquad 0 < t < \infty, \tag{17}$$

which is then a rate function.

Proof. Under the stated assumptions the function $\bar{\psi}$ is a well-defined rate function. Namely, we can write $\frac{1}{\sqrt{t\Psi^{-1}(1/t)}} = \sqrt{\frac{\Psi(u)}{u}}$ when using the substitution $u := \Psi^{-1}(1/t)$. The variable $u > 0$ is strictly decreasing with respect to $t > 0$ such that $u \to \infty$ corresponds with $t \to +0$ and vice versa $t \to \infty$ corresponds with $u \to +0$, because Ψ^{-1} is also strictly increasing and we have $\lim_{\lambda\to\infty} \Psi^{-1}(\lambda) = \infty$ and $\lim_{\lambda\to+0} \Psi^{-1}(\lambda) = 0$ for the functions Ψ under consideration. Now by (6) we have $\lim_{u\to\infty} \frac{\Psi(u)}{u} = 0$ and with Lemma 2.3 (a) the quotient $\frac{\Psi(u)}{u}$ is monotonically decreasing in $u > 0$. This, however, implies that $\bar{\psi}(t)$ is monotonically increasing for $t > 0$ with limit condition $\lim_{t\to+0} \bar{\psi}(t) = 0$. Hence, $\bar{\psi}$ is a rate function.

Moreover, we have

$$f \in X_{\psi\theta} \quad \Longleftrightarrow \quad (f, \Psi^{-1}(T)T^{-1}f) < \infty$$

and

$$f = \bar{\psi}(A^*A)v, \ \text{ for } \ v \in X \quad \Longleftrightarrow \quad ([\bar{\psi}(A^*A)]^{-1}f, [\bar{\psi}(A^*A)]^{-1}f) < \infty\,.$$

One has equivalence if and only if

$$[\Psi^{-1}((A^*A)^{-1})](A^*A) = [\bar{\psi}(A^*A)]^{-2}$$

and the claim follows. This proves the proposition. □

3. **Modulus of continuity of A^{-1}.** The *modulus of continuity* of A^{-1} restricted to the set AM with $M \subseteq X$ is

$$\omega(M, \delta) = \sup\{\|x\| : \ x \in M, \ \|Ax\| \le \delta\}.$$

The impact of the modulus of continuity on error bounds in regularisation has recently been discussed in the paper [19, §4]. Here we will prove a bound for the modulus of continuity where M is a ball in a variable Hilbert scale. It turns out that in many cases the bounds using more traditional source sets (see for example [27, 29, 19, 33]) are equivalent to the ones considered here. This is shown explicitly in the following for the bounds from [29, 19]. We found that the conditions (and proofs) simplify in the variable Hilbert scale framework compared to the traditional source set framework.

From [19, Lemma 4.2] one can find a minimax-expression for the modulus of continuity in the case of centrally symmetric and convex *source sets* for M of the form

$$M = G[B(R)] := \{x \in X : \ x = Gv, \ v \in X, \ \|v\| \le R\}\,, \tag{18}$$

corresponding to condition (38). To obtain an explicit upper bound we consider the special case $G = \bar{\psi}(A^*A)$ and

$$M = \bar{\psi}(A^*A)[B(R)] := \{x \in X : \ x = \bar{\psi}(A^*A)v, \ v \in X, \ \|v\| \le R\}$$

associated with the source condition (16). Note that the rate function $\bar{\psi}(t)$ is only of interest here for arguments $0 < t \le \|A\|^2$, but without loss of generality (cf. [20, Theorem 1 (b)]) we can extend $\bar{\psi}$ to be a monotonically increasing index function defined on $(0,\infty)$. Then by using the strictly increasing auxiliary function

$$\Theta(t) := \sqrt{t}\,\bar{\psi}(t)\,, \qquad 0<t<\infty \tag{19}$$

satisfying the limits conditions $\lim_{t\to+0}\Theta(t)=0$ and $\lim_{t\to\infty}\Theta(t)=\infty$ one obtains for $M=\bar{\psi}(A^*A)[B(R)]$

$$\omega(M,\delta)\,\le\, R\,\bar{\psi}\left(\Theta^{-1}\left(\frac{\delta}{R}\right)\right), \qquad \delta>0 \tag{20}$$

provided that

$$\bar{\psi}^2((\Theta^2)^{-1}(t)) \quad \text{is concave for} \quad 0<t<\infty\,. \tag{21}$$

This result can be derived from Corollary 3.7 and Theorem 2.1(c) in [19] (see also Theorem 1 in the earlier paper [29]). A similar assertion was already mentioned in a rudimentary form in a paper by Ivanov and Korolyuk in 1969 [24].

The following proposition yields an upper bound for the modulus of continuity based on a variable Hilbert scale interpolation inequality using Lemma 2.1 or Corollary 2.4. For the proof we use Lemma 2.3 (a).

Proposition 3.1. *Let $\Psi(\lambda)$, for $0<\lambda<\infty$, be a concave and strictly increasing index function satisfying the limit conditions $\lim_{\lambda\to+0}\Psi(\lambda)=0$, $\lim_{\lambda\to\infty}\Psi(\lambda)=\infty$ and (6), for which an index function χ exists that satisfies*

$$\chi(\lambda)\ge\Psi^{-1}(\lambda)/\lambda, \qquad 0<\lambda<\infty.$$

Furthermore let X_χ be an element of a Hilbert scale generated by $T=(A^*A)^{-1}$ where A is injective. Then

$$\omega(M,\delta)\le\delta\sqrt{\Psi\left(\frac{R^2}{\delta^2}\right)}, \qquad \delta>0\,, \tag{22}$$

for $M=B_\chi(R)$.

Proof. Under the assumptions stated on Ψ and χ Corollary 2.4 applies. We can conclude (from the proof of Corollary 2.4) that $\|h\|\le\|Ah\|\sqrt{\Psi\left(\frac{\|h\|_\chi^2}{\|Ah\|^2}\right)}$ for all $0\ne h\in X_\chi$. As Ψ is monotonically increasing one then gets $\|h\| \le\|Ah\|\sqrt{\Psi\left(\frac{R^2}{\|Ah\|^2}\right)}$. Now by Lemma 2.3 (a) the function $\Xi(\zeta)=\Psi(\zeta)/\zeta$ is monotonically increasing and so $\Xi(\zeta_1)\ge\Xi(\zeta_2)$ for $0<\zeta_1\le\zeta_2<\infty$. This gives with $\zeta_1:=\frac{R^2}{\delta^2}$ and $\zeta_2:=\frac{R^2}{\|Ah\|^2}$ the estimate $\|h\|\le\delta\sqrt{\Psi\left(\frac{R^2}{\delta^2}\right)}$ for all $h\in B_\chi(R)$ satisfying the additional condition $\|Ah\|\le\delta$. Thus the proposition is proven. $\square$

We note that for centrally symmetric and convex sets M, $f\in M$, and regularised solutions $f^\delta_{\alpha_{\rm dis}}\in M$ obtained from the discrepancy principle of form (14) mentioned in Remark 2 we easily derive along the lines of [19, Lemma 2.2] that

$$\|f-f^\delta_{\alpha_{\rm dis}}\|\le\omega(2M,(C_{\rm dis}+1)\delta) \tag{23}$$

with $2M:=\{u\in X:\ u=2v,\ v\in M\}$. In the case $M=B_\chi(R)$ with $2M=B_\chi(2R)$ the estimate (23) yields with (22) a convergence rate of the form (15) with constant $\bar{K}=4R^2/(C_{\rm dis}+1)^2$. With more generality such rates were verified above directly from Corollary 2.4.

Under weak additional assumptions (see [19, Corollary 3.7]) there is also a constant $\underline{C} > 0$ such that

$$\omega(B_\chi(R), \delta) \geq \underline{C}\, \delta \sqrt{\Psi\left(\frac{R^2}{\delta^2}\right)}, \qquad \delta > 0\,.$$

Then a convergence rate of the form (15) is *order optimal* independent of the constant $\bar{K} > 0$ because of $\sqrt{\Psi\left(\frac{C\,R^2}{\delta^2}\right)} \leq \max\{C, 1\}\sqrt{\Psi\left(\frac{R^2}{\delta^2}\right)}$ for all $C > 0$. On the other hand, Corollary 2.6 yields an error estimate of best order just for $\xi(\delta) \sim \delta$, hence the discrepancy principle is order optimal in that sense.

Evidently, under the assumptions of Proposition 2.7 with the interdependencies

$$\chi(\lambda) := \frac{\Psi^{-1}(\lambda)}{\lambda} = \frac{1}{\bar{\psi}(1/\lambda)^2}, \qquad 0 < \lambda < \infty \tag{24}$$

one has

$$\bar{\psi}(A^*A)[B(R)] = B_\chi(R)$$

where $B_\chi(R)$ denotes the ball (11) of radius R in X_χ, an element of the Hilbert scale generated by $T = (A^*A)^{-1}$ expressed through the index function χ. We emphasise that the upper bound in (22) for the modulus of continuity from Proposition 3.1 needing only one function Ψ has a *much simpler structure* than the nested upper bound in (20) composing the functions $\bar{\psi}$ and Θ^{-1}. Also the required concavity of Ψ for obtaining (22) *looks much simpler* than the needed concavity of the composite function

$$\bar{\psi}^2((\Theta^2)^{-1}(t)) \equiv \bar{\psi}^2(\Theta^{-1}(\sqrt{t})), \qquad 0 < t < \infty,$$

for obtaining (20).

Owing to the correspondence (17) between the concave index function Ψ and the rate function $\bar{\psi}$ it is of some interest to compare the quality of the estimates (20) and (22) as well as the strength of conditions which have to imposed in order to ensure those bounds for ω.

Proposition 3.2. *Let $\Psi(\lambda)$, for $0 < \lambda < \infty$, be a concave and strictly increasing index function satisfying the limit conditions $\lim\limits_{\lambda \to +0} \Psi(\lambda) = 0$, $\lim\limits_{\lambda \to \infty} \Psi(\lambda) = \infty$ and (6). Then for the rate function $\bar{\psi}(t) := 1/\sqrt{t\Psi^{-1}(1/t)}$ (cf. (17)) and by setting $\Theta(t) := \sqrt{t}\bar{\psi}(t)$, $0 < t < \infty$, we have the following assertions: The error bounds in (22) and in (20) and the corresponding concavity conditions required for obtaining those bounds coincide, i.e., we have*

$$\delta \sqrt{\Psi\left(\frac{R^2}{\delta^2}\right)} = R\,\bar{\psi}\left(\Theta^{-1}\left(\frac{\delta}{R}\right)\right), \qquad R > 0, \quad \delta > 0\,. \tag{25}$$

Moreover, the function $\bar{\psi}^2((\Theta^2)^{-1}(t))$ is concave for all $0 < t < \infty$.

Conversely, any rate function $\bar{\psi}(t)$, $0 < t < \infty$, determines by equation (17) in a unique manner a strictly increasing index function $\Psi(\lambda)$, $0 < \lambda < \infty$, satisfying the limit conditions $\lim\limits_{\lambda \to +0} \Psi(\lambda) = 0$, $\lim\limits_{\lambda \to \infty} \Psi(\lambda) = \infty$ and (6) which is concave for all $0 < \lambda < \infty$ if $\bar{\psi}^2((\Theta^2)^{-1}(t))$ is concave for all $0 < t < \infty$ which again implies the coincidence (25) of the error bounds.

Proof. First we find from Proposition 2.7 that $\bar{\psi}(t)$, $t > 0$, is a rate function if $\Psi(\lambda)$, $0 < \lambda < \infty$ is a concave and strictly increasing index function satisfying the limit conditions $\lim\limits_{\lambda \to +0} \Psi(\lambda) = 0$, $\lim\limits_{\lambda \to \infty} \Psi(\lambda) = \infty$ and (6). Then from the right equation in (24) (cf. (17)) we have $\Psi^{-1}(\lambda) = \lambda/\bar{\psi}^2(1/\lambda)$. By using the bijective substitution $u = 1/\lambda$ in $(0,\infty)$ this yields $\Psi^{-1}(1/u) = \frac{1}{\Theta^2(u)}$ and $\frac{1}{u} = \Psi\left(\frac{1}{\Theta^2(u)}\right)$ for $0 < u < \infty$. Multiplying the last equation by the factor $u\,\bar{\psi}^2(u)$ we derive

$$\bar{\psi}^2(u) = u\,\bar{\psi}^2(u)\Psi\left(\frac{1}{\Theta^2(u)}\right) = \Theta^2(u)\Psi\left(\frac{1}{\Theta^2(u)}\right)$$

and $\bar{\psi}(u) = \Theta(u)\sqrt{\Psi\left(\frac{1}{\Theta^2(u)}\right)}$. By exploiting the bijection $t = \Theta(u)$ of $(0,\infty)$ into itself this provides us with the equation $\bar{\psi}\left(\Theta^{-1}(t)\right) = t\sqrt{\Psi\left(\frac{1}{t^2}\right)}$ which implies the required identity (25) by inserting $t := \delta/R$ and multiplying the arising equation by R.

In a second step we note that by using the monotonically increasing bijection $s = \Theta^2(u)$ between $s \in (0,\infty)$ and $u \in (0,\infty)$ and once more by exploiting the right equation in (24) we can write as follows for all $s > 0$:

$$\bar{\psi}^2((\Theta^2)^{-1}(s)) = \bar{\psi}^2(u) = \frac{\Theta^2(u)}{u} = \Theta^2(u)\,\Psi\left(\frac{1}{\Theta^2(u)}\right) = s\,\Psi\left(\frac{1}{s}\right) = [\mathcal{S}(\Psi)](s).$$

Hence, by Lemma 2.3 (b) we immediately see that as required $\bar{\psi}^2((\Theta^2)^{-1}(s))$, $s > 0$, is concave if $\Psi(\lambda)$, $\lambda > 0$, is concave.

Since the involution $\mathcal{S}$ (cf. Remark 1) preserves concavity, the reverse assertion formulated in Proposition 3.2 becomes immediately clear, since (17) represents a one-to-one correspondence between index functions $\bar{\psi}$ and strictly increasing functions Ψ with the limit conditions under consideration. □

We now investigate the concavity condition for the function $\bar{\psi}^2((\Theta^2)^{-1}(s))$ in more detail. For this a characterisation of the concavity of index functions is given in terms of the monotonicity of certain divided differences.

Lemma 3.3. *Let ψ be an index function. Then the three following statements are equivalent:*

1. *ψ is concave*
2. *$(\psi(s_0+s) - \psi(s_0))/s$ is a decreasing index function for all $s_0 > 0$*
3. *$(\psi(s_0) - \psi(s_0-s))/s$ is an increasing continuous function $(0,s_0) \to \mathbb{R}_+$ for all $s_0 > 0$.*

Proof. If ψ is a concave index function then by Lemma 2.2 ψ is increasing and so both $(\psi(s_0+s)-\psi(s_0))/s$ and $(\psi(s_0)-\psi(s_0-s)/s$ are positive continuous functions for $s > 0$ and $s \in (0,s_0)$, respectively. Furthermore by definition

$$(t_2 - t_0)\psi(t_1) \geq (t_2 - t_1)\psi(t_0) + (t_1 - t_0)\psi(t_2)$$

and by simple algebraic manipulations and the right choice of $t_0 < t_1 < t_2$ one gets the second and third statement from the first.

Conversely, if $(\psi(s_0+s) - \psi(s_0))/s$ is a decreasing index function for all $s_0 > 0$ one has for all $t_0 < t_1 < t_2$

$$\frac{\psi(t_1) - \psi(t_0)}{t_1 - t_0} \geq \frac{\psi(t_2) - \psi(t_0)}{t_2 - t_0}$$

and thus ψ is concave. A similar argument shows that ψ is concave if the third statement holds. □

A direct consequence of this lemma is that for concave rate functions $\bar\psi$ one has

$$\frac{\bar\psi(s_0)-\bar\psi(s_0-s)}{s}\le\frac{\bar\psi(s_0)}{s_0}$$

as $\lim_{s\to 0}\bar\psi(s)=0$. Another consequence is

Proposition 3.4. *If $\psi(t)$ is a concave rate function then so is $\psi(\sqrt{t})^2$.*

Proof. By lemma 3.3 we have to show that for all $t_0>0$ the function $(\psi(\sqrt{t+t_0})^2-\psi(t_0)^2)/t$ is a decreasing index function. As the mapping $s\to(s+s_0)^2$ is monotone it is sufficient to show that

$$\omega(s)=\frac{\psi(s+s_0)^2-\psi(s_0)^2}{(s+s_0)^2-s_0^2}$$

is monotonically decreasing.

As ψ is assumed to be concave, Lemma 3.3 implies that

$$\sigma(s)=\frac{\psi(s+s_0)-\psi(s_0)}{s}$$

is monotonically decreasing. Furthermore

$$\omega(s)=\sigma(s)\left(\frac{\psi(s+s_0)+\psi(s_0)}{s+2s_0}\right)=\sigma(s)\frac{s\sigma(s)+2\psi(s_0)}{s+2s_0}.$$

Now let $s_1<s_2$. As $\sigma(s)$ is monotonically decreasing one has

$$\omega(s_1)\ge\sigma(s_2)\frac{s_1\sigma(s_2)+2\psi(s_0)}{s_1+2s_0}=\sigma(s_2)^2\frac{s_1+2\psi(s_0)/\sigma(s_2)}{s_1+2s_0}.$$

The right-hand side is a decreasing function of s_1 if $2s_0\le 2\psi(s_0)/\sigma(s_2)$, i.e., $\sigma(s_2)\le\psi(s_0)/s_0$. This is a consequence of Lemma 3.3 as stated in the remark after the lemma. Replacing s_1 by s_2 thus gives a lower bound for $\omega(s_1)$ and thus

$$\omega(s_1)\ge\sigma(s_2)^2\frac{s_2+2\psi(s_0)/\sigma(s_2)}{s_2+2s_0}=\omega(s_2).$$

It follows that ω is monotonically decreasing. □

A consequence of this lemma is that for the concavity of the function $\bar\psi^2((\Theta^2)^{-1}(s))=\bar\psi^2(\Theta^{-1}(\sqrt{s}))$ it is thus sufficient to show that $\bar\psi\circ\Theta^{-1}$ is concave.

Finally we conjecture that a similar result to the proposition above also holds more generally, i.e., that a sufficient condition for concavity of $g\circ\psi\circ g^{-1}$ is the concavity of ψ where g belongs to a class of suitably chosen functions.

4. **Linear regularisation approaches.**

Our goal in this section is to draw conclusions from Corollary 2.4 for *linear regularisation methods*. Taking into account the setting of Corollary 2.4 we assume throughout this section that the index function $\Psi(\lambda)$ is concave and strictly increasing for all $0<\lambda<\infty$ satisfying the limit conditions $\lim\limits_{\lambda\to+0}\Psi(\lambda)=0$, $\lim\limits_{\lambda\to\infty}\Psi(\lambda)=\infty$, and (6). Moreover, we have the interdependencies (24) between Ψ and the index functions χ and $\bar\psi$. Then χ is an increasing index function with $\lim\limits_{\lambda\to\infty}\chi(\lambda)=\infty$ and $\bar\psi$ is an increasing index function with $\lim\limits_{t\to+0}\bar\psi(t)=0$, hence a rate function. As outlined in Section 3 under these assumptions we have $\bar\psi(A^*A)[B(R)]=B_\chi(R)$ and the best case for regularised solutions f_α^δ approximating the exact solution $f\in X_\chi$ based on data g^δ satisfying (2) by using an a priori choice $\alpha = \alpha(\delta)$ or a posteriori choice $\alpha = \alpha(\delta, g^\delta)$ is to achieve the order optimal convergence rate (15). It is a specific consequence of interpolation theory and can be seen easily by inspection of Corollary 2.4 that a successful use requires the focus on regularisation methods which yield regularised solutions of appropriate smoothness. Precisely, there must be a ball $B_\chi(R)$ to which the elements f_α^δ belong for all $\alpha > 0$ attributed to sufficiently small $\delta > 0$ and g^δ satisfying (2).

4.1. **General linear regularisation schemata.** In a first approach we are going to consider *linear regularisation schemes* as described in many textbooks on linear regularisation theory (see, e.g., [9, Chap. 4], [11, Chap. 2] and [2, 3, 25, 26, 36]). We consider approximate solutions

$$f_\alpha^\delta := h_\alpha(A^*A)A^*g^\delta. \tag{26}$$

to f based on a family of piecewise continuous real functions $h_\alpha(t),\ 0 < t \le \|A\|^2$, to which we assign bias functions

$$r_\alpha(t) := t\, h_\alpha(t) - 1,\ \ 0 < t \le \|A\|^2\,.$$

These functions depend on a regularisation parameter $\alpha \in (0, \alpha_{\max}]$, where $\alpha_{\max}$ may be a finite real number or ∞. Small $\alpha > 0$ characterise good approximation of the original problem (1), whereas larger values α are connected with more stability. Hence, an appropriate trade-off between the two conflicting goals approximation and stability can be controlled by the choice of α. We say that such a function h_α describes a linear regularisation method if the properties

$$\lim_{\alpha\to+0} r_\alpha(t) = 0, \qquad 0 < t \le \|A\|^2, \tag{27}$$

and

$$\sup_{0<\alpha\le\alpha_{\max}}\ \sup_{0<t\le\|A\|^2} t\,|\,h_\alpha(t)\,| \,\le\, C_1 \tag{28}$$

with a constant $C_1 > 0$ hold. Because of (28) we have another constant $C_2 > 0$ such that

$$\sup_{0<\alpha\le\alpha_{\max}}\ \sup_{0<t\le\|A\|^2} |\,r_\alpha(t)\,| \,\le\, C_2$$

and hence for all $0 < \alpha \le \alpha_{\max}$ the estimate

$$\|Af_\alpha^\delta - g^\delta\| = \|(Ah_\alpha(A^*A)A^* - I)g^\delta\| \le \left[\sup_{0<t\le\|A\|^2} |\,r_\alpha(t)\,|\right] \|g^\delta\| \le C_2\,\|g^\delta\|.$$

This implies the limit condition $\lim\limits_{\alpha\to+0} \|Af_\alpha^\delta - g^\delta\| = 0$ for all data $g^\delta \in Y$. As a consequence we have that there is always a parameter choice $\alpha = \alpha(\delta, g^\delta),\ 0 < \delta \le \delta_{\max}$, such that

$$\|Af^\delta_{\alpha(\delta,g^\delta)} - g^\delta\| \le C_{\rm dis}\,\delta \qquad (0 < \delta \le \delta_{\max})$$

for some prescribed constant $C_{\rm dis} > 0$. If the mapping $\alpha \mapsto \|Af_\alpha^\delta - g^\delta\|$ is even continuous, then the discrepancy principle can be realised by a parameter choice $\alpha_{\rm dis} = \alpha_{\rm dis}(\delta, g^\delta)$ satisfying the equation (14).

Here we call a rate function $\bar\varphi$ a *qualification* of the regularisation method generated by h_α if there is a constant $C_{\rm quali} > 0$ such that

$$\sup_{0<t\le\|A\|^2} |r_\alpha(t)|\bar\varphi(t) \,\le\, C_{\rm quali}\,\bar\varphi(\alpha), \qquad 0 < \alpha \le \alpha_{\max}\,. \tag{29}$$

Now we are going to study under what conditions the inequality (12) in Corollary 2.6 can be fulfilled here with $\bar{\xi}(\delta) = \delta$. First we obtain

$$\|Af_\alpha^\delta - g\| = \|Ar_\alpha(A^*A)f + Ah_\alpha(A^*A)A^*(g^\delta - Af)\| \le \|Ar_\alpha(A^*A)f\| + C_1\delta. \tag{30}$$

In order to apply that corollary for obtaining a convergence rate (15) we assume $f \in B_\chi(R_1) = \bar{\psi}(A^*A)[B(R_1)]$ taking into account the cross-connection (24). So let $f = \bar{\psi}(A^*A)v,\ \|v\| \le R_1$. Provided that $\Theta(t) := \sqrt{t}\bar{\psi}(t)$ is a qualification of the method with constant $C_{\text{quali}} > 0$ this gives with (30)

$$\|Af_\alpha^\delta - g\| \le \left[\sup_{0<t\le\|A\|^2} r_\alpha(t)\Theta(t)\right] R_1 + C_1\delta \le C_{\text{quali}} R_1\, \Theta(\alpha) + C_1\delta \tag{31}$$

and hence an estimate of type (12) is fulfilled with $\bar{\xi}(\delta) = \delta$ when an a priori parameter choice $\alpha = \Theta^{-1}(\delta)$ is used.

Next we will check whether $f_\alpha^\delta \in B_\chi(R_2)$ for some $0 < R_2 < \infty$. We have

$$f_\alpha^\delta = h_\alpha(A^*A)A^*(g^\delta - Af) + h_\alpha(A^*A)A^*Af$$

and after some reformulation

$$f_\alpha^\delta = \bar{\psi}(A^*A)\left[h_\alpha(A^*A)(\psi(A^*A))^{-1}(A^*A)^{1/2}\tilde{g} + h_\alpha(A^*A)A^*Av\right]$$

with $\|\tilde{g}\| \le \delta$, since the different functions of A^*A are commutable. Now let the interplay of the regularisation method expressed by $h_\alpha(t)$ and the parameter choice $\alpha = \alpha(\delta, g^\delta)$ be such that there is a constant $C_{\text{para}} > 0$ with

$$\sup_{0<t\le\|A\|^2} \frac{\sqrt{t}\,|h_{\alpha(\delta,g^\delta)}(t)|\,\delta}{\bar{\psi}(t)} \le C_{\text{para}}, \qquad 0 < \delta \le \delta_{\max}\,. \tag{32}$$

The upper bound C_{para} in (32) must hold for all data $g^\delta \in Y$ associated with the noise level $\delta > 0$ and satisfying (2), where the case of an a priori parameter choice $\alpha = \alpha(\delta)$ should be included as a special case. Under (32) we have with (28)

$$\|h_{\alpha(\delta,g^\delta)}(A^*A)(\psi(A^*A))^{-1}(A^*A)^{1/2}\tilde{g} + h_{\alpha(\delta,g^\delta)}(A^*A)A^*Av\| \le R_2 := C_{\text{para}} + C_1\, R_1,$$

in other terms $f^\delta_{\alpha(\delta,g^\delta)} \in \bar{\psi}(A^*A)[B(R_2)] = B_\chi(R_2)$.

If there is a function $\Gamma(\alpha)$ satisfying for sufficiently small $\alpha > 0$ the inequality

$$\left[\sup_{0<t\le\|A\|^2} \frac{\sqrt{t}|h_\alpha(t)|}{\bar{\psi}(t)}\right] \le \Gamma(\alpha) \tag{33}$$

such that

$$\delta\,\Gamma(\alpha(\delta, g^\delta)) \le C_{\text{para}}, \qquad 0 < \delta \le \delta_{\max}\,, \tag{34}$$

this represents a sufficient condition for (32). In particular, if moreover the a priori parameter choice $\alpha(\delta, g^\delta) := \Theta^{-1}(\delta)$ satisfies (34) we have an estimate of type (12) with $\bar{\xi}(\delta) = \delta$ for that a priori parameter choice whenever Θ is a qualification of the regularisation method under consideration.

Hence the considerations above gave a sketch of the proof for the following proposition as a consequence of Corollary 2.6:

Proposition 4.1. *Under the standing assumptions of this section let* $f \in X_\chi = \text{range}(\bar{\psi}(A^*A))$ *and consider regularised solutions (26) with a generator function* h_α *that determines the regularisation method and satisfies (27) – (28) as well as (33) with some function* Γ *such that* $\Theta(t) := \sqrt{t}\bar{\psi}(t)$ *satisfies (34) with some constant* $C_{para} > 0$ *and is a qualification of the method (cf. (29)). Then for the a priori* *regularisation parameter choice* $\alpha = \alpha(\delta) := \Theta^{-1}(\delta) \to +0$ *as* $\delta \to +0$ *we have the convergence rate*

$$\|f - f_\alpha^\delta\| = \mathcal{O}\left(\delta\sqrt{\Psi(\bar{K}/\delta^2)}\right) \qquad as \qquad \delta \to +0 \tag{35}$$

with some constant $\bar{K} > 0$.

Note that in Proposition 4.1 the rate (35) also holds for any other parameter choice $\alpha = \alpha(\delta, g^\delta)$ that fulfils the inequalities (34) and

$$\|Af^\delta_{\alpha(\delta,g^\delta)} - g\| \le \bar{C}\,\delta, \qquad 0 < \delta \le \delta_{\max}\,, \tag{36}$$

with some constant $\hat{C} > 0$.

Example 1. The most prominent example of a linear regularisation method (26) is the Tikhonov regularisation with the generator function $h_\alpha(t) = \frac{1}{t+\alpha}$ and with the bias function $r_\alpha(t) = \frac{\alpha}{t+\alpha}$, where the requirements (27) and (28) are satisfied for the constants $C_1 = C_2 = 1$. It is well known that all concave rate functions $\bar{\varphi}$ are qualifications of the method satisfying (29) with the constant $C_{\text{quali}} = 1$. From that class we consider the monomials $\bar{\varphi}(t) = t^\nu$ for exponents $0 < \nu \le 1$. Then $\Theta(t) = \sqrt{t}\bar{\psi}(t)$ is a qualification with the same constant for the Tikhonov regularisation in case of a rate function $\bar{\psi}(t) = t^\mu$ with $0 < \mu \le 1/2$. Taking into account (24) this rate function is associated with $\chi(\lambda) = \lambda^{2\mu}$ and the strictly concave function $\Psi(\lambda) = \lambda^{\frac{1}{2\mu+1}}$. By the estimate (31) we have then (36) with $\bar{C} = R_1 + 1$ for $f = (A^*A)^\mu v,\ \|v\| \le R_1$ and for the a priori parameter choice

$$\alpha = \Theta^{-1}(\delta) = \delta^{\frac{2}{2\mu+1}}\,. \tag{37}$$

To derive a function Γ such that (33) is valid, we exploit the inequality

$$\frac{t^\kappa}{t+\alpha} \le (1-\kappa)^{1-\kappa}\kappa^\kappa\,\alpha^{\kappa-1}\,,$$

which holds for all $t > 0$, $\alpha > 0$ and $0 < \kappa < 1$. In the limit case $\kappa = 0$ we also have the inequality $1/(t+\alpha) \le 1/\alpha$. Thus there is a constant $\hat{c} > 0$ depending on $\kappa \in [0,1)$ such that $\frac{t^\kappa}{t+\alpha} \le \frac{\hat{c}}{\alpha^{1-\kappa}}$. By setting $\kappa := 1/2 - \mu$ we obtain for $\bar{\psi}(t) = t^\mu,\ 0 < \mu \le 1/2$ the inequality (33) with the function

$$\Gamma(\alpha) = \frac{\hat{c}}{\alpha^{\mu+\frac{1}{2}}}\,.$$

Then one easily verifies that $\delta\,\Gamma(\Theta^{-1}(\delta)) \le \frac{\hat{c}\delta}{\left(\delta^{\frac{2}{2\mu+1}}\right)^{\mu+\frac{1}{2}}} = \hat{c}$ and that (34) is fulfilled with $C_{\text{para}} = \hat{c}$. Hence Proposition 4.1 applies and we obtain for the parameter choice (37) and all $0 < \mu \le 1/2$ the optimal convergence rate

$$\|f - f_\alpha^\delta\| = \mathcal{O}\left(\delta^{\frac{2\mu}{2\mu+1}}\right) \quad \text{as} \quad \delta \to +0\,.$$

The best possible rate obtained in that way is $\|f - f_\alpha^\delta\| = \mathcal{O}\left(\sqrt{\delta}\right)$ for $\mu = 1/2$. For $\mu > 1/2$ the function Ψ remains strictly concave, but a finite function $\Gamma(\alpha)$ in (33) fails to exist, since we have $\sup\limits_{0<t\le\|A\|^2} \frac{\sqrt{t}}{\bar{\psi}(t)(t+\alpha)} = +\infty$. The limitation of Proposition 4.1 to lower Hölder rates than the saturation of Tikhonov's method admits seems to be a consequence of the fact that our approach based on Corollary 2.6 and the construction (26) do not interact good enough in case of higher smoothness

of f. In order to overcome that effect, we will consider another approach in the following subsection.

4.2. **Regularisation with unbounded operators and range inclusions.** In a second approach we suppose a non-standard source condition

$$f = Gw \tag{38}$$

with source element $w \in X$ and with an injective bounded self-adjoint positive definite linear operator $G : X \to X$ possessing a non-closed range. Under this condition which characterises the available a priori knowledge on the solution smoothness, we exploit a variant of the Tikhonov regularisation with regularised solutions

$$f_\alpha^\delta := G(GA^*AG + \alpha I)^{-1} GA^* g^\delta \,. \tag{39}$$

Since the unbounded linear operator with $B = G^{-1} : \mathrm{range}(G) \subseteq X \to X$ is frequently a differential operator, this approach is sometimes called *regularisation with differential operators*, see also HANKE [12]. Precisely, by construction the element $f_\alpha^\delta \in \mathrm{range}(G)$ is well-defined for all $\alpha > 0$ as the minimiser of the extremal problem

$$T_\alpha(\tilde f) := \|A\tilde f - g^\delta\|^2 + \alpha \|B\tilde f\|^2 \to \min, \quad \text{subject to} \quad \tilde f \in \mathrm{range}(G),$$

and then the penalty term in T_α contains derivatives of the function $\tilde f$.

To apply Corollary 2.6 under (24) we assume $f \in G[B(R_1)]$, with $G[B(R)]$ from (18), and a *range inclusion*

$$\mathrm{range}(G) \subseteq X_\chi \; = \mathrm{range}(\bar\psi(A^*A))\,, \tag{40}$$

which is equivalent to

$$\|Gw\| \le C\, \|\bar\psi(A^*A)w\|, \qquad \text{for all} \quad w \in X, \tag{41}$$

with some $C > 0$ and links the operators A and G. Then from [18, Lemma 6.2] we obtain that $f \in G[B(R_1)]$ implies $f \in B_\chi(CR_1) = \bar\psi(A^*A)[B(CR_1)]$. For more details on range inclusions we refer to [6, 21].

Along the lines of the paper [7] by CHENG and YAMAMOTO we consider an a priori parameter choice $\alpha = \alpha(\delta)$ as

$$\underline{c}\, \delta^2 \le \alpha(\delta) \le \overline{c}\, \delta^2, \qquad 0 < \delta \le \delta_{\max}\,, \tag{42}$$

with constants $0 < \underline{c} \le \overline{c} < \infty$, for which we obtain from $T_\alpha(f_\alpha^\delta) \le T_\alpha(f)$ the inequalities

$$\|Af_{\alpha(\delta)}^\delta - g^\delta\|^2 + \alpha(\delta)\|G^{-1}f_{\alpha(\delta)}^\delta\|^2 \le \|Af - g^\delta\|^2 + \alpha(\delta)\|G^{-1}f\|^2 \le \delta^2 + \overline{c}\,\delta^2\, R_1^2\,.$$

Now we have

$$\|Af_{\alpha(\delta)}^\delta - g\| \le \bar C\,\delta, \quad \text{with} \quad \bar C = \sqrt{1 + \overline{c}\, R_1^2} + 1$$

satisfying condition (12) with $\bar\xi(\delta) = \delta$ and

$$\|G^{-1}f_{\alpha(\delta)}^\delta\| \le \sqrt{\frac{\delta^2}{\alpha(\delta)} + \|G^{-1}f\|^2} \le \sqrt{\frac{1}{\underline{c}} + R_1^2} =: R_2\,.$$

This yields $f_{\alpha(\delta)}^\delta \in G[B(R_2)]$, thus $f_{\alpha(\delta)}^\delta \in B_\chi(CR_2) = \bar\psi(A^*A)[B(CR_2)]$ and consequently an estimate of type (13) with $\bar\xi(\delta) = \delta$ and CR_1, CR_2 instead of R_1, R_2. With the above considerations we have shown the convergence rate result of the following proposition again as a consequence of Corollary 2.6:

Proposition 4.2. *Under the standing assumptions of this section let f satisfy (38), where the link condition (40) is valid. Then for the a priori regularisation parameter choice (42) we have the convergence rate (35) with some constant $\bar{K} > 0$.*

Due to [19, Corollary 4.5] for all concave Ψ fulfilling the standing assumptions of this section the rate (35) is even order optimal in the sense of

$$\mathcal{O}\left(\delta\sqrt{\Psi(\bar{K}/\delta^2)}\right) = \mathcal{O}\left(\omega(G[B(R_1)],\delta)\right) \quad \text{as} \quad \delta \to +0\,.$$

Note that the requirement (40) gets stronger for higher rates in (35). In many applications (see as an illustration the examples in [21]) instead of (40) one can only verify range inclusions of the form

$$\text{range}(\bar{\varrho}(G)) \subseteq \text{range}(A^*) = \text{range}((A^*A)^{1/2}) \tag{43}$$

with some rate function $\bar{\varrho}$. Under operator monotonicity of the function $[\bar{\rho}^{-1}(\sqrt{t})]^2$ (43) implies (40) with $\bar{\psi}(t) = \bar{\rho}^{-1}(\sqrt{t})$ and $\chi(\lambda) = \left[\bar{\rho}^{-1}\left(1/\sqrt{\lambda}\right)\right]^{-2}$.

In order to verify in general for which index functions χ a range inclusion (40) with $\text{range}(G) \subseteq X_\chi$ is fulfilled, one can use the *spectral theorem* for unbounded self-adjoint operators T (see [42, Chapter VII.3] and also [35, Chapter VIII]). In the Hilbert space X, the injective, densely defined, self-adjoint, positive definite, and unbounded linear operator T is unitarily invariant to a multiplication operator $\mathcal{M}$ expressed by a real multiplier function m. This means that there are a measure space $(\Sigma, \mathcal{A}, \mu)$ with finite measure μ, a unitary operator $\mathcal{U}\colon X \to L_2(\Sigma, \mathcal{A}, \mu)$ and a real measurable function $m(t)$, $t \in \Sigma$, such that $[\mathcal{M}h](t) := m(t)h(t)$ a.e., where $\mathcal{M}$ maps in $L_2(\Sigma, \mathcal{A}, \mu)$, and

$$\mathcal{U}\,T\,\mathcal{U}^*\,h = \mathcal{M}\,h = m \cdot h$$

for all h from the domain of $\mathcal{M}$. We note that the closure of the range range(m) of the multiplier function m and the spectrum $\text{spec}(T) \subseteq [\|T\|^{-1}, \infty)$ of the operator T, possessing $+\infty$ as an accumulation point, coincide. Moreover, we have for index functions $\psi \in \mathcal{I}$ and h from the domain of $\psi(\mathcal{M})$

$$\mathcal{U}\,\psi(T)\,\mathcal{U}^*\,h = \psi(M)\,h = \psi(m)\cdot h\,.$$

Then by using the notations $\hat{f} := \mathcal{U}f \in L_2(\Sigma, \mathcal{A}, \mu)$ and $\widehat{(Gw)} := \mathcal{U}\,G\,w \in L_2(\Sigma, \mathcal{A}, \mu)$ by definition we immediately find that $\text{range}(G) \subseteq X_\chi$ is equivalent to the condition that

$$(Gw, \chi(T)Gw) = (\widehat{(Gw)}, \chi(\mathcal{M})\widehat{(Gw)})_{L_2(\Sigma,\mathcal{A},\mu)} = \int\limits_{\Sigma} \chi(m(t))|\widehat{(Gw)}(t)|^2 dt < \infty \tag{44}$$

holds for all $w \in X$. In Example 2 with background in imaging (cf. [37]) we will consider the special case that $\mathcal{U}$ denotes the two-dimensional Fourier transform and that the corresponding measure space is $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \mu)$ with the associated Borel σ-algebra and measure. In that example, T and G are commuting operators, both non-compact with a non-closed range.

On the other hand, in Example 3 we will exploit the one-dimensional Fourier transform to formulate sufficient conditions such that classical source conditions are satisfied for linear compact integral operators.

5. **Further examples.** In the remaining examples we illustrate the theory. All the occurring operators A are linear integral operators. First the Example 2 refers to convolution operators A which occur, for example, when the deblurring of noisy images is under consideration. Then the Example 3 illustrates the *low rate* case where an integral equation with a smooth kernel is solved and it is known that the solution is in a Sobolev space. The situation here is similar to the case of elliptic partial differential equations and has been discussed in [6]. In contrast to the PDE situation here convergence rates are low, typically of the form $O(|\log(\delta)|^{-k})$. The final Example 4 illustrates the *high rate* case where a derivative of data in the range of an integral operator with smooth kernel is considered. The high convergence rates are here of the form $O(\delta|\log(\delta)|^k)$.

In the examples we consider functions over $\mathbb{R}^d$ ($d = 1, 2$) and Sobolev spaces $H^l(\mathbb{R}^d)$ ($l = 1, 2, ...$) of Hilbert type will be used with norms $\|\cdot\|_l$ defined by

$$\|x\|_l^2 = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (1 + |\omega|^2 + \cdots + |\omega|^{2l})|\hat{x}|^2 \, d\omega,$$

where $\hat{x} = \hat{x}(\omega)$, $\omega \in \mathbb{R}^d$, is the Fourier transform of x. Now let $E_l : H^l(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$ denote the embedding and E_l^* the adjoint of E_l. Then $E_l E_l^* : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$ is an integral operator and

$$\widehat{E_l E_l^* y}(\omega) = \frac{\hat{y}(\omega)}{1 + |\omega|^2 + \cdots + |\omega|^{2l}}.$$

Example 2. In this example with $X = Y = L_2(\mathbb{R}^2)$ we are interested in deblurring, that means in finding a true picture which is characterised by a function $f = f(t) \in L_2(\mathbb{R}^2)$, $t = (t_1, t_2)^T$, that satisfies a linear operator equation (1) of convolution type

$$Af\,(s) = \int_{\mathbb{R}^2} k(s - t) f(t) \, dt = g(s), \quad s = (s_1, s_2)^T \in \mathbb{R}^2, \tag{45}$$

where $g \in L_2(\mathbb{R}^2)$ is a blurred image of f which is additionally contaminated with noise such that only the noisy blurred image $g^\delta \in L_2(\mathbb{R}^2)$ satisfying (2) is available as data. Following [4, Chapter 3] the kernel function $k(\tau)$, $\tau = (\tau_1, \tau_2)^T \in \mathbb{R}^2$, is called *point spread function* of a space invariant imaging system under consideration. We assume that the kernel is such that its Fourier transform $\hat{k} = \hat{k}(\omega)$, $\omega = (\omega_1, \omega_2)^T$, called *transfer function* is bounded. Different variants of such deblurring problems are presented and analysed in [4]. As a reference situation we exploit for illustration a variant of an out-of-focus blur for which

$$\hat{k}(\omega) = 2\frac{J_1(D|\omega|)}{D|\omega|}$$

where J_1 is the Bessel function of order one and D is the radius of the circle of confusion (cf. [4, formula (3.25) on p.60]). The linear convolution operator $A : L_2(\mathbb{R}^2) \to L_2(\mathbb{R}^2)$ in this example has a non-closed range but it is non-compact and the kernel is not square integrable.

In order to apply our theory to this example one needs to find an index function θ and a symmetric positive definite operator T such that $A^*A = \theta(T)$. A natural choice in this context is $T = -\Delta$ and in this case θ needs to satisfy $|\hat{k}(\omega)|^2 = \theta(|\omega|^2)$. This, however, is not possible, as $\hat{k}(\omega)$ is zero for some finite ω but an index function has to satisfy $\theta(\lambda) > 0$ for all $\lambda > 0$ and it can only be zero asymptotically at zero or infinity. It is thus not possible to get error bounds for the deblurring problem using the variable Hilbert scale theory and $T = -\Delta$.

One does not have this problem if one chooses $T = (A^*A)^{-1}$. Let us define the solution smoothness as $f \in H^l(\mathbb{R}^2)$. Then we have the operator $G = E_l^* E_l$ in (38) characterising the associated non-standard source condition. To find index functions χ that satisfy the link condition (40) we can make use of formula (44) taking into account that $m(\omega) = 1/|\hat{k}(\omega)|^2$ and

$$|\widehat{Gw}(\omega)|^2 = (1 + |\omega|^2 + \cdots + |\omega|^{2l})^{-1} |\hat{w}(\omega)|^2.$$

Then the range inclusion range$(G) \subseteq X_\chi$ takes the form

$$\chi\left(\frac{1}{|\hat{k}(\omega)|^2}\right) \frac{1}{(1 + |\omega|^2 + \cdots + |\omega|^{2l})} \leq \bar{C} < \infty \quad \text{for all} \quad \omega \in \mathbb{R}^2 . \tag{46}$$

Now as $\hat{k}(\omega)$ can be zero for finite ω, this range condition can only be satisfied if χ is bounded. Thus $\chi(\lambda) \leq C < \infty$ and consequently $X_\chi \supset X = L_2(\mathbb{R})$. The "source condition" then reduces to $f \in L_2(\mathbb{R})$ which does not lead to an error bound.

The failure of the above attempts to get error bounds clearly illustrates the need to extend the variable Hilbert scale theory to be able to cope with the deblurring problem. One can, however, deal with a partial deblurring problem. Observe that one has the asymptotics

$$2 \left| \frac{J_1(D|\omega|)}{D|\omega|} \right| \asymp |\omega|^{-3/2}$$

for large $|\omega|$ (cf. [4, formula (3.29) on p.60]). It follows that $\hat{k}(\omega) = |\omega|^{-3/2} \kappa(\omega)$ for some bounded $\kappa(\omega)$. The first factor $|\omega|^{3/2}$ relates to a "smoothing component" of the out-of-focus blur situation. We now consider inversion of this smoothing component only. For this we introduce an integral operator A with kernel k which satisfies

$$\hat{k}(\omega) = |\omega|^{3/2}. \tag{47}$$

For the "partial" out-of-focus blur situation (47) and monomials $\chi(\lambda) = \lambda^\kappa$, $\kappa > 0$, we have (46) if and only if $\kappa \leq \frac{2l}{3}$. With the relation $\chi(\lambda) = \Psi^{-1}(\lambda)/\lambda$ this corresponds with $\Psi(\lambda) \leq \lambda^{3/(2l+3)}$. Hence based on Proposition 4.2 for the situation (47) and under $f \in H^l(\mathbb{R}^2)$ a best possible convergence rate

$$\|f - f_\alpha^\delta\|_{L_2(\mathbb{R}^2)} = \mathcal{O}\left(\delta^{\frac{2l}{2l+3}}\right) \quad \text{as} \quad \delta \to +0$$

can be obtained by Tikhonov regularisation with H^l-penalty term.

Example 3. In this example we consider compact forward operators A in Equation (1) with $X = Y = L_2(\mathbb{R})$ and linear operators $A : L_2(\mathbb{R}) \to L_2(\mathbb{R})$, for which the range of the operator $K := \bar{\psi}(A^*A)$ is a subset of X_ϕ with some index function ϕ and some rate function $\bar{\psi}$; and where the generating operator of the Hilbert scales is $T = -d^2/dt^2$. It follows that range$(K) \subseteq X_\phi$ and a source condition of the form of Equation (16) leads to the condition $f \in X_\phi$ implying the corresponding convergence rates in regularisation. In this context, let K be a linear Fredholm integral operator of Hilbert-Schmidt type. For such operators one can provide conditions on the kernel which guarantee this range condition.

Lemma 5.1. *Let $K : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ be a Hilbert-Schmidt operator with kernel $k(t,s) \in L_2(\mathbb{R}^2)$. Furthermore, let $\tilde{K} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ be an integral operator with kernel $\tilde{k}(\omega,s) = \int_R e^{-i\omega t} k(t,s)\,dt$. Then $\tilde{K}$ is a Hilbert-Schmidt operator and*

$$\tilde{K}x = \widehat{Kx}, \quad x \in L_2(\mathbb{R}).$$

Proof. The adjoint operator K^* of K is an integral operator with kernel $k^*(s,t) = \overline{k(t,s)}$ as a consequence of the theorem of Fubini. By Plancherel's theorem one has

$$K^* u = \frac{1}{2\pi} \tilde{K}^* \hat{u}.$$

An application of Parseval's identity several times gives for $u, v \in L_2(\mathbb{R})$:

$$\begin{aligned} \frac{1}{2\pi}(\hat{u}, \widehat{Kv}) &= (u, Kv) \\ &= (K^* u, v) \\ &= \frac{1}{2\pi}(\tilde{K}^* \hat{u}, v) \\ &= \frac{1}{2\pi}(\hat{u}, \tilde{K} v). \end{aligned}$$

□

Proposition 5.2. *Let $K : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ be a Hilbert-Schmidt operator where the Fourier transform $\tilde{k}(\omega,s) = \int_{\mathbb{R}} e^{-i\omega t} k(t,s)\,dt$ of the kernel of K satisfies*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \phi(\omega^2) |\tilde{k}(\omega,s)|^2 \,ds\,d\omega < \infty$$

for some index function ϕ. Then $\operatorname{range}(K) \subseteq X_\phi$ where X_ϕ is generated by $T = -d^2/dt^2$.

Proof. By Lemma 5.1, if $y = Kx$, then

$$\hat{y}(\omega) = \int_{\mathbb{R}} \tilde{k}(\omega,s) x(s)\,ds.$$

Therefore, from the Cauchy-Schwarz inequality:

$$\left| \int_{\mathbb{R}} \tilde{k}(\omega,s) x(s)\,ds \right|^2 \le \int_{\mathbb{R}} |\tilde{k}(\omega,s)|^2 \,ds\, \|x\|^2,$$

it follows that

$$\begin{aligned} \|y\|_\phi^2 &= \frac{1}{2\pi} \int_{\mathbb{R}} \phi(\omega^2) |\hat{y}(\omega)|^2 \,d\omega \\ &\le \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \phi(\omega^2) |\tilde{k}(\omega,s)|^2 \,ds\,d\omega\, \|x\|^2 \end{aligned}$$

and consequently $y \in X_\phi$. □

Example 4. As a concrete application example we consider a problem from derivative spectroscopy [39]. Here numerical derivatives are used to enhance the resolution of measured spectra in order to separate close peaks. An instance is the Eddington correction formula. The approach determines

$$f = Lg := g - \frac{g^{(2)}}{2}$$

from observed g_δ where $g^{(2)}$ is the second derivative of g. We now apply the theory developed so far to determine how well $f = Lg$ can be determined from spectral data g_δ.

For $f \in H^2(\mathbb{R})$ and $f = Lg$ the Fourier transforms $\hat{f}$ and $\hat{g}$ satisfy

$$\hat{f}(\omega) = (1 + \omega^2/2)\,\hat{g}(\omega), \quad \text{a.e.}$$

Using Plancherel's theorem, one obtains from this the bounds

$$\frac{1}{2}\|f\|_2 \le \|Lf\| \le \|f\|_2, \quad f \in H^2(\mathbb{R})$$

which means in particular that $\|Lf\|$ is an equivalent norm for $H^2(\mathbb{R})$. Using standard arguments, one can then show that $L : H^2(\mathbb{R}) \to L_2(\mathbb{R})$ is a Hilbert space isomorphism. Using the convolution theorem one sees that $A = E_2 L^{-1} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ is an integral operator with

$$Af\,(t) = \frac{1}{\sqrt{2}} \int_{\mathbb{R}} \exp(-\sqrt{2}|t-s|) f(s)\, ds \quad t \in \mathbb{R}$$

where E_2 denotes the embedding $H^2(\mathbb{R}) \to L_2(\mathbb{R})$. As L^{-1} maps $L_2(\mathbb{R})$ onto $H^2(\mathbb{R})$ the range of A can be identified with $H^2(\mathbb{R})$.

In addition to the Sobolev spaces, which form a classical Hilbert scale, we will use a *variable Hilbert scale* of function spaces X_ϕ generated by the operator $T = -d^2/dt^2$ which thus have norms $\|\cdot\|_\phi$ defined by

$$\|x\|_\phi^2 = \frac{1}{(2\pi)} \int_{-\infty}^{\infty} \phi(\omega^2)\, |\hat{x}(\omega)|^2\, d\omega$$

where ϕ are *index functions*. The index functions

$$\nu_k(\lambda) = 1 + \lambda + \cdots + \lambda^k = \frac{\lambda^k - 1}{\lambda - 1}$$

define the Sobolev spaces, in particular, one has $X_{\nu_k} = H^k(\mathbb{R})$ and furthermore, the Sobolev norm is equal to the norm of the corresponding variable Hilbert scale:

$$\|f\|_k = \|f\|_{\nu_k}, \quad f \in X_{\nu_k}.$$

In this framework, we now get error bounds analogue to the ones in Corollary 2.4 which are again a consequence of Lemma 2.1, see Corollary 2.5. For the application of 2.5 to the case of the Eddington correction formula one chooses $\theta(\lambda) = 1/(1+\lambda/2)$ and so $\phi(\lambda) = 1 + \lambda/2$.

In contrast to the usual case, where the source condition is stated as a condition on f, here the source condition is stated as a condition on (the original spectrum) g. This source condition results from physical models for the spectrum, and, in particular for the so-called spectral broadening. A variety of models are used, the most common ones are the Gaussian, Lorenz and Voigt spectra where a Voigt spectrum is a combination of a Lorenz and a Gaussian spectrum. Here we consider Gaussian spectra defined by

$$g(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(-(t-s)^2/2) v(s)\, ds$$

for some $v \in L_2(\mathbb{R})$. For a different discussion and more background on the problem, the reader may consult the paper by Hegland [15].

It follows that $g \in X_\psi$ with $\psi(\lambda) = \exp(\lambda)$. The concave function Ψ can then be chosen as

$$\Psi(\lambda) = \begin{cases} \lambda, & \text{for } \lambda \leq 1 \\ (1 + \log(\lambda)/2)^2, & \text{for } \lambda \geq 1. \end{cases}$$

It follows that Ψ is concave and that $\phi(\lambda) \leq \Psi(\psi(\lambda))$. As a consequence one gets the error bounds

$$\|f - f_\alpha^\delta\| \leq \delta(1 + \log(\eta/\delta))$$

for $\delta < \eta$ and $\|f - f_\alpha^\delta\| \leq \eta$ if $\delta \geq \epsilon$. The stabilisation guarantees that even if the errors are very large, the error of the approximation does not grow to infinity. In fact, the solution $f_\alpha = 0$ would probably be a good choice for the large data error case.

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