Curious ill-posedness phenomena in the composition of non-compact linear operators in Hilbert spaces

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Abstract

We consider the composition of operators with non-closed range in Hilbert spaces and how the nature of ill-posedness is affected by their composition. Specifically, we study the Hausdorff-, Cesàro-, integration operator, and their adjoints, as well as some combinations of those. For the composition of the Hausdorff- and the Cesàro-operator, we give estimates of the decay of the corresponding singular values. As a curiosity, this provides also an example of two practically relevant non-compact operators, for which their composition is compact. Furthermore, we characterize those operators for which a composition with a non-compact operator gives a compact one.

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1   Introduction

Let $\mathcal{X}, \mathcal{Y}$, and $\mathcal{Z}$ denote infinite-dimensional real separable Hilbert spaces. In this note, we consider composite operators T that are factorized as

$$T : \mathcal{X} \xrightarrow{\ B\ } \mathcal{Z} \xrightarrow{\ A\ } \mathcal{Y},$$

where $B : \mathcal{X} \to \mathcal{Z}$, $A : \mathcal{Z} \to \mathcal{Y}$, and consequently, $T = A\,B : \mathcal{X} \to \mathcal{Y}$ are bounded linear operators with non-closed range, which means that zero belongs to the spectrum of the operators A, B, and T. So the composite equation

$$T\,x = A\,(B\,x) = y \qquad (x \in \mathcal{X},\, y \in \mathcal{Y}), \tag{1}$$

but also the outer equation

$$A\,z = y \qquad (z \in \mathcal{Z},\, y \in \mathcal{Y}) \tag{2}$$

and the inner equation

$$B\,x = z \qquad (x \in \mathcal{X},\, z \in \mathcal{Z}) \tag{3}$$

represent ill-posed linear operator equations and can serve as models for inverse problems characterized by forward operators A, B, and T with non-closed dense ranges $\mathcal{R}(T) \subset \mathcal{Y}$, $\mathcal{R}(B) \subset \mathcal{Z}$,

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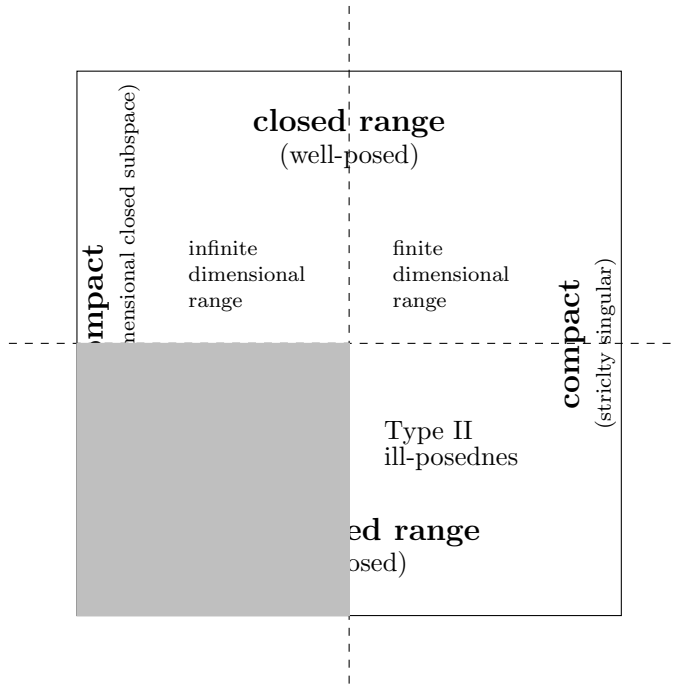
Figure 1: Case distinction for bounded linear operators mapping between infinite-dimensional Hilbert spaces.

and $\mathcal{R}(A) \subset \mathcal{Y}$, respectively. The stated assumptions furthermore imply that the corresponding adjoint operators T^*, A^*, and B^* are also bounded linear operators. Note that when only (2) is considered together with the assumption $z = Bx$ and B being compact, then this problem is not ill-posed as the solution is searched in the compact set $\mathcal{R}(B)$. This follows from the well-known Tikhonov theorem on the inverses on compact sets.

In this context we recall the paper [23], where Nashed distinguished for such operator equations *ill-posedness of type I* when the forward operator is non-compact and, as alternative, *ill-posedness of type II* when the forward operator is compact. See Figure 1 for an illustration of the concepts in Hilbert spaces. The operators of special interest in this article are from the gray-shaded part.

Unfortunately, only for type II the *strength and degree of ill-posedness* caused by the forward operator can be simply expressed by the decay rate of the associated singular values of this operator; see Definitions 1 and 2 below. For discussions about the degree of ill-posedness of equations (1) with non-compact operators that are ill-posed of type I in the sense of Nashed, we refer to the articles [15, 16, 22]. Here, however, we assume in the sequel that the composite operator $T : \mathcal{X} \to \mathcal{Y}$ in (1) is compact and possesses the singular system

$$\{\sigma_i(T) > 0, u_i \in \mathcal{X}, v_i \in \mathcal{Y}\}_{i=1}^{\infty},$$

with decreasingly ordered singular values

$$\|T\|_{\mathcal{L}(\mathcal{X}, \mathcal{Y})} = \sigma_1(T) \geq \sigma_2(T) \geq \ldots \geq \sigma_i(T) \geq \sigma_{i+1}(T) \geq \ldots$$

tending to zero as $i \to \infty$ and complete orthonormal systems $\{u_i\}_{i=1}^{\infty}$ in $\mathcal{X}$ and $\{v_i\}_{i=1}^{\infty}$ in $\mathcal{Y}$ obeying $T u_i = \sigma_i(T) v_i$ as well as $T^* v_i = \sigma_i(T) u_i$, for all $i \in \mathbb{N}$.

Definition 1 (Mild, moderate, and severe ill-posedness)**.** *Let the bounded linear operator $T : \mathcal{X} \to \mathcal{Y}$ be compact. Then we call the operator equation* (1)

- *mildly ill-posed whenever the decay rate of $\sigma_i(T) \to 0$, as $i \to \infty$, is slower than any polynomial rate.*

- *moderately ill-posed whenever the decay rate of $\sigma_i(T) \to 0$, as $i \to \infty$, is polynomial.*

- *severely ill-posed whenever the decay rate of $\sigma_i(T) \to 0$, as $i \to \infty$, is higher than any polynomial rate.*

Along the lines of [18] (see also [16]), one can define in more detail an interval and a degree of ill-posedness as follows.

Definition 2 (Interval and degree of ill-posedness). *We denote, for an ill-posed operator equation* (1) *with a compact forward operator T, the well-defined interval of the form*

$$[\underline{\kappa}, \overline{\kappa}] = \left[\liminf_{i \to \infty} \frac{-\log(\sigma_i(T))}{\log(i)}, \limsup_{i \to \infty} \frac{-\log(\sigma_i(T))}{\log(i)} \right] \subset [0, \infty] \tag{4}$$

as interval of ill-posedness. If $\underline{\kappa}$ and $\overline{\kappa}$ from $[0, \infty]$ are both finite positive, then we have moderate ill-posedness, and if they even coincide as $\underline{\kappa} = \overline{\kappa} = \kappa$, then we call the equation ill-posed of degree $\kappa > 0$. Severe ill-posedness occurs if the interval degenerates as $\underline{\kappa} = \overline{\kappa} = \infty$, and vice versa, mild ill-posedness is characterized by a degeneration as $\underline{\kappa} = \overline{\kappa} = 0$.

2 A selection of linear operators with non-closed range

To investigate the ill-posedness behaviour of composite operators $T = A\,B$ in the equation (1), we present a selection of bounded compact and non-compact operators with non-closed range that can be exploited for A and B. We start with the *simple integration operator* $J : L^2(0,1) \to L^2(0,1)$ and its adjoint operator $J^* : L^2(0,1) \to L^2(0,1)$ defined as

$$[J\,x](s) := \int_0^s x(t)\,dt \qquad (0 \le s \le 1, \quad x \in L^2(0,1)), \tag{5}$$

and

$$[J^*x](t) := \int_t^1 x(s)\,ds \qquad (0 \le t \le 1, \quad x \in L^2(0,1)), \tag{6}$$

respectively. Both operators are *compact* and so is the self-adjoint specific *diagonal operator* $D : \ell^2 \to \ell^2$, which appears here as

$$[D\,y]_j := \frac{y_j}{j} \qquad (j = 1, 2, \ldots, \quad y \in \ell^2). \tag{7}$$

It is well-known for J and J^* and evident for D that the degree of ill-posedness is *one*. The singular system of J and J^* can be written down in an explicit manner, where we have $\sigma_i(J) \asymp i^{-1}$, for $i \to \infty$, as singular value asymptotics. The singular system of D is of the form $\{i^{-1}, e^{(i)}, e^{(i)}\}_{i=1}^{\infty}$, where $e^{(i)}$ denotes the i-th unit vector in ℓ^2.

The *Cesàro operator* $C : L^2(0,1) \to L^2(0,1)$ and its adjoint operator $C^* : L^2(0,1) \to L^2(0,1)$, which attain the form

$$[C\,x](s) := \frac{1}{s} \int_0^s x(t)\,dt \qquad (0 \le s \le 1, \quad x \in L^2(0,1)) \tag{8}$$

and

$$[C^*x](t) := \int_t^1 \frac{x(s)}{s}\,ds \qquad (0 \le t \le 1, \quad x \in L^2(0,1)), \tag{9}$$

respectively, are non-compact operators with non-closed range; see [2, 21]. A further interesting non-compact operator with non-closed range connecting the spaces $L^2(0,1)$ and ℓ^2 is the *Hausdorff moment operator* $H : L^2(0,1) \to \ell^2$ defined as

$$[H\,x]_j := \int_0^1 x(t)\,t^{j-1}dt \qquad (j = 1, 2, \ldots, \quad x \in L^2(0,1)), \tag{10}$$

with the corresponding adjoint operator $H^* : \ell^2 \to L^2(0,1)$ of the form

$$[H^* y](t) := \sum_{j=1}^{\infty} y_j \, t^{j-1} \qquad (0 \le t \le 1, \quad y \in \ell^2), \tag{11}$$

and we refer for details to [10] (see also [9, 12, 17]).

For the last four non-compact operators, a degree or interval of ill-posedness in the sense of Definition 2 does not make sense. But if those operators occur as A or B in a composition $T = A\,B$, where T is compact, then they can substantially influence the degree of ill-posedness for T. This is also the case for non-compact *multiplication operators* $M : L^2(0,1) \to L^2(0,1)$ with non-closed range

$$[M\,x](t) := m(t)\,x(t) \qquad (0 \le t \le 1, \quad x \in L^2(0,1)), \tag{12}$$

for which the multiplier functions $m \in L^\infty(0,1)$ possess essential zeros in $(0,1)$. We refer in this context also to the papers [14, 19].

3 Can a non-compact operator in composition destroy the degree of ill-posedness of a compact operator?

In the past years, equations (1) with compact composite operators $T = A\,B$ have been studied under the assumption that B is compact and A is a non-compact operator with non-closed range. It had been an open question whether the non-compact operator A can amend the degree of ill-posedness of the compact operator B in the composition T.

The first studies in [7, 19, 20] investigated the case $T = M\,J : L^2(0,1) \to L^2(0,1)$ with multiplication operators $A := M : L^2(0,1) \to L^2(0,1)$ from (12) and the simple integration operator $B := J : L^2(0,1) \to L^2(0,1)$ from (5). Indeed, all these studies indicated the asymptotics $\sigma_i(T) \asymp i^{-1}$ as $i \to \infty$, even for multiplier functions m with strong (exponential-type) zeros that occur in inverse problems of option pricing; see [13]. This means that along the lines of those studies, the non-compact multiplication operator M does not destroy the degree of ill-posedness *one* of the compact operator J in such composition.

However, the situation changed when for $B := J$, the multiplication operator M was replaced with the non-compact Hausdorff moment operator $A := H : L^2(0,1) \to \ell^2$ from (10). In the article [17], the assertion of the following proposition could be shown in the context of Corollary 2 and Theorem 3 ibid.

Proposition 1. *The operator $T = H\,J : L^2(0,1) \to \ell^2$ with $H : L^2(0,1) \to \ell^2$ from (10) and $J : L^2(0,1) \to L^2(0,1)$ from (5) obeys for some positive constants $\underline{c}$ and $\overline{c}$ the inequalities*

$$\exp(-\underline{c}\,i) \le \sigma_i(T) \le \frac{\overline{c}}{i^{3/2}} \tag{13}$$

for sufficiently large indices $i \in \mathbb{N}$.

As a consequence of Proposition 1, the interval of ill-posedness for the composition $H\,J$ is a subset of the interval $[\frac{3}{2}, \infty]$. This was the first example in the literature to demonstrate with respect to $B := J$ the degree-destroying potential of a non-compact operator A in such a composition. Unfortunately, by now it could not be cleared if $T = H\,J$ really leads to an exponentially (severely) ill-posed problem or whether it leads to a moderate ill-posed problem.

So it was exciting to replace the Hausdorff moment operator H (used as A) with the non-compact Cesàro operator $A := C : L^2(0,1) \to L^2(0,1)$ from (8) in the composition $T = C\,J$. The recent paper [4] has proven that we have, for such T, the asymptotics $\sigma_i(T) \asymp i^{-2}$ as $i \to \infty$, which means that the degree of ill-posedness is *two* for $T = C\,J$. Hence, C increases in that composition the degree of ill-posedness of J just by one. Taking into account that $J^2 = M\,T$ with the multiplication operator $M : L^2(0,1) \to L^2(0,1)$ and the multiplier function $m(t) = t$, one can see again that such multiplication operator does not amend the degree of ill-posedness, because the asymptotics $\sigma_i(J^2) \asymp i^{-2}$, as $i \to \infty$, is well-known; see for example [26].

4 The curious case that the composition of two non-compact operators is compact

It was surprising for the authors that also two *non-compact* operators A and B with non-closed range can generate a *compact* operator by composition, $T = AB$. Indeed, let $B := C^* : L^2(0,1) \to L^2(0,1)$ from (9) and $A := H : L^2(0,1) \to \ell^2$ from (10). Then we have such a situation as the next proposition indicates.

Proposition 2. *The operator $T : L^2(0,1) \to \ell^2$ defined as $T := HC^*$ with the non-compact operators H from* (10) *and C^* from* (9) *is compact and even a Hilbert-Schmidt operator.*

Proof. We have that
$$T = HC^* = DH,$$
with the *compact* diagonal operator $D : \ell^2 \to \ell^2$ from (7). This can be seen by inspection of the j-th component of Tx, which can be written as
$$[H(C^*x)]_j = \int_0^1 \left(\int_t^1 \frac{x(s)}{s} \, ds \right) t^{j-1} \, dt \, .$$

Integration by parts yields moreover
$$[H(C^*x)]_j = \frac{1}{j} \int_0^1 x(t) \, t^{j-1} \, dt = \frac{1}{j} \, [Hx]_j = [D(Hx)]_j \, .$$

Since D is a compact operator, this property carries over to the composition $T = DH$ of D with the bounded linear operator H. In the same manner, the Hilbert-Schmidt operator D with the Hilbert-Schmidt norm $\|D\|_{HS} = \sqrt{\sum_{i=1}^\infty \frac{1}{i^2}} < \infty$ leads to a Hilbert-Schmidt property of T by favour of the inequality $\|T\|_{HS} \le \|D\|_{HS} \|H\|_{\mathcal{L}(L^2(0,1),\ell^2)}$. □

Remark 1. We note that of course the same fact can also be formulated for the adjoint operator $T^* = CH^* : \ell^2 \to L^2(0,1)$, where C from (8) and H^* from (11) are again non-compact operators with non-closed range, but $T^* = H^*D$ is compact. Let us recall that the adjoint of an operator completely mirrors the ill-posedness of the original operator; see, for instance, [10]: T is ill-posed of type I (resp. II) if and only if T^* is ill-posed of type I (resp. II). Moreover, it follows by the properties of the singular values that the degree of ill-posedness of T is identical to that of T^*. On the other hand, calculating the singular values seems to be equally difficult for both operators, T, T^*. □

Remark 2. Although the phenomenon Non-compact ∘ Non-compact = Compact is well-known in the literature for "academic examples"—see the Section 6 below—we found the above situation "curious", since it occurs for two operators (Cesàro- and Hausdorff operator) arising from practically relevant problems. We will show in Section 6 that this is actually a characteristic of type I ill-posedness.

Along the lines of the proof of [17, Theorem 3] one can prove the following theorem.

Theorem 1. *For the composition $T = DH = HC^* : L^2(0,1) \to \ell^2$ with the operators $H : L^2(0,1) \to \ell^2$ from* (10)*, $D : \ell^2 \to \ell^2$ from* (7) *and $C^* : L^2(0,1) \to L^2(0,1)$ from* (9)*, there exists a positive constant c such that*
$$\sigma_i(T) \le \frac{c}{i^{3/2}} \quad (i = 1, 2, \ldots), \tag{14}$$

and the degree of ill-posedness of T is at least 3/2.

Proof. A main tool for the proof is the system $\{L_j\}_{j=1}^\infty$ of shifted Legendre polynomials which represent a complete orthonormal system in the Hilbert space $L^2(0,1)$. This system is the result of the Gram-Schmidt orthonormalization process of the system $\{t^{j-1}\}_{j=1}^\infty$ of monomials. Consequently, we have

$$\text{span}(1, t, \ldots, t^{j-1}) = \text{span}(L_1, L_2, \ldots, L_j).$$

Hence, we have for $m \geq 2$ that $L_m \perp t^{j-1}$, for all $1 \leq j < m$. As has been proven by [17, Proposition 3], we have for the Hilbert-Schmidt operator $T = DH$

$$\sum_{i=n+1}^\infty \sigma_i^2(T) \leq \|T(I - Q_n)\|_{HS}^2,$$

where Q_n denotes the projection onto $\text{span}(L_1, ..., L_n)$. From that we derive here the estimates

$$\sum_{i=n+1}^\infty \sigma_i^2(T) \leq \sum_{i=n+1}^\infty \|T(I - Q_n)L_i\|_{\ell^2}^2 = \sum_{i=n+1}^\infty \|T L_i\|_{\ell^2}^2 = \sum_{i=n+1}^\infty \sum_{j=1}^\infty \langle D(HL_i), e^{(j)}\rangle_{\ell^2}^2. \quad (15)$$

By exploiting the system of normalized functions

$$h_j(s) := \sqrt{2j+1}\, s^j \in L^2(0,1) \quad (j = 0, 1, 2, \ldots),$$

we can rewrite the terms of the form $\langle D(HL_i), e^{(j)}\rangle_{\ell^2}^2$ in (15) as

$$\langle D(HL_i), e^{(j)}\rangle_{\ell^2} = \frac{1}{j^2}\langle HL_i, e^{(j)}\rangle_{\ell^2} = \frac{1}{j^2}\left(\int_0^1 \frac{h_{j-1}(s)\, L_i(s)\, ds}{\sqrt{2j-1}}\right)^2 = \frac{1}{j^2(2j-1)}\langle h_{j-1}, L_i\rangle_{L^2(0,1)}^2.$$

Taking into account $\|h_j\|_{L^2(0,1)} = 1$ and the orthogonality relations between h_j and L_i we derive now from (15) the estimate

$$\sum_{i=n+1}^\infty \sigma_i^2(T) \leq \sum_{j=n+2}^\infty \frac{1}{j^2(2j-1)}\sum_{i=n+1}^\infty \langle h_{j-1}, L_i\rangle_{L^2(0,1)}^2 = \sum_{j=n+2}^\infty \frac{1}{j^2(2j-1)}\|(I - Q_n)h_{j-1}\|_{L^2(0,1)}^2,$$

and with $\|(I - Q_n)h_{j-1}\|_{L^2(0,1)} \leq 1$, we can further estimate as

$$\sum_{i=n+1}^\infty \sigma_i^2(T) \leq \sum_{j=n+2}^\infty \frac{1}{j^2(2j-1)} \leq c_1\, n^{-2}$$

for some constant $c_1 > 0$. We recall now from [17, Lemma 4] the fact that an estimate

$$\sum_{i=n+1}^\infty \sigma_i^2(T) \leq c_1\, n^{-2\gamma} \ (n \in \mathbb{N}), \ \text{ for } \ \gamma > 0 \ \text{ and } \ c_1 > 0,$$

implies the existence of a constant $c_2 > 0$ such that $\sigma_i^2(T) \leq c_2\, i^{-(2\gamma+1)}$ $(i \in \mathbb{N})$. Applying this fact with $\gamma = 1$ yields the inequality (14), which completes the proof. $\square$

Remark 3. The singular system of the compact operator $D : \ell^2 \to \ell^2$ mentioned above indicates the asymptotics $\sigma_i(D) \asymp i^{-1}$ as $i \to \infty$. From Theorem 1, however, we see that there is some constant $K > 0$ such that

$$\sigma_i(DH)/\sigma_i(D) \leq \frac{K}{i^{1/2}} \quad (i = 1, 2, \ldots).$$

Consequently, as for the composition HJ (see Proposition 1) also here for DH the non-compact operator H has the power to increase the decay rate of the singular values of the respective compact operators by an exponent of at least $1/2$. $\square$

$\begin{smallmatrix} & A \\ B & \end{smallmatrix}$	H	C	M
J	$e^{-ci} \lesssim \sigma_i \lesssim i^{-\frac{3}{2}}$	$\sigma_i \sim i^{-2}$	$\sigma_i \sim i^{-1}$
C^*	$i^{-1}e^{-2i} \lesssim \sigma_i \lesssim i^{-\frac{3}{2}}$		

Table 1: Overview of known bounds for the singular values of compositions of certain operators.

As in Proposition 1 (cf. [17, Corollary 3.6]) for the composition HJ, one can also here verify for the composition $DH = HC^*$ lower bounds of exponential type for the singular values. We make this explicit by the following theorem.

Theorem 2. *Consider the operator $T = HC^* = DH$ from Theorem 1. Then we have the lower bound*

$$\frac{c_0}{i} \exp(-2i) \leq \sigma_i(T) \quad (i = 1, 2, \ldots), \tag{16}$$

with some constant $c_0 > 0$.

Proof. Consider the operator $TT^* : \ell^2 \to \ell^2$. Since HH^* is the infinite Hilbert matrix $\mathcal{H}$, we observe that

$$TT^* = D\mathcal{H}D$$

with D the diagonal operator $D = \operatorname{diag}((\frac{1}{i})_i)$ from (7). Let $P_N : \ell^2 \to \ell^2$ be the projection onto the first N components and let $\mathcal{H}_N = P_N \mathcal{H} P_N$ be the first $n \times n$-segment of the Hilbert matrix. Then the estimate

$$\sigma_N(P_N TT^* P_N) \leq \|P_N\|^2 \sigma_N(TT^*) = \sigma_N(TT^*)$$

holds. On the other hand, P_N commutes with D. Now let $D_N = P_N D P_N$ be the $N \times N$-segment of D, which means that $D_n = \operatorname{diag}((1/i)_{i=1,N})$. Under such setting we consequently have

$$P_N TT^* P_N = D_N \mathcal{H}_N D_N.$$

It is well-known from [31] that there is a constant $c > 0$ in the context of an estimate from above for the norm of the inverse of the finite Hilbert matrix $\mathcal{H}_N$ as

$$\|\mathcal{H}_N^{-1}\| \leq c \exp(4N).$$

This gives

$$\sigma_N(P_N TT^* P_N) = \frac{1}{\|(D_N \mathcal{H}_N D_N)^{-1}\|_{\mathbb{R}^N \to \mathbb{R}^N}} = \frac{1}{\|D_N^{-1} \mathcal{H}_N^{-1} D_N^{-1}\|_{\mathbb{R}^N \to \mathbb{R}^N}}$$
$$\geq \frac{1}{\|D_N^{-1}\|^2 \|\mathcal{H}_N^{-1}\|_{\mathbb{R}^N \to \mathbb{R}^N}} \geq \frac{1}{N^2 c \exp(4N)}$$

and yields the claimed result (16) by taking into account that $\sigma_N(T)^2 = \sigma_N(TT^*)$. $\qquad\square$

Remark 4. Table 1 gives an overview of known estimates for the singular values of the composition of certain operators. By inspecting the estimates (13) as well as (14) and (16), it is a really challenging question whether the compositions HJ and DH may lead to moderately ill-posed problems, although the character of the Hausdorff moment operator H seems to be severely ill-posed as the paper [27] indicates. If the answer is *yes*, then the moderate decay of the singular values of J and D has the power to stop in such compositions the severe ill-posedness character of H expressed by an exponential decay of the corresponding multiplier function in the spectral decomposition of HH^* (infinite Hilbert matrix). $\qquad\square$

By the above example of compactness of the composition two non-compact operators, the following issue is raised that seems to be trivial only at first glimpse: When T is a non-compact operator between Hilbert spaces, is the selfadjoint operator T^*T also non-compact? Clearly, T is non-compact if and only if T^* is (by Schauder's theorem). However, as we have seen, this does not necessarily imply non-compactness of the composition. Using polar decomposition, the following lemma can be shown:

Lemma 1. *Let $T : \mathcal{X} \to \mathcal{Y}$ be a bounded linear operator between Hilbert spaces $\mathcal{X}, \mathcal{Y}$. Then*

$$T \text{ is compact} \Leftrightarrow T^* \text{ is compact} \Leftrightarrow T^*T \text{ is compact}. \tag{17}$$

Proof. As mentioned, the first equivalence is Schauder's theorem (see, e.g. [28, Thm 4.19]), and since compact operators form an ideal, we only have to show that if T^*T is compact, then T^* is compact. Compactness of T^*T implies compactness of the square root $\sqrt{T^*T}$, as can be shown by a spectral decomposition. Now the polar decomposition $T^* = \sqrt{T^*T}U$, e.g., [28, Thm 12.35] or [3, Thm 3.9], with a bounded (unitary) operator U implies compactness of T^*. □

5 Numerical illustration of the decay of the singular values

In order to illustrate the findings of Theorems 1 and 2, we provide some numerical calculations of the singular values of $T = H\,C^* = D\,H$, and, for comparison, also for the operator in Proposition 1, namely $\tilde{T} = H\,J$. Both involve the Hausdorff operator and are composed with either the Cesàro or the integration operator. The results for $\tilde{T}$ are for comparison only as similar numerics was performed in [8]. Let us spoil the story by noting that the conclusions from these numerical calculations do not give evidence of a certain decay of the singular values for the original operators $T, \tilde{T}$ because of the exponential ill-posedness of the discretized operators and the corresponding huge margins of uncertainty left.

We discretized T and $\tilde{T}$ each in two ways:

1.) The first one is obtained by discretizing from the left by using $P_N : \ell^2 \to \ell^2$ the projection onto the first N components. Then, we numerically calculate the eigenvalues of $(P_N T)(P_N T)^* = P_N T T^* P_N$ and similar for $\tilde{T}$. The resulting finite-dimensional matrix in both cases involves discrete Hilbert-type matrices, for which an exponential decay of the singular values is known; cf. $\mathcal{H}_N$ and the statements in the proof of Theorem 2. But, and this is the main point, this decay depends non-uniformly on the discretization, which leads to the above-mentioned uncertainty.

2.) The second discretization is done from the right by discretizing the integral in the Hausdorff operator by a midpoint rule. This can be seen as using the projector $Q_N : L^2(0,1) \to L^2(0,1)$ onto piecewise constant functions. The corresponding matrix $(TQ_N)^*(TQ_N)$ has entries which can be calculated using the Li_2 polylogarithmus function. (Note that more advanced quadrature rules do not lead to simple expressions for the corresponding matrices, which justifies out choice of the midpoint rule).

For both discretizations, the eigenvalues of the matrices are then calculated using Matlab's eig-function. The results are illustrated in Figure 2. The figures provide a plot of the logarithm of the singular values for different discretization levels $N = 5, 10, 15, 20$. The top row corresponds to the operator $\tilde{T} = H\,J$ (as considered in [8]) and the bottom row to $T = D\,H$, and the left column corresponds to a discretization by P_N from the left and the right column to a discretization by Q_N from the right.

The two different discretizations give qualitatively similar results. Since an exponential decay in this semi-logarithmic plot corresponds to a linear decay, we find that in any case the discretized operators show exponential decay. However, the asympotics come with a huge error margin such that no conclusion about the decay for the original operators can be drawn.

The next figure, Figure 3 further illustrates this issue: her we simultaneously plot the log of the singular values for the two discretization procedures for $N = 20$ (for each operator separately). The dashed lines correspond to the theoretical upper and lower bounds from the analysis above.
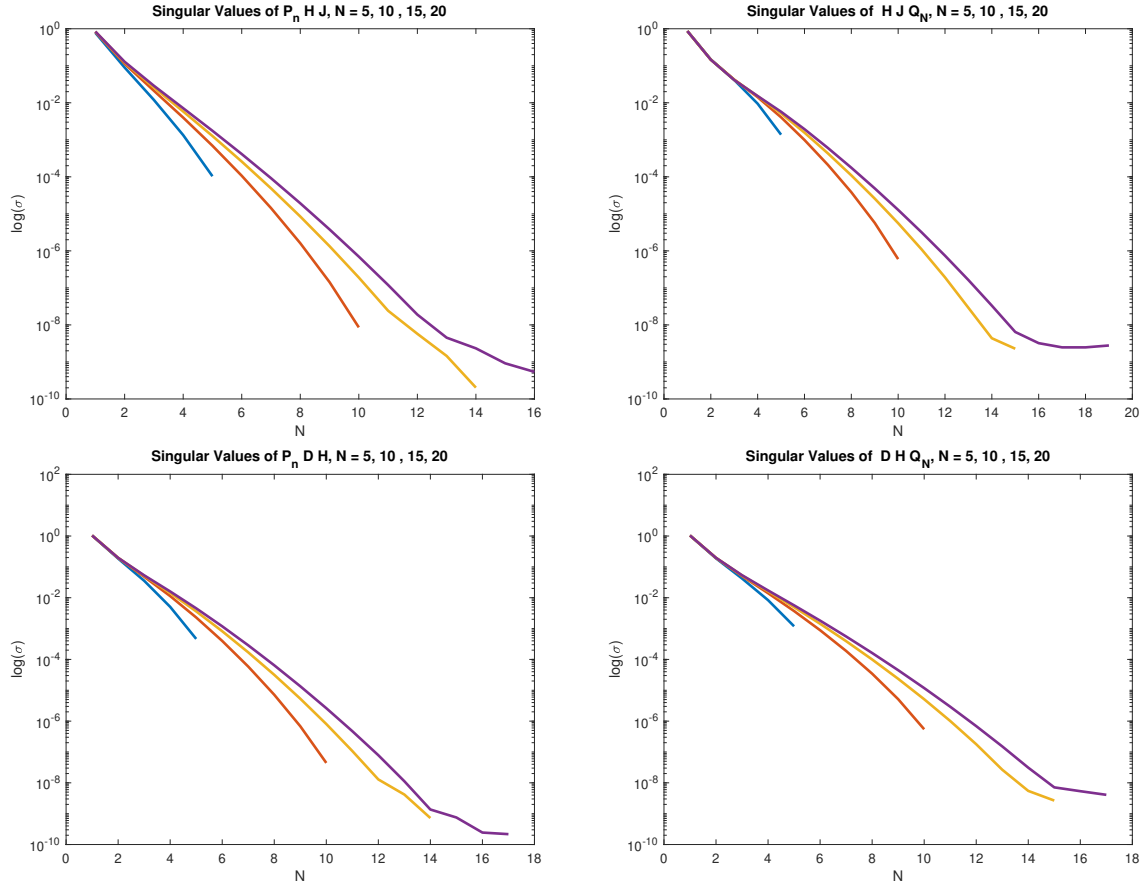
Figure 2: Plot of $\log(\sigma)$ for the two discretizations of the operators $H\,J$ (top row) and $D\,H = H\,C^*$ (bottom row). Left: discretization by the left using P_N. Right: discretization from the right using Q_N
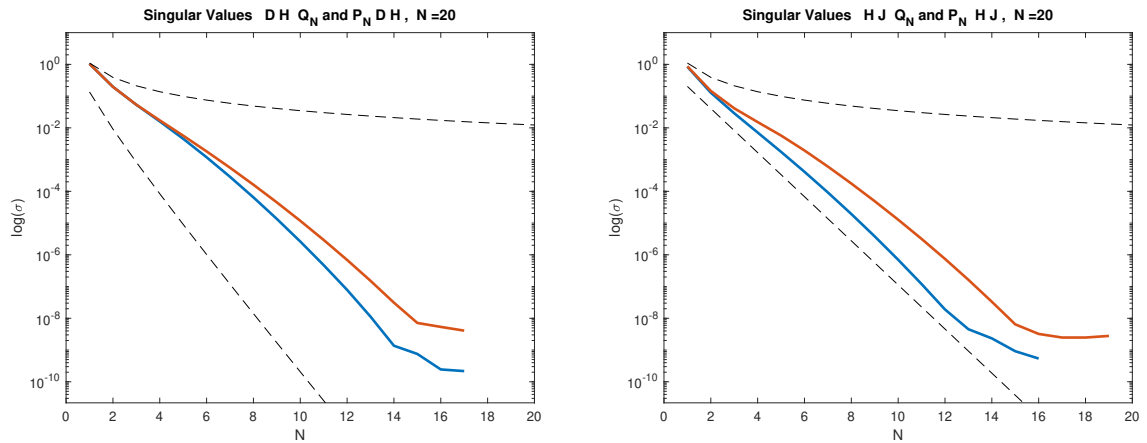

Figure 3: Plot of $\log(\sigma)$ for the two discretizations of the operators $D\,H = H\,C^*$ (left) and $H\,J$ (right) for $N = 20$. Dashed lines: Upper and lower theoretical estimates, $i^{-3/2}$ and $\exp(-ci)$ with $c = 1.6$ for $H\,J$ (right) and $i^{-3/2}$ and $\exp(-2i)/i$ for $D\,H$ (left).

On the left we illustrate the results for DH and on the right for HJ. Note that the lower bound in Theorem 2 for DH seem to be too pessimistic, and one might conjecture that a lower bound $\exp(-ci)$ as for HJ holds. What can be observed is that the singular values σ_i of the different discretization methods agree only for indices $i = 1 \ldots, 4$. In other words every singular value higher than the 4th one is here discretization-dependent and does not give any clue about the true values of the original operator. In particular, the question about the degree of ill-posedness (moderate or severely) is still open and cannot be answered by such numerical calculations.

6 When is the composition AB compact for non-compact operators A and B?

In this section, we would further investigate the above curious observation of a compact product of non-compact operators. There are some studies of this phenomenon in operator theory. For instance, there is the notion of a *power compact operator* [1, 30], that is, an operator T (not necessarily compact), where some power T^k is a compact operator. In these references, a characterization of power-compactness was given. In [5, pp 579–580], it is remarked that such operators have similar spectral properties as compact operators and that weakly compact operators in the space of continuous functions are instances of such power compact operators. All these are examples of compact products of non-compact operators.

Related to this are polynomially compact operators, where $p(T)$ is compact for some polynomial [24]. Also in this reference, there is even a characterization when the product of two operators is compact; see Theorem 2.3 in [24]:

Theorem 3. *Let A, B be two bounded operators on a separable Hilbert space. Then the product AB is compact if and only if there is a projection E such that AE and $(I - E)B$ are both compact.*

The question when the composition AB is compact for non-compact A and B can also be stated in terms of abstract operator theory. Recall that the set of compact operators $K : \mathcal{X} \to \mathcal{X}$ constitute an ideal $I(\mathcal{X})$ in the algebra of bounded operators $B(\mathcal{X})$. Thus, one can construct the quotient algebra $B(\mathcal{X})/I(\mathcal{X})$, called the Calkin algebra. The case that AB is compact (and hence is an element in $I(\mathcal{X})$) for A, B non-compact means in the Calkin algebra that $[A][B] = 0$, for $[A], [B] \neq 0$, where $[.]$ denotes equivalence classes. In other words, $[A]$ (and $[B]$ as well) is a zero divisor.

The zero divisors of the Calkin algebra have been characterized by Pfaffenberger in [25] as the complement of the semi-Fredholm operators; cf. [25, Corollary 3.7], where semi-Fredholm means closed range together with finite-dimensional nullspace or finite-dimensional codimension of the range. Essentially this means that for all type I ill-posed operators, we can find a non-compact operator such that their product is compact (for type II operators this is trivial).

Hence, the identity "Non-compact ∘ Non-compact = Compact" is almost a characterization of type I ill-posedness. For a better understanding of these results, we will detail and prove (following the lines of [25] and the references therein) the corresponding results below for our simplified case of Hilbert space operators.

In the following we assume throughout that $\mathcal{X}, \mathcal{Y}$ are Hilbert spaces. Let us start with a simple well-known lemma:

Lemma 2. *Let $A : \mathcal{X} \to \mathcal{Y}$ be a bounded operator with non-closed range. Then for any $\epsilon > 0$ there exists a $x \in N(A)^\perp$ such that*

$$\|Ax\| \leq \epsilon \|x\|.$$

Proof. If the conclusion would not hold, then we had an ϵ such that for all $x \in N(A)^\perp$, with $\|Ax\| \geq \epsilon \|x\|$. This means that the pseudo-inverse restricted to $\mathcal{R}(A)$, $A^\dagger : \mathcal{R}(A) \to N(A)^\perp$ satisfies

$$\|A^\dagger y\| \leq \epsilon^{-1} \|AA^\dagger y\| = \epsilon^{-1} \|Q_\mathcal{R} y\| \leq \epsilon^{-1}$$

and hence is bounded, where $Q_{\mathcal{R}}$ denotes the orthogonal projection onto $\overline{\mathcal{R}(A)}$ (see the Moore-Penrose inverse, [6, Prop. 2.3]. Thus, $A^\dagger$ has a bounded extension to $\overline{\mathcal{R}(A)}$, which contradicts the assumption of a non-closed range of A via Nashed's characterization of ill-posedness [23]. □

Next, we follow a construction of [11].

Proposition 3. *Let $A : \mathcal{X} \to \mathcal{Y}$ have non-closed range. Then there exists an infinite-dimensional closed subspace $\tilde{\mathcal{X}} \subset \mathcal{X}$ such that A restricted to $\tilde{\mathcal{X}}$ is compact:*

$$A|_{\tilde{\mathcal{X}}} = K,$$

where $K : \tilde{\mathcal{X}} \to \mathcal{Y}$ is compact.

Proof. Let us define $\mathcal{Z} := N(A)^\perp$ and the operator $\tilde{A} : \mathcal{Z} \to \mathcal{Y}$ such that $\tilde{A} = A|_{N(A)^\perp}$. Since the ranges of A and $\tilde{A}$ agree, $\tilde{A}$ has non-closed range as well. Note that $N(\tilde{A}) = 0$.

We inductively construct $\tilde{\mathcal{X}}$ through an orthonormal basis $(x_n)_{n=1}^\infty$ in $\mathcal{Z}$ such that

$$\|\tilde{A}x_n\| \leq \frac{1}{2^n}.$$

Let us start with $n = 1$: Let x_1 be such that $\|x_1\| = 1$ and $\|\tilde{A}x_1\| \leq \frac{1}{2}$. By Lemma 2 such an x_1 exists. We proceed by the induction step. Assume that we have defined an orthonormal basis $x_1, \ldots x_n$ in $\mathcal{Z}$ such that

$$\|\tilde{A}x_i\| \leq \frac{1}{2^i}, \qquad i = 1, \ldots n,$$

and denote by $\mathcal{X}_n$ the span of the $(x_i)_{i=1}^n$, by P_n the orthogonal projector onto $\mathcal{X}_n$, and by $Q_n = I - P_n$ the orthogonal projector onto $\mathcal{X}_n^\perp$ with complements taken with respect to $\mathcal{Z}$.

Note that $\mathcal{X}_n^\perp$ is an infinite-dimensional space. Because otherwise, if it were finite-dimensional with $\mathcal{X}_n^\perp = \text{span}\{z_i, i = 1, k\} =: \mathcal{W}$, then $\mathcal{W} + \mathcal{X}_n = \mathcal{Z}$, and thus the image of $\mathcal{Z}$ under $\tilde{A}$ is the span of Ax_i, Az_j $i = 1, \ldots, n$, $j = 1, \ldots, k$ and hence finite-dimensional, contradicting the assumption for A. Thus Q_n is a nonzero projection onto an infinite-dimensional subspace.

Note that we have $\tilde{A} = \tilde{A}P_n + \tilde{A}Q_n$. Since the sum of two closed subspaces is closed if one of them is finite-dimensional, we note that, as $\tilde{A}P_n$ is finite-dimensional, the range of $\tilde{A}Q_n$ cannot be closed as otherwise the range of $\tilde{A}$ would be closed as well.

By Lemma 2 applied to $\tilde{A}Q_n$, we thus find a normalized y_{n+1} with $\|\tilde{A}Q_n y_{n+1}\| \leq \frac{1}{2^{n+1}}\|y_{n+1}\|$ and $y_{n+1} \in N(\tilde{A}Q_n)^\perp$. Since $N(\tilde{A}Q_n)^\perp = \overline{R(Q_n \tilde{A}^*)}$, it follows that $y_{n+1} \in \mathcal{X}_n^\perp$ and thus $\|y_{n+1}\| = \|Q_n y_{n+1}\|$. Take $x_{n+1} = \frac{Q_n y_{n+1}}{\|Q_n y_{n+1}\|}$. Then $x_1, \ldots, x_{n+1}$ is orthonormalized and

$$\|\tilde{A}x_{n+1}\| \leq \frac{1}{2^{n+1}}\frac{\|y_{n+1}\|}{\|Q_n y_{n+1}\|} = \frac{1}{2^{n+1}}\frac{\|Q_n y_{n+1}\|}{\|Q_n y_{n+1}\|} = \frac{1}{2^{n+1}}.$$

Now take the operator $K_n := \tilde{A}P_n$. Since $\mathcal{R}(P_n) \subset N(A)^\perp$ we also have $K_n = AP_n$. This operator can be written with help of the orthonormal basis x_i as

$$K_n x = \sum_{i=1}^n Ax_i (x, x_i)_{\mathcal{X}}$$

and obviously has finite-dimensional range. Then,

$$\|(K_n - K_m)x\| = \|\sum_{i=n}^m Ax_i (x, x_i)_{\mathcal{X}}\| \leq \sum_{i=n}^m \|Ax_i\|\|x\| \leq \sum_{i=n}^m \frac{1}{2^i}\|x\| \leq \frac{1}{2^n}\|x\|.$$

Thus K_n is a Cauchy-sequence of finite-range operators which therefore converge to a compact limit operator $K_n \to K$. By construction, we have that $Kx_i = Ax_i$ and thus $\tilde{\mathcal{X}} = \text{span}\{x_i\}$ and K provides the subspace and compact operator that we were looking for. □

As mentioned above, we have adapted the proof of the proposition from [11, Thm. 4.1, p.72], where it was done in a more general Banach-space setting with biorthogonal sequences. The similar construction was used in [29], the result of which was employed in the proof of [25].

We now state the first main result of this section:

Theorem 4. *Let $A : \mathcal{X} \to \mathcal{Y}$ be a bounded operator.*

1. *Assume that A has a non-closed range. Then there exists a non-compact operator $B : \mathcal{X} \to \mathcal{X}$ such that AB is compact.*

2. *Conversely, suppose that $N(A)$ is finite-dimensional and assume that there exists a non-compact $B : \mathcal{X} \to \mathcal{X}$ such that AB is compact. Then A has a non-closed range.*

Proof. Ad 1: By taking B as the projector onto the infinite-dimensional space $\tilde{\mathcal{X}}$ from Proposition 3, we have that $AB = K$ is compact. Since P projects onto an infinite-dimensional space, it cannot be compact.

Ad 2: Assume that A has a closed range. Then, by classical results in regularization theory [6], there exists a bounded pseudo-inverse $A^{\dagger}$. The Moore-Penrose equations give $A^{\dagger}A = Q_{N(A)^{\perp}}$, where $Q_{N(A)^{\perp}}$ is the orthogonal projector onto $N(A)^{\perp}$. Let $P = I - Q_{N(A)^{\perp}}$ be the corresponding projector onto $N(A)$, which has thus finite-dimensional range by assumption. Then, since AB is compact, $A^{\dagger}AB = Q_{N(A)^{\perp}}B$ is a compact operator as well, and hence $B = PB + Q_{N(A)^{\perp}}B$ is the sum of a compact operator and one with finite-dimensional range, hence compact as well. This is a contradiction to B being non-compact. Thus A cannot have a closed range. $\square$

Remark 5. The assumption on the finite-dimensionality of $N(A)$ is necessary since otherwise A could indeed have a closed range: For instance, if A has closed range with infinite-dimensional nullspace, then we take as B the projector onto $N(A)$, which is non-compact. The product $AB = 0$ is thus compact.

We can collect the findings into a theorem, which is the second main result of this section:

Theorem 5. *Let $A : \mathcal{X} \to \mathcal{Y}$ be a bounded operator. Then the following is equivalent:*

1. *A has non-closed range or $N(A)$ is infinite-dimensional.*

2. *There exists a non-compact $B : \mathcal{X} \to \mathcal{X}$ such that AB is compact.*

Proof. $1 \Rightarrow 2$: In case A has non-closed range, we have shown this in Theorem 4 and in case of an infinite-dimensional $N(A)$ in the above remark.

The case $2 \Leftarrow 1$ follows from Theorem 4, item 2. $\square$

Remark 6. Theorem 5 reflects essentially a result of [25], which was shown in an even more general setting of Banach spaces but stated in rather abstract operator algebra form. The case of BA being compact for a given A and non-compact B can be easily treated by taking adjoints: $(BA)^{*} = A^{*}B^{*}$. Since BA is compact if and only if $(BA)^{*}$ is, we could express the analogous results of Theorem 5 with BA in place of AB and $N(A^{*})$ in place of $N(A)$.

We note that we did not require the operator B in Theorem 5 to be ill-posed in the sense of Nashed. However, under certain injectivity conditions, the property $AB = $ compact is *equivalent* to type I ill-posedness of *both* operators, as it is the case for our example with the Cesàro- and Hausdorff operator ($A = H, B = C^{*}$):

Corollary 1. *Let $A : \mathcal{X} \to \mathcal{Y}$ and $B : \mathcal{Y} \to \mathcal{Z}$ be noncompact. Assume that A and B^{*} are injective, and assume that AB is compact. Then both A and B have non-closed range, i.e., they are ill-posed of type I.*

Acknowledgment

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