

Convergence rates of a penalized variational inequality method for nonlinear monotone ill-posed equations in Hilbert spaces

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Abstract

We consider perturbed nonlinear ill-posed equations in Hilbert spaces, with operators that are monotone on a given closed convex subset. A simple stable approach is Lavrentiev regularization, but existence of solutions of the regularized equation on the given subset can be guaranteed only under additional assumptions that are not satisfied in some applications.

Lavrentiev regularization of the related variational inequality seems to be a reasonable alternative then. For the latter approach, in this paper we present new error estimates for suitable a priori parameter choices, if the considered operator is cocoercive and if in addition the solution admits an adjoint source representation. Some numerical experiments are included.

1 Introduction

In this paper we consider nonlinear equations of the form

$$Fu = f_*, \quad (1.1)$$

where $F : \mathcal{H} \supset \mathcal{D}(F) \rightarrow \mathcal{H}$ is a nonlinear operator in a real separable Hilbert space \mathcal{H} with inner product $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$, and $f_* \in \mathcal{R}(F) = F(\mathcal{D}(F))$. It is assumed that equation (1.1) is ill-posed in one of the concepts considered in [13], i.e., it is unstable solvable at f_* or locally ill-posed at each solution of (1.1); see also [5]. If not specified otherwise, throughout the paper we restrict the considerations to the following class of operators.

Definition 1.1. The operator $F : \mathcal{H} \supset \mathcal{D}(F) \rightarrow \mathcal{H}$ is called *monotone* on a set $\mathcal{M} \subset \mathcal{D}(F)$ if

$$\langle Fu - Fv, u - v \rangle \geq 0 \quad \text{for each } u, v \in \mathcal{M}. \quad (1.2)$$

In the following we assume that equation (1.1) has a solution $u_* \in \mathcal{M}$. Moreover, we suppose that the right-hand side of (1.1) is only approximately given as $f^\delta \in \mathcal{H}$ satisfying

$$\|f_* - f^\delta\| \leq \delta, \quad (1.3)$$

where $\delta \geq 0$ is a given noise level. For the regularization of the considered equation (1.1) with noisy data as in (1.3), Lavrentiev regularization

$$(F + \alpha I)u = f^\delta, \quad (1.4)$$

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may be considered, where $\alpha > 0$ is a regularization parameter. Solvability of equation (1.4) on \mathcal{M} is a critical issue and can only be guaranteed under additional assumptions on the operator F and the set \mathcal{M} , e.g.,

- (a) F is hemicontinuous and $\mathcal{D}(F) = \mathcal{M} = \mathcal{H}$, or
- (b) F is maximal monotone on \mathcal{M} , or
- (c) F is hemicontinuous, \mathcal{M} is a closed ball, centered at a solution of (1.1) and with sufficiently large radius, and $\frac{\delta}{\alpha}$ is sufficiently small.

For (b) we refer e.g. to Deimling [9, Theorem 12.5] and note that (a) is a special case of (b) (cf. e.g. Showalter [23, p. 39]). The case (c) is considered in Tautenhahn [24], with some clarification given by Neubauer [20].

There exist examples, however, where none of these conditions (a) – (c) on F and \mathcal{M} is necessarily satisfied. For other examples, maximal monotonicity in (b) is hard to verify, e.g. for operators on $\mathcal{H} = L^2(\Omega)$ with $\Omega \subset \mathbb{R}^n$, and $\mathcal{M} \subset \{f \in \mathcal{H} \mid f \geq 0 \text{ a.e.}\}$.

In such cases, a variational formulation (see formula (2.1) below) seems to be a reasonable alternative for (1.4). To prove this fact, is one of the goals of the present paper.

We conclude this section with some references on the regularizing properties of (1.4): see, e.g., Alber and Ryazantseva [1], Boş and Hofmann [5], Hofmann, Kaltenbacher and Resmerita [12], Janno [14], Liu and Nashed [17], Tautenhahn [24], as well as Mahale and Nair [19].

2 Penalized variational inequality method – Basic notations

We introduce the following assumptions and notations.

Assumption 2.1. Let $F : \mathcal{H} \supset \mathcal{D}(F) \rightarrow \mathcal{H}$ be a demicontinuous bounded operator in the real separable Hilbert space \mathcal{H} which is monotone on a given closed convex subset $\mathcal{M} \subset \mathcal{H}$, with $\mathcal{M} \subset \mathcal{D}(F)$. In addition, let $f_*, f^\delta \in \mathcal{H}$ satisfy the noise model (1.3). Furthermore, we suppose that the equation $Fu = f_*$ has a solution which belongs to \mathcal{M} .

Throughout the present paper, we assume that Assumption 2.1 holds. Recall that the operator F is, by definition in the sense of Deimling [9, Definition 11.2] and Showalter [23, p. 36],

- *demicontinuous*, if for each $v \in \mathcal{D}(F)$ and for each sequence $(v_n) \subset \mathcal{D}(F)$ with $v_n \rightarrow v$ as $n \rightarrow \infty$, we have weak convergence $Fv_n \rightharpoonup Fv$ as $n \rightarrow \infty$,
- *bounded*, if for each bounded set $\mathcal{N} \subset \mathcal{D}(F)$, the set $F(\mathcal{N}) \subset \mathcal{H}$ is bounded.

Instead of Lavrentiev regularization (1.4), in what follows we consider the following penalized variational inequality method (2.1). Let, for $\alpha > 0$, $u_\alpha^\delta \in \mathcal{M}$ satisfy

$$\langle Fu_\alpha^\delta + \alpha u_\alpha^\delta - f^\delta, v - u_\alpha^\delta \rangle \geq 0 \quad \text{for each } v \in \mathcal{M}. \quad (2.1)$$

For technical purposes, we use for $\alpha > 0$ the notation

$$u_\alpha = u_\alpha^0 \quad (2.2)$$

for the noise-free case $\delta = 0$, which means that the approximation obtained by the penalized variational inequality method has been derived on the basis of exact data $f^\delta = f_*$. An approach (2.1) can be considered as a variational inequality formulation of Lavrentiev regularization.

A solution to the variational inequality (2.1) with the penalized operator always exists on \mathcal{M} and depends stably of f^δ :

Theorem 2.2. *Let Assumption 2.1 be satisfied. Then for each parameter $\alpha > 0$, the penalized variational inequality (2.1) has a unique solution $u_\alpha^\delta \in \mathcal{M}$. In addition, the following stability estimate is satisfied,*

$$\|u_\alpha^\delta - u_\alpha\| \leq \frac{\delta}{\alpha}, \quad (2.3)$$

where $u_\alpha \in \mathcal{M}$ is given by (2.2).

Proof. Consider, for $\alpha > 0$ fixed, the nonlinear operator $F_\alpha : \mathcal{H} \supset \mathcal{D}(F) \rightarrow \mathcal{H}$ which maps as $u \mapsto Fu + \alpha u$. Then we obviously have $\langle F_\alpha u - F_\alpha v, u - v \rangle \geq \alpha \|u - v\|^2$ for each $u, v \in \mathcal{M}$, i.e., the nonlinear operator F_α is strongly monotone on the set \mathcal{M} . Existence thus follows, e.g., from Showalter [23, proof of Theorem 2.3 in Chapter II]. The mentioned proof in that reference may be applied, using the notations from there, with $\mathcal{A} = F_\alpha$ and $v_0 = u_*$, where again $u_* \in \mathcal{M}$ satisfies $Fu_* = f_*$. Notice that the operator \mathcal{A} considered in [23] is assumed to be pseudo-monotone all over the considered Hilbert space. However, the proof in that paper can be employed straightforward under the assumptions made in the present paper.

We next verify estimate (2.3). For notational convenience we introduce the notation

$$\chi_\alpha^\delta = u_\alpha^\delta - u_\alpha \quad \text{for each } \alpha > 0.$$

We have

$$\langle Fu_\alpha + \alpha u_\alpha - f_*, \chi_\alpha^\delta \rangle \geq 0, \quad \langle Fu_\alpha^\delta + \alpha u_\alpha^\delta - f^\delta, -\chi_\alpha^\delta \rangle \geq 0.$$

Summation of those two inequalities gives

$$\begin{aligned} 0 &\leq \langle Fu_\alpha + \alpha u_\alpha - f_*, \chi_\alpha^\delta \rangle - \langle Fu_\alpha^\delta + \alpha u_\alpha^\delta - f^\delta, \chi_\alpha^\delta \rangle \\ &= -\langle Fu_\alpha^\delta - Fu_\alpha, \chi_\alpha^\delta \rangle - \alpha \langle \chi_\alpha^\delta, \chi_\alpha^\delta \rangle + \langle f^\delta - f_*, \chi_\alpha^\delta \rangle \leq 0 - \alpha \|\chi_\alpha^\delta\|^2 + \delta \|\chi_\alpha^\delta\|, \end{aligned}$$

and the statement of the theorem follows by rearranging terms. \square

Remark 2.3. Existence results for variational inequalities (either for similar, more general or more specific situations) may also be found in others papers and monographs. See, e.g. Barbu and Precupanu [3, Theorem 2.67 and subsequent remark] and Kinderlehrer and Stampacchia [16, Corollary 1.8 of Chapter III]. In Bakushinsky, Kokurin and Kokurin [2, Lemma 6.1.3] and Brézis [6, Proposition 31], a simple proof is given for the special case that the operator F satisfies a Lipschitz condition on the monotonicity set \mathcal{M} .

Using more refined arguments, it is possible to weaken the assumptions of Theorem 2.2 without changing the statement of the theorem: the condition “separable” on the Hilbert space \mathcal{H} can be removed in fact, and the assumption “demicontinuous, bounded” on the operator F can be replaced by the weaker property “hemicontinuous”; cf. Browder [7] or Brézis [6, Theorem 24]. \triangle

Below we consider the overall regularization error $u_\alpha^\delta - u_*$, where $u_* \in \mathcal{M}$ denotes a classical or generalized solution of the equation $Fu = f_*$, cf. (1.1) above or (3.1) below. This overall

error can be decomposed into regularization error $u_\alpha - u_*$ and noise amplification term $u_\alpha^\delta - u_\alpha$. The latter term has already been estimated in (2.3), and we thus have

$$\|u_\alpha^\delta - u_*\| \leq \|u_\alpha - u_*\| + \frac{\delta}{\alpha} \quad \text{for each } \alpha > 0. \quad (2.4)$$

Below we thus may focus on the estimation of the bias norm $\|u_\alpha - u_*\|$.

3 Convergence of regularized solutions

In this section, we consider strong convergence of the elements u_α generated by the penalized variational inequality method (2.1) as $\alpha \rightarrow 0$. We continue to assume that the conditions stated in Assumption 2.1 are satisfied.

As a preparation, we consider the unperturbed, unpenalized version of the penalized variational inequality method (2.1), i.e. the determination of an $u_* \in \mathcal{M}$ which satisfies the variational inequality

$$\langle Fu_* - f_*, v - u_* \rangle \geq 0 \quad \text{for each } v \in \mathcal{M}. \quad (3.1)$$

Remark 3.1. (a) We note that any classical solution of (1.1) obviously satisfies the variational inequality (3.1), so the set of solutions of (3.1) is by assumption non-empty.

(b) The variational inequality (3.1) is equivalent with $\langle Fv - f_*, v - u_* \rangle \geq 0$ for each $v \in \mathcal{M}$, cf. e.g., Showalter [23, Corollary 2.4]), or Browder [7, Lemma 1]. This in particular means that the set of solutions satisfying the variational inequality (3.1) is closed and convex, and thus it has a unique element with minimal norm u_{**} .

(c) Let the operator F be strictly monotone on \mathcal{M} , i.e., in (1.2) we may replace “ \geq ” by strict inequality “ $>$ ” for each $u, v \in \mathcal{M}$ with $u \neq v$. Then (3.1) and also (1.1) have at most one solution, respectively.

(d) Any element $u_* \in \mathcal{M}$ solves the variational inequality (3.1) if and only if the identity $u_* = \mathcal{P}_\mathcal{M}(u_* - \mu(Fu_* - f_*))$ holds for each $\mu \geq 0$, where $\mathcal{P}_\mathcal{M} : \mathcal{H} \rightarrow \mathcal{H}$ denotes the convex projection onto the set \mathcal{M} . This follows from a standard variational formulation for convex projections, see e.g., Kinderlehrer and Stampacchia [16, Theorem 2.3 of Chapter I]. A similar statement holds for the penalized variational inequality method (2.1). \triangle

Theorem 3.2. *Let Assumption 2.1 be satisfied. We have $u_\alpha \rightarrow u_{**}$ as $\alpha \rightarrow 0$, where $u_{**} \in \mathcal{M}$ denotes the minimum norm solution of the variational inequality (3.1).*

Proof. This easily follows, e.g., by a compilation of the steps considered in the proof of Theorem 3 in Ryzantseva [22]. \square

Remark 3.3. Convergence of the penalized variational inequality method is in fact the subject of many research papers and monographs, see e.g., Alber and Ryzantseva [1, Theorem 4.1.1], Bakushinsky, Kokurin and Kokurin [2, Lemma 6.1.4], Khan, Tammer and Zalinescu [15], Liu and Nashed [18], and Ryzantseva [22], and the references therein.

Quite frequently in the literature, more general situations than in the present paper are considered, e.g., perturbation of the considered convex set \mathcal{M} in (3.1), or set-valued operators F in Banach spaces. On the other hand, the assumptions made in Theorem 3.2 are weaker in some aspects. For example, we allow the monotonicity set in (1.2) to be a nontrivial subset of

\mathcal{H} , with a possibly empty interior, and in addition no Lipschitz continuity of the operator F is required in Theorem 3.2. \triangle

As an immediate consequence of Theorem 3.2 and estimate (2.4), we obtain the following result.

Corollary 3.4. *Let Assumption 2.1 be satisfied. For any a priori parameter choice $\alpha = \alpha(\delta)$ with $\alpha(\delta) \rightarrow 0$ and $\frac{\delta}{\alpha(\delta)} \rightarrow 0$ as $\delta \rightarrow 0$, we have*

$$u_{\alpha(\delta)}^\delta \rightarrow u_{**} \quad \text{as } \delta \rightarrow 0, \quad (3.2)$$

where u_{**} is as in Theorem 3.2.

4 Convergence rates for regularized solutions

In this section, we provide convergence rates of u_α as $\alpha \rightarrow 0$ under adjoint source conditions. We continue to assume that the conditions stated in Assumption 2.1 are satisfied. In addition, the following class of operators will be of importance, cf. Bauschke and Combettes [4, Definition 4.4].

Definition 4.1. An operator $F : \mathcal{H} \supset \mathcal{D}(F) \rightarrow \mathcal{H}$ in a Hilbert space \mathcal{H} is called *cocoercive* on a subset $\mathcal{M} \subset \mathcal{D}(F)$ if, for some constant $\tau > 0$, we have

$$\langle Fu - Fv, u - v \rangle \geq \tau \|Fu - Fv\|^2 \quad \text{for each } u, v \in \mathcal{M}. \quad (4.1)$$

A cocoercive operator is sometimes called inverse strongly monotone. For $\tau > 0$ fixed, an operator F is cocoercive on \mathcal{M} with constant τ if and only if $I - \mu F$ is nonexpansive for each $0 \leq \mu \leq 2\tau$. Cocoerciveness obviously implies monotonicity. An example of a cocoercive operator may be found in Liu and Nashed [18, Example 3]. Another example is given in section 6.1 of the present paper.

Below, frequently we make use of the following Lipschitz condition.

Assumption 4.2. Let $\mathcal{D}(F) \subset \mathcal{H}$ be an open subset, and let F be Fréchet differentiable on $\mathcal{D}(F)$. In addition, let the following Lipschitz condition be satisfied on a given subset $\mathcal{M} \subset \mathcal{D}(F)$,

$$\|F'(u) - F'(v)\| \leq L \|u - v\| \quad \text{for each } u, v \in \mathcal{M}, \quad (4.2)$$

where $L \geq 0$ denotes some finite constant.

The following proposition provides a useful tool for the verification of cocoerciveness of a nonlinear operator.

Proposition 4.3. Let Assumptions 2.1 and 4.2 be satisfied. Let $F'(u)$ be cocoercive on \mathcal{H} , uniformly for $u \in \mathcal{M}$, i.e., there exists some constant $\tau > 0$ such that for each $u \in \mathcal{M}$

$$\langle F'(u)h, h \rangle \geq \tau \|F'(u)h\|^2 \quad \forall h \in \mathcal{H}, \quad (4.3)$$

holds. Then F is cocoercive on \mathcal{M} , with constant τ .

Proof. From uniform cocoerciveness of F' , we obtain for any $u \in \mathcal{M}$ and $h \in \mathcal{H}$ with $u + h \in \mathcal{M}$ that $F(u + h) - F(u) = \int_0^1 F'(u + th)h dt$, and thus

$$\begin{aligned} \langle F(u + h) - F(u), h \rangle &= \int_0^1 \langle F'(u + th)h, h \rangle dt \geq \tau \int_0^1 \|F'(u + th)h\|^2 dt \\ &\geq \tau \left(\int_0^1 \|F'(u + th)h\| dt \right)^2 \geq \tau \left\| \int_0^1 F'(u + th)h dt \right\|^2 = \tau \|F(u + h) - F(u)\|^2. \quad \square \end{aligned}$$

Remark 4.4. (a) If $F'(u)$ is a monotone operator on \mathcal{H} for each $u \in \mathcal{M}$, then F is monotone on \mathcal{M} . This immediately follows from the proof of Proposition 4.3 by considering the case $\tau = 0$ there.

(b) It is evident from the proof of Proposition 4.3 that in (4.3), “ $\forall h \in \mathcal{H}$ ” can be replaced by the weaker condition “ $\forall h \in \mathcal{H}$ satisfying $u + th \in \mathcal{M}$ for $t > 0$ sufficiently small”, without changing the statement of the proposition. One can show that this in fact yields an equivalent condition for cocoerciveness. \triangle

For ill-posed problems, convergence rates can only be obtained under additional conditions on the solution. In this section we assume that there exists a solution of equation (1.1) which belongs to \mathcal{M} and satisfies an adjoint source condition, i.e.,

$$u_* \in \mathcal{M}, \quad Fu_* = f_*, \quad u_* = F'(u_*)^* z, \quad \|z\| =: \varrho, \quad (4.4)$$

for some $z \in \mathcal{H}$. This completes the formulation of the basic assumptions needed in this section.

For the proof of the main result of this section, cf. Theorem 4.6 below, we need the following lemma. For any element $u \in \mathcal{M}$ consider

$$\Delta_\alpha := \Delta_\alpha(u) = u_\alpha - u, \quad r_\alpha = Fu_\alpha - f_*, \quad e_\alpha = \Delta_\alpha(u_*) \quad \text{for } \alpha > 0, \quad (4.5)$$

where $u_\alpha \in \mathcal{M}$ is introduced in (2.2).

Lemma 4.5. *Let Assumption 2.1 be satisfied. For any $u \in \mathcal{M}$ we have, with the notations from (4.5),*

$$\langle r_\alpha, \Delta_\alpha \rangle + \alpha \|\Delta_\alpha\|^2 \leq -\alpha \langle u, \Delta_\alpha \rangle \quad \text{for } \alpha > 0. \quad (4.6)$$

Proof. We consider (2.1) with $\delta = 0$, which means $f^\delta = f_*$ in fact:

$$\langle Fu_\alpha - f_* + \alpha u_\alpha, u_\alpha - u \rangle = \langle r_\alpha + \alpha u_\alpha, \Delta_\alpha \rangle = \langle r_\alpha, \Delta_\alpha \rangle + \alpha \langle u_\alpha, \Delta_\alpha \rangle \leq 0.$$

From this we obtain

$$\langle r_\alpha, \Delta_\alpha \rangle + \alpha \|\Delta_\alpha\|^2 = \langle r_\alpha, \Delta_\alpha \rangle + \alpha \langle u_\alpha, \Delta_\alpha \rangle - \alpha \langle u, \Delta_\alpha \rangle \leq -\alpha \langle u, \Delta_\alpha \rangle,$$

which is (4.6). \square

We are now in a position to formulate the main result of this section.

Theorem 4.6. *Let Assumptions 2.1 and 4.2 be fulfilled. If F is cocoercive on \mathcal{M} , and if in addition the adjoint source condition (4.4) is satisfied with $\varrho L < 2$, then*

$$\|u_\alpha - u_*\| = \mathcal{O}(\alpha^{1/2}), \quad \|Fu_\alpha - f_*\| = \mathcal{O}(\alpha) \quad \text{as } \alpha \rightarrow 0. \quad (4.7)$$

Proof. We proceed with (4.6) for $u = u_*$. From (4.4) we obtain, with the notations introduced in (4.5),

$$-\langle u_*, e_\alpha \rangle = -\langle F'(u_*)^* z, e_\alpha \rangle = -\langle z, F'(u_*) e_\alpha \rangle \leq \varrho \|F'(u_*) e_\alpha\|. \quad (4.8)$$

For a further estimation of (4.8), we need to consider the first order remainder $\mathcal{R} = \mathcal{R}_{u_*}$ of a Taylor expansion at $u_* \in \mathcal{D}(F)$:

$$\mathcal{R}(u) = F(u) - F(u_*) - F'(u_*)(u - u_*), \quad u \in \mathcal{D}(F).$$

For $h \in \mathcal{H}$ such that the line segment from u_* to $u_* + h$ belongs to $\mathcal{D}(F)$, we have $\mathcal{R}(u_* + h) = \int_0^1 (F'(u_* + th) - F'(u_*))h dt$ and thus $\|\mathcal{R}(u_* + h)\| \leq \frac{L}{2}\|h\|^2$. This gives $F'(u_*)e_\alpha = F(u_\alpha) - F(u_*) - \mathcal{R}(u_\alpha) = r_\alpha - \mathcal{R}(u_\alpha)$ with $\|\mathcal{R}(u_\alpha)\| \leq \frac{L}{2}\|e_\alpha\|^2$. We are now in a position to proceed with the upper bound in (4.8):

$$\|F'(u_*)e_\alpha\| \leq \|r_\alpha\| + \|\mathcal{R}(u_\alpha)\| \leq \|r_\alpha\| + \frac{L}{2}\|e_\alpha\|^2. \quad (4.9)$$

The estimates (4.6) for $u = u_*$ and (4.8) – (4.9) finally give

$$\langle r_\alpha, e_\alpha \rangle + \alpha \|e_\alpha\|^2 \leq -\alpha \langle u_*, e_\alpha \rangle \leq \varrho \alpha \|F'(u_*)e_\alpha\| \leq \varrho \alpha (\|r_\alpha\| + \frac{L}{2}\|e_\alpha\|^2),$$

and thus

$$\langle r_\alpha, e_\alpha \rangle + \alpha(1 - \frac{\varrho L}{2})\|e_\alpha\|^2 \leq \varrho \alpha \|r_\alpha\|. \quad (4.10)$$

This in particular means $\langle r_\alpha, e_\alpha \rangle \leq \varrho \alpha \|r_\alpha\|$, and cocoerciveness, cf. (4.1), moreover means $\langle r_\alpha, e_\alpha \rangle \geq \tau \|r_\alpha\|^2$. We thus obtain

$$\tau \|r_\alpha\| \leq \varrho \alpha, \quad (4.11)$$

i.e., $\|r_\alpha\| = \mathcal{O}(\alpha)$ as $\alpha \rightarrow 0$. From (4.10) and (4.11) we finally obtain

$$\tau(1 - \frac{\varrho L}{2})\|e_\alpha\|^2 \leq \tau \varrho \|r_\alpha\| \leq \varrho^2 \alpha,$$

which is the first statement in (4.7). \square

Remark 4.7. (a) From Theorem 4.6 and Theorem 3.2 it follows that any u_* satisfying the conditions in (4.4) is the minimum norm solution of the variational inequality (3.1).

(b) Theorem 4.6 improves results in Liu and Nashed [18, Theorem 6], where $\|u_\alpha - u_*\| = \mathcal{O}(\alpha^{1/3})$ as $\alpha \rightarrow 0$ is obtained only (under more general assumptions, however, e.g., possible set perturbations).

(c) The first error estimate in Theorem 4.6 remains valid if in (4.4), the identity $Fu_* = f_*$ is replaced by the weaker assumption that $u_* \in \mathcal{M}$ satisfies the variational inequality (3.1). In the proof of Theorem 4.6, then one only has to make additional use of the fact that the inequality $\langle Fu_\alpha - Fu_*, u_\alpha - u_* \rangle \leq \langle Fu_\alpha - f_*, u_\alpha - u_* \rangle$ holds. The second error estimate in Theorem 4.6 has to be replaced by $\|Fu_\alpha - Fu_*\| = \mathcal{O}(\alpha)$ then.

(d) Using some ideas of Tautenhahn [24] and Janno [14], one may obtain convergence rates for source conditions of the form $u_* = F'(u_*)z$, i.e., the adjoint source condition is replaced by a classical one. This topic, however, goes beyond the scope of the present study and will be considered elsewhere.

(e) For recent results on adjoint source conditions for linear problems, see Plato, Hofmann, and Mathé [21]. \triangle

Corollary 4.8. *Under the conditions of Theorem 4.6 we have, for any a priori parameter choice $\alpha(\delta) \sim \delta^{2/3}$, the convergence rate result*

$$\|u_{\alpha(\delta)}^\delta - u_*\| = \mathcal{O}(\delta^{1/3}) \quad \text{as } \delta \rightarrow 0. \quad (4.12)$$

Remark 4.9. (a) The rate (4.12) is identical with rates obtained in [12, Theorem 3, Remark 4] for Lavrentiev regularization (1.4) with variational source conditions.

(b) The rate of convergence in (4.12) is higher than those obtained by Liu and Nashed [18], Thuy [25], and Buong [8] for the penalized variational inequality method under similar source conditions. Note that, on the other hand, the results in those papers are established in a more general framework, respectively, e.g., in Banach spaces or allowing set perturbations, and for a posteriori parameter choice strategies. \triangle

5 Modified penalized variational inequality method

Occasionally it may be useful to consider a modified version of the penalized variational inequality method (2.1). For this purpose let $\bar{u} \in \mathcal{H}$ be fixed. For $\alpha > 0$ let $u_\alpha^\delta \in \mathcal{M}$ satisfy

$$\langle Fu_\alpha^\delta + \alpha(u_\alpha^\delta - \bar{u}) - f^\delta, v - u_\alpha^\delta \rangle \geq 0 \quad \text{for each } v \in \mathcal{M}. \quad (5.1)$$

We denote by $u_\alpha = u_\alpha^0$ the approximation obtained by the modified penalized variational inequality method (5.1) with exact data $f^\delta = f_*$. Method (5.1) can be considered as variational inequality formulation of the translated Lavrentiev regularization $Fu + \alpha(u - \bar{u}) = f^\delta$. The results of sections 2–4 can be easily applied to the modified penalized variational inequality method by considering translation: replace the operator F and the monotonicity set \mathcal{M} there by

$$\tilde{F} : \mathcal{H} \supset -\bar{u} + \mathcal{D}(F) \rightarrow \mathcal{H}, \quad v \mapsto F(\bar{u} + v), \quad \tilde{\mathcal{M}} = -\bar{u} + \mathcal{M},$$

respectively. We briefly formulate the relevant results under the general assumption that the conditions stated in Assumption 2.1 are satisfied.

- (a) The modified penalized variational inequality method (5.1) has a unique solution $u_\alpha^\delta \in \mathcal{M}$ which satisfies $\|u_\alpha^\delta - u_\alpha\| \leq \frac{\delta}{\alpha}$ for each $\alpha > 0$.
- (b) We have $u_\alpha \rightarrow u_{**}$ as $\alpha \rightarrow 0$, where $u_{**} \in \mathcal{M}$ denotes the solution of the variational inequality (3.1) having minimal distance to \bar{u} . In addition, for any a priori parameter choice $\alpha = \alpha(\delta)$ with $\alpha(\delta) \rightarrow 0$ and $\frac{\delta}{\alpha(\delta)} \rightarrow 0$ as $\delta \rightarrow 0$, we have $u_{\alpha(\delta)}^\delta \rightarrow u_{**}$ as $\delta \rightarrow 0$.
- (c) If Assumption 4.2 is fulfilled and F is cocoercive on \mathcal{M} , and if in addition the adjoint source condition

$$u_* \in \mathcal{M}, \quad Fu_* = f_*, \quad u_* - \bar{u} = F'(u_*)^* z, \quad \varrho := \|z\|, \quad (5.2)$$

is satisfied with some $z \in \mathcal{H}$ and $\varrho L < 2$, then

$$\|u_\alpha - u_*\| = \mathcal{O}(\alpha^{1/2}) \quad \text{as } \alpha \rightarrow 0, \quad \|u_{\alpha(\delta)}^\delta - u_*\| = \mathcal{O}(\delta^{1/3}) \quad \text{as } \delta \rightarrow 0,$$

for any a priori parameter choice $\alpha(\delta) \sim \delta^{2/3}$.

An appropriate choice of \bar{u} guarantees that $u_* - \bar{u}$ belongs to the range of $F'(u_*)^*$, which typically requires, besides sufficient smoothness, that appropriate conditions on a subset of the boundary of the domain of definition $\mathcal{D}(F)$ are satisfied.

6 An example, and numerical illustrations

6.1 A parameter estimation problem

We consider the estimation of the coefficient $u \in L^2(0, 1)$ in the following initial value problem:

$$f' + uf = 0 \text{ a.e. on } [0, 1], \quad f(0) = -c_0 < 0,$$

where $f \in H^1(0, 1)$; cf. Groetsch [10], Hofmann [11], or Tautenhahn [24]. The initial value $-c_0$ with $c_0 > 0$ is assumed to be known exactly. This problem can be written as $Fu = f$, with

$$(Fu)(t) := -c_0 e^{-U(t)}, \quad U(t) = \int_0^t u(s) ds, \quad 0 \leq t \leq 1. \quad (6.1)$$

The operator $F : L^2(0, 1) \rightarrow L^2(0, 1)$ is bounded and Fréchet differentiable on $L^2(0, 1)$, with Fréchet derivative

$$[F'(u)h](t) = -(Fu)(t)H(t) \quad \text{for } h \in L^2(0, 1), \quad H(t) = \int_0^t h(s) ds, \quad 0 \leq t \leq 1. \quad (6.2)$$

Let

$$\mathcal{M}_\theta = \{ u \in L^2(0, 1) \mid u \geq \theta \text{ a.e. on } [0, 1] \}, \quad (6.3)$$

where $\theta \in \mathbb{R}$.

Proposition 6.1. The operator F in (6.1) is monotone on \mathcal{M}_0 . For any $\theta > 0$, it is cocoercive on \mathcal{M}_θ , with constant $\tau = \frac{\theta}{2c_0}$.

Proof. We shall make use of Proposition 4.3 and Remark 4.4. Let $u \in L^2(0, 1)$, $g := -Fu$, and $h \in L^2(0, 1)$. From (6.2) it follows that

$$\langle F'(u)h, h \rangle = \int_0^1 (gH)H' dt = gH^2|_0^1 - \int_0^1 (gH)'H dt \geq - \int_0^1 g'H^2 dt - \langle F'(u)h, h \rangle,$$

and thus

$$2\langle F'(u)h, h \rangle \geq - \int_0^1 g'H^2 dt = \int_0^1 guH^2 dt, \quad (6.4)$$

where the properties $g(1)H^2(1) \geq 0$ and $H(0) = 0$ have been used. Estimate (6.4) implies for each $u \in \mathcal{M}_0$ that $\langle F'(u)h, h \rangle \geq 0$ for each $h \in \mathcal{H}$, and the monotonicity statement for F immediately follows from Remark 4.4.

Now let $\theta > 0$ be fixed. For any $u \in \mathcal{M}_\theta$ we proceed with (6.4):

$$2\langle F'(u)h, h \rangle \geq \theta \int_0^1 gH^2 dt \geq \frac{\theta}{c_0} \int_0^1 (gH)^2 dt = \frac{\theta}{c_0} \int_0^1 (F'(u)h)^2 dt = \frac{\theta}{c_0} \|F'(u)h\|^2,$$

where the estimate $g \leq c_0$ has been applied. The cocoerciveness statement for F now follows from Proposition 4.3. \square

6.2 Numerical experiments

The theoretical results are finally illustrated by some numerical experiments for the operator $F : L^2(0, 1) \rightarrow L^2(0, 1)$ considered in (6.1), with $c_0 = 1$ there. We give a few preparatory notes on the numerical tests first.

- In each of our numerical experiments we choose a convex closed subset $\mathcal{M} = \mathcal{M}_\theta$ of the form (6.3) with some lower bound $\theta > 0$. The setting (6.3) guarantees cocoerciveness (cf. Proposition 6.1), and Lipschitz continuity (4.2) of the operator F' on \mathcal{M} holds with $L = c_0 = 1$. We consider some $u_* \in H^1(0, 1)$ with $u_* \in \mathcal{M}$, and then the adjoint source condition (5.2) is satisfied for $\bar{u} \equiv u_*(1)$. The solution u_* and the set \mathcal{M} are always chosen in such a way that the condition $\rho L < 2$ is satisfied, cf. (5.2) and the subsequent conclusion there.
- We consider the a priori parameter choice $\alpha(\delta) = \delta^{2/3}$, for different values of δ .
- The modified penalized variational inequality (5.1) is approximately solved by using a fixed point iteration for the corresponding fixed point equation

$$u_\alpha^\delta = \mathcal{P}_\mathcal{M}(u_\alpha^\delta - \mu(Fu_\alpha^\delta + \alpha(u_\alpha^\delta - \bar{u}) - f^\delta)), \quad \text{with } \alpha = \alpha(\delta),$$

and the initial guess is the function \bar{u} . Notice that the underlying fixed point operator is contractive, with contraction constant $1 - \mu\alpha$, provided that the step size satisfies $0 < \mu < 2\tau = \theta$, cf. the remarks following Definition 4.1, and Proposition 4.3. In addition, the regularization parameter must satisfy $0 < \alpha \leq \frac{1}{\mu} - \frac{1}{\theta}$. In our numerical experiments we always choose $\mu = \frac{\theta}{2}$.

Iteration is stopped if the norm difference of two consecutive iterates satisfies, for the first time, an estimate of the form $\leq c\delta$, with some constant $c > 0$. This stopping criterion ensures that the resulting approximation $\tilde{u}_{\alpha(\delta)}^\delta \in \mathcal{M}$ satisfies $\|\tilde{u}_{\alpha(\delta)}^\delta - u_{\alpha(\delta)}^\delta\| = \mathcal{O}(\delta^{1/3})$, which is of sufficient accuracy.

- The problem is discretized using a backward rectangular rule for the integrals, and replacing each considered (continuous) function $\psi : [0, 1] \rightarrow \mathbb{R}$ by $(\psi(nh))_{n=0, \dots, N}$, with step size $h = \frac{1}{N}$ for $N = 200$. This leads to a fully discretized nonlinear problem in \mathbb{R}^{N+1} .
- In the numerical experiments we consider perturbations of the form $f_n^\delta = f(nh) + \Delta_n$, $n = 0, 1, \dots, N$, with uniformly distributed random values Δ_n satisfying $|\Delta_n| \leq \delta$.

Example 6.2. We first consider the equation $Fu = f_*$, with right-hand side

$$f_*(t) = -\exp(-\frac{a}{2}t^2 - bt) \quad \text{for } 0 \leq t \leq 1,$$

with $a = b = \frac{1}{2}$. The exact solution is then given by

$$u_*(t) = at + b \quad \text{for } 0 \leq t \leq 1.$$

We may consider the set $\mathcal{M} = \mathcal{M}_\theta$ in (6.3) with $\theta = b$. The numerical results are given in Table 1. \triangle

Example 6.3. We next consider the equation $Fu = f_*$ with right-hand side

$$f_*(t) = -\exp(\frac{a}{\pi}(\cos \pi t - 1) - bt) \quad \text{for } 0 \leq t \leq 1,$$

δ	$100 \cdot \delta / \ f\ $	$\ u_{\alpha(\delta)}^\delta - u_*\ $	$\ u_{\alpha(\delta)}^\delta - u_*\ / \delta^{1/3}$
$1.0 \cdot 10^{-2}$	$1.33 \cdot 10^0$	$9.87 \cdot 10^{-2}$	0.46
$5.0 \cdot 10^{-3}$	$6.66 \cdot 10^{-1}$	$8.23 \cdot 10^{-2}$	0.48
$2.5 \cdot 10^{-3}$	$3.33 \cdot 10^{-1}$	$6.72 \cdot 10^{-2}$	0.50
$1.2 \cdot 10^{-3}$	$1.67 \cdot 10^{-1}$	$5.42 \cdot 10^{-2}$	0.50
$6.2 \cdot 10^{-4}$	$8.33 \cdot 10^{-2}$	$4.17 \cdot 10^{-2}$	0.49
$3.1 \cdot 10^{-4}$	$4.16 \cdot 10^{-2}$	$3.26 \cdot 10^{-2}$	0.48
$1.6 \cdot 10^{-4}$	$2.08 \cdot 10^{-2}$	$3.26 \cdot 10^{-2}$	0.61
$7.8 \cdot 10^{-5}$	$1.04 \cdot 10^{-2}$	$2.72 \cdot 10^{-2}$	0.64
$3.9 \cdot 10^{-5}$	$5.21 \cdot 10^{-3}$	$2.53 \cdot 10^{-2}$	0.75

Table 1: Numerical results for Example 6.2

with $a = \frac{1}{4}$, $b = \frac{1}{3}$. The exact solution is then given by

$$u_*(t) = a \sin \pi t + b \quad \text{for } 0 \leq t \leq 1.$$

We may consider the set $\mathcal{M} = \mathcal{M}_\theta$ in (6.3) with $\theta = b$. The numerical results are shown in Table 2. \triangle

δ	$100 \cdot \delta / \ f\ $	$\ u_{\alpha(\delta)}^\delta - u_*\ $	$\ u_{\alpha(\delta)}^\delta - u_*\ / \delta^{1/3}$
$1.0 \cdot 10^{-2}$	$1.25 \cdot 10^0$	$7.00 \cdot 10^{-2}$	0.32
$5.0 \cdot 10^{-3}$	$6.25 \cdot 10^{-1}$	$4.66 \cdot 10^{-2}$	0.27
$2.5 \cdot 10^{-3}$	$3.12 \cdot 10^{-1}$	$3.87 \cdot 10^{-2}$	0.29
$1.2 \cdot 10^{-3}$	$1.56 \cdot 10^{-1}$	$3.01 \cdot 10^{-2}$	0.28
$6.2 \cdot 10^{-4}$	$7.81 \cdot 10^{-2}$	$2.22 \cdot 10^{-2}$	0.26
$3.1 \cdot 10^{-4}$	$3.90 \cdot 10^{-2}$	$1.60 \cdot 10^{-2}$	0.24
$1.6 \cdot 10^{-4}$	$1.95 \cdot 10^{-2}$	$1.08 \cdot 10^{-2}$	0.20
$7.8 \cdot 10^{-5}$	$9.76 \cdot 10^{-3}$	$7.54 \cdot 10^{-3}$	0.18
$3.9 \cdot 10^{-5}$	$4.88 \cdot 10^{-3}$	$4.70 \cdot 10^{-3}$	0.14

Table 2: Numerical results for Example 6.3

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