Regularization of autoconvolution and other ill-posed quadratic equations by decomposition

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Abstract

Standard methods for regularizing ill-posed nonlinear equations rely on derivatives of the nonlinear forward mapping. Thereby stronger structural properties of the concrete problem are neglected and the derived algorithms only show mediocre efficiency.

We concentrate on nonlinear mappings with quadratic structure and develop a derivative-free regularization method that allows us to apply classical techniques known from linear inverse problems to quadratic equations. In fact, regularization of a quadratic problem can be reduced to regularization of one linear problem and a downstream inversion of a well-posed quadratic mapping.

The motivation for considering problems with quadratic structure in more detail comes from applications in laser optics where kernel-based autoconvolution-type equations have to be solved.

1 Introduction

In the early nineties of the previous century regularization of nonlinear ill-posed equations $F(x) = y$ became an important and widely discussed topic in mathematical research [4]. Many methods and algorithms have been

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suggested and are passably understood now [14, 18, 19]. Most of these regularization techniques linearize the problem, regularize the linearization by classical methods designed for linear problems, then linearize at a new point, and so on. For example this is the case for Landweber and Gauss-Newton methods. But also nonlinear Tikhonov regularization, which at first sight is derivative-free, finally gets linearized for the sake of minimization [17]. All these techniques have proven useful but nonetheless we sould ask: Can we do better?

First we quickly fix the setting. We seek for a stable approximate solution of

$$F(x) = y \tag{1.1}$$

with $F : X \to Y$ mapping between real separable Hilbert spaces X and Y. Typically, the right-hand side y is only known approximately, that is, we have y^δ from Y such that $\|y - y^\delta\| \leq \delta$ for some noise level $\delta \geq 0$. Note that for our purposes we may assume that the nonlinear mapping F is defined on the whole space X.

All well-established derivative-based regularization techniques have a common weak point. To ensure convergence of regularized solutions to an exact solution of (1.1) and to derive corresponding convergence rates one has to assume that the derivative $F'[x_0]$ at some point x_0 (often the unknown exact solution) represents the nonlinear mapping F in a sufficiently precise way [14, 19]. In mathematical terms one assumes that

$$\|F(x) - F(x_0) - F'[x_0](x - x_0)\| \leq c\|F(x) - F(x_0)\|^{\gamma_1}\|x - x_0\|^{\gamma_2} \tag{1.2}$$

for sufficiently many x, see [10, 13] and references therein. Depending on γ_1 and γ_2 the derived convergence rate results are more or less strong.

Unfortunately, there are nonlinear mappings which do not satisfy estimates (1.2) or the corresponding constants γ_1 and γ_2 have to be chosen in a way that only provides quite weak conclusions from such a nonlinearity condition. This is the case for autoconvolution problems [6, 7, 9]: for $X = L^2(0, 1)$ and $Y = L^2(0, 2)$ we define F by

$$\big(F(x)\big)(s) := \int_{\mathbb{R}} x(t)x(s - t)\,\mathrm{d}t, \tag{1.3}$$

where x is set to zero outside $(0, 1)$. For this concrete ill-posed nonlinear mapping the only known combination of constants in (1.2) is $\gamma_1 = 0$ and $\gamma_2 = 2$, which constitutes a very weak connection between F and its derivative.

If the derivative does not carry enough information about F we have to look for other, derivative-free, regularization techniques. Especially autoconvolution-type mappings posses a strong structure of quadratic type. Below we will give a precise definition of the term *quadratic mapping*. Although the term *quadratic* sounds very familiar handling quadratic mappings is not a trivial task. To give the reader an idea of what a quadratic mapping can do with the three-dimensional space we provide Figure 1.

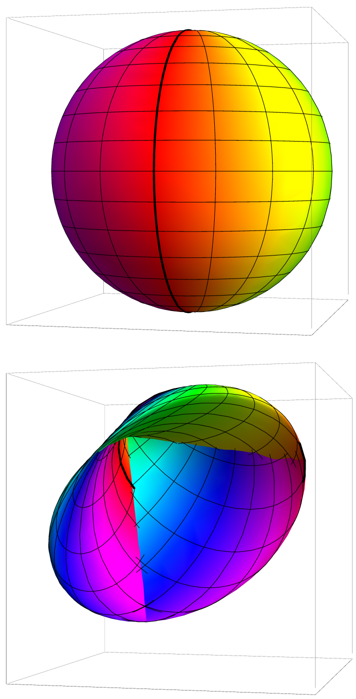

Figure 1: Example of a quadratic mapping acting in $\mathbb{R}^3$. Upper left: unit sphere. Upper right: image set of unit sphere under $(x_1, x_2, x_3) \mapsto (x_1^2 + \sqrt{2}x_2x_3, x_2^2 + \sqrt{2}x_1x_3, x_3^2 + \sqrt{2}x_1x_2)$. Lower left: same as upper right, but from opposite direction. Lower right: same as upper right but with cap removed to see the overlap.

Our idea is to decompose an ill-posed quadratic mapping into a well-posed quadratic part and an ill-posed linear operator. The latter can be inverted by classical regularization techniques and the quadratic part will turn out to be invertible without serious trouble.

Other approaches can be found in the literature. In [3] a specific problem of autoconvolution-type is solved by so called local regularization. General quadratic equations are solved in [17] by Tikhonov regularization with emphasis on gradient-based minimization of the Tikhonov functional.

The idea to decompose nonlinear mappings into a well-posed nonlinear part and an ill-posed linear one is not totally new. For example so called state space regularization [2] uses decomposition techniques and the degree of ill-posedness can be studied via suitable decompositions [11].

Quadratic autoconvolution-type mappings appear for example in laser optics. For measuring ultra-short laser pulses physicists from Max Born Institute for Nonlinear Optics and Short Pulse Spectroscopy in Berlin, Germany, developed the so called SD SPIDER method [8,15]. The data of interest and the measurable data are connected by a kernel-based autoconvolution-type mapping

$$\big(F(x)\big)(s) := \int_{\mathbb{R}} k(s,t)x(t)x(s-t)\,\mathrm{d}t, \tag{1.4}$$

with a complexvalued kernel function $k : (0,2) \times (0,1) \to \mathbb{C}$. The square-integrable functions x and $F(x)$ with supports $(0,1)$ and $(0,2)$ are complexvalued, too. As a first step we restrict our attention to realvalued functions. But our decomposition approach in principle works also for the complex case. Another possible application are problems in Schlieren tomography [16] where the pressure field of ultrasound transducers has to be reconstructed and the technique of appearence potential spectroscopy [1].

The remaining part of the article is organized as follows. The next section contains material on quadratic mappings. Most things are quite standard, but up to now not prevalent in the inverse problems community. The decomposition approach is presented and analyzed in Section 3, and Section 4 contains some hints on discretizing quadratic equations.

2 Quadratic mappings and their properties

First we provide the definition of quadratic mappings. Remember that X and Y are real separable Hilbert spaces throughout this article and that a bilinear mapping $B : X \times X \to Y$ is bounded if $\|B(x,u)\| \leq c\|x\|\|u\|$ for all x and u from X and for some nonnegative constant c. The smallest constant c in this estimate will be denoted by $\|B\|$. As known for linear operators boundedness and continuity of bilinear mappings are equivalent.

Definition 2.1. A mapping $F : X \to Y$ is *quadratic* if there is a bounded bilinear mapping $B : X \times X \to Y$ such that $F(x) = B(x,x)$ for all x in X.

In the definition we assume that the underlying bilinear mapping is bounded. This assumption guarantees that the corresponding quadratic mapping is continuous. Before we prove this fact we have to show that there is a one-to-one correspondence between quadratic mappings and *symmetric* bounded bilinear mappings.

Proposition 2.2. *For each quadratic mapping F there is a uniquely determined symmetric bounded bilinear mapping B_F such that $F(x) = B_F(x,x)$ for all x in X.*

Proof. One easily verifies that B_F defined by

$$B_F(x,u) := F\left(\frac{x+u}{2}\right) - F\left(\frac{x-u}{2}\right) \tag{2.1}$$

for x and u from X is bilinear, bounded, and symmetric.

If there is another symmetric bounded bilinear mapping B with $F(x) = B(x,x)$ for all x, then exploiting symmetry and bilinearity of B we immediately see

$$\begin{aligned} B_F(x,u) &= F\left(\frac{x+u}{2}\right) - F\left(\frac{x-u}{2}\right) \\ &= B\left(\frac{x+u}{2}, \frac{x+u}{2}\right) - B\left(\frac{x-u}{2}, \frac{x-u}{2}\right) = B(x,u) \end{aligned}$$

for all x and u, that is, $B_F = B$. □

Throughout this article B_F denotes the underlying symmetric bilinear mapping for F.

Proposition 2.3. *Quadratic mappings are Lipschitz continuous on bounded sets. In more detail, each quadratic mapping F satisfies*

$$\|F(x) - F(u)\| \leq 2r\|B_F\|\|x-u\| \tag{2.2}$$

for all x and u with $\|x\| \leq r$ and $\|u\| \leq r$.

Proof. Exploiting the symmetry of B_F we obtain

$$\|F(x)-F(u)\| = \|B_F(x+u, x-u)\| \leq \|B_F\|\|x+u\|\|x-u\| \leq 2r\|B_F\|\|x-u\|.$$

□

To specify the term *ill-posed* for quadratic mappings we use the well-known subsequential formulation [5].

Definition 2.4. A quadratic mapping $F : X \to Y$ is *well-posed* if for each convergent sequence $(y_k)_{k\in\mathbb{N}}$ in Y with limit y in Y each sequence $(x_k)_{k\in\mathbb{N}}$ of preimages x_k from $F^{-1}(y_k)$ has a convergent subsequence and the corresponding limit belongs to $F^{-1}(y)$. Otherwise F is *ill-posed.*

The formulation of our decomposition approach in Section 3 requires a notion of quadratic isometry. It is well known that a linear operator preserves inner products if and only if it preserves norms. In the quadratic case there are (strong) isometries, which preserve both inner products and norms, and there are (weak) isometries, which only preserve norms. Below we provide a simple example of a weak isometry which is not strong. But first the exact definitions.

Definition 2.5. A quadratic mapping $F : X \to Y$ is a *strong isometry* if $\langle F(x), F(u)\rangle = \langle x, u\rangle^2$ for all x and u from X and a *weak isometry* if $\|F(x)\| = \|x\|^2$ for all x.

Obviously, each strong isometry is also weak.

Example 2.6. Define $F : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F(x) = \begin{bmatrix} x_1^2 - x_2^2 \\ 2x_1x_2 \end{bmatrix}.$$

Then $\|F(x)\|^2 = (x_1^2 - x_2^2)^2 + 4x_1^2x_2^2 = \|x\|^2$ for all x, but

$$\left\langle F\left(\begin{bmatrix}1\\0\end{bmatrix}\right), F\left(\begin{bmatrix}0\\1\end{bmatrix}\right)\right\rangle = -1 \neq 0 = \left\langle \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\rangle.$$

Thus, F is a weak isometry but not a strong one.

An example of a strong quadratic isometry in infinite-dimensional spaces will be given in Section 3. Figure 2 visualizes a strong quadratic isometry mapping between $\mathbb{R}^2$ and $\mathbb{R}^3$.

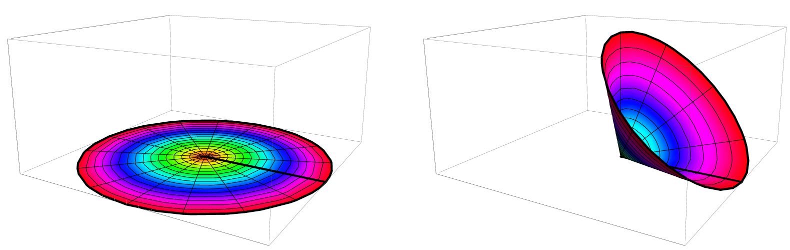

Figure 2: Example of a strong quadratic isometry acting between $\mathbb{R}^2$ and $\mathbb{R}^3$. The element (x_1, x_2) is mapped to $(x_1^2, \sqrt{2}x_1x_2, x_2^2)$. Left-hand side: unit disc in $\mathbb{R}^2$. Right-hand side: image set of unit disc.

For checking isometric properties of quadratic mappings we provide the following criterion.

Proposition 2.7. *Let $(e_i)_{i\in\mathbb{N}}$ be an orthonormal basis in X. A quadratic mapping $F : X \to Y$ is a strong isometry if and only if the following two conditions hold:*

(i) $\|B_F(e_i, e_j)\| = \begin{cases} 1, & j = i, \\ \frac{1}{\sqrt{2}}, & j < i. \end{cases}$

(ii) The set $\{B_F(e_i, e_j) : i \in \mathbb{N},\, j \le i\}$ is an orthogonal system.

Proof. Necessity follows from calculation of $\langle B_F(e_i, e_j), B_F(e_k, e_l)\rangle$. With (2.1) we obtain

$$\begin{aligned}\langle B_F(e_i, e_j), B_F(e_k, e_l)\rangle &= \frac{1}{2}\langle e_i, e_k\rangle\langle e_j, e_l\rangle + \frac{1}{2}\langle e_i, e_l\rangle\langle e_j, e_k\rangle \\ &= \begin{cases} 1, & i = j = k = l, \\ \frac{1}{2}, & i = k \neq l = j \text{ or } i = l \neq k = j, \\ 0, & \text{else,} \end{cases}\end{aligned}$$

which directly yields the two condions in the proposition.

For sufficiency we observe

$$\begin{aligned}&\left\langle F\left(\sum_{i=1}^{\infty} x_i e_i\right), F\left(\sum_{k=1}^{\infty} u_k e_k\right)\right\rangle \\ &\quad= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} x_i x_j u_k u_l \langle B_F(e_i, e_j), B_F(e_k, e_l)\rangle \\ &\quad= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x_i x_j u_i u_j = \left(\sum_{i=1}^{\infty} x_i u_i\right)^2 = \left\langle \sum_{i=1}^{\infty} x_i e_i, \sum_{k=1}^{\infty} u_k e_k \right\rangle^2. \qquad \square\end{aligned}$$

As one might expect from an isometry, each strong quadratic isometry is continuously invertible. Note that quadratic mappings cannot be injective because $F(x) = F(-x)$ for all x. Thus, we have to use a slightly generalized notion of continuous invertibility. In view of Definition 2.4 strong quadratic isometries always are well-posed.

Proposition 2.8. *Let $F : X \to Y$ be a strong quadratic isometry and denote by $F^{-1}(y)$ the full preimage of F at some point y. If a sequence $(y_k)_{k\in\mathbb{N}}$ in Y converges to some y in Y and if $(x_k)_{k\in\mathbb{N}}$ is a sequence of corresponding preimages x_k from $F^{-1}(y_k)$, then $(x_k)_{k\in\mathbb{N}}$ decomposes into two subsequences $(x_k^+)_{k\in\mathbb{N}}$ and $(x_k^-)_{k\in\mathbb{N}}$ such that $x_k^+ \to x$ and $x_k^- \to -x$, where x is a preimage of y.*

Proof. Define index sets

$$I^+ := \{k \in \mathbb{N} : \langle x_k, x\rangle \geq 0\} \quad \text{and} \quad I^- := \{k \in \mathbb{N} : \langle x_k, x\rangle < 0\}.$$

We only consider the case that both sets have infinitely many elements. Then $(x_k^+)_{k\in\mathbb{N}}$ is the subsequence $(x_k)_{k\in I^+}$ and $(x_k^-)_{k\in\mathbb{N}}$ is the subsequence $(x_k)_{k\in I^-}$.

Since F is a strong isometry we have

$$\|x_k^+ - x\|^2 = \|x_k^+\|^2 - 2\langle x_k^+, x\rangle + \|x\|^2 = \|y_k\| - 2\sqrt{\langle y_k, y\rangle} + \|y\|.$$

The first summand converges to $\|y\|$ and the second to $-2\|y\|$. Thus,

$$\|x_k^+ - x\|^2 \to 0.$$

Analogously, we obtain

$$\|x_k^- - (-x)\|^2 = \|x_k^-\|^2 + 2\langle x_k^-, x\rangle + \|x\|^2 = \|y_k\| - 2\sqrt{\langle y_k, y\rangle} + \|y\| \to 0.$$

The second equality follows from $\langle x_k^-, x\rangle^2 = \langle y_k, y\rangle$ and $\langle x_k^-, x\rangle < 0$. □

We close our short run-through of quadratic mappings with the introduction of adjoint bilinear mappings.

Definition 2.9. A bounded bilinear mapping $B^* : X\times Y \to X$ is the *adjoint* of the symmetric bounded bilinear mapping $B : X \times X \to Y$ if

$$\langle x, B^*(u, y)\rangle = \langle B(x, u), y\rangle = \langle u, B^*(x, y)\rangle$$

for all x and u in X and all y in Y.

Proposition 2.10. *Each symmetric bounded bilinear mapping $B : X\times X \to Y$ has a uniquely determined adjoint.*

Proof. For y in Y and u in X the mapping $x \mapsto \langle B(x, u), y\rangle$ is a bounded linear functional on X. Thus, there is some $\bar{u}$ in X with $\langle x, \bar{u}\rangle = \langle B(x, u), y\rangle$ for all x. Setting $B^*(u, y) := \bar{u}$ and exploiting symmetry of B we see

$$\langle x, B^*(u, y)\rangle = \langle B(x, u), y\rangle = \langle B(u, x), y\rangle = \langle u, B^*(x, y)\rangle$$

for all x, u, y. Obviously, B^* is bilinear and bounded, thus, an adjoint of B. Uniqueness of the adjoint follows by construction. □

3 Regularization of quadratic mappings by decomposition

Here is our main theorem.

Theorem 3.1. *Each quadratic mapping $F : X \to Y$ can be decomposed into a strong quadratic isometry $Q : X \to \ell^2(\mathbb{N})$ and a densely defined linear operator $A : \ell^2(\mathbb{N}) \to Y$ such that*

$$F(x) = AQ(x) \tag{3.1}$$

for all x in X.

The proof is constructive and will be given in the following. The two lemmas provide a possible choice of the quadratic part Q and the linear part A. But as we will discuss later, other choices are possible and maybe advantageous.

For easier handling of indices we define the mapping $\kappa : \{(i,j) \in \mathbb{N}\times\mathbb{N} : 1 \le j \le i\} \to \mathbb{N}$ by

$$\kappa(i,j) := j + \frac{i(i-1)}{2}. \tag{3.2}$$

This is a bijection.

Lemma 3.2. *Let $(e_i)_{i\in\mathbb{N}}$ be an orthonormal basis of X. The mapping $Q : X \to \ell^2(\mathbb{N})$ defined by*

$$\big(Q(x)\big)_{\kappa(i,j)} := \begin{cases} \sqrt{2}\langle x, e_i\rangle\langle x, e_j\rangle, & j < i, \\ \langle x, e_i\rangle^2, & j = i \end{cases} \tag{3.3}$$

for (i,j) in $\mathbb{N}\times\mathbb{N}$ with $1 \le j \le i$ and x in X is a strong quadratic isometry.

Proof. The underlying symmetric bilinear mapping of Q is given by

$$\big(B_Q(x,u)\big)_{\kappa(i,j)} := \begin{cases} \frac{1}{\sqrt{2}}\big(\langle x, e_i\rangle\langle u, e_j\rangle + \langle x, e_j\rangle\langle u, e_i\rangle\big), & j < i, \\ \langle x, e_i\rangle\langle u, e_i\rangle, & j = i. \end{cases} \tag{3.4}$$

Thus, $B_Q(e_i, e_j)$ is one or $\frac{1}{\sqrt{2}}$ at position $\kappa(i,j)$ if $i = j$ or $i \neq j$, respectively, and zero at all other positions. The assertion of the lemma now follows from Proposition 2.7. □

Lemma 3.3. *Let $(e_i)_{i\in\mathbb{N}}$ be an orthonormal basis of X and let $F : X \to Y$ be quadratic. The mapping $A : \ell^2(\mathbb{N}) \to Y$ defined by*

$$Az := \sum_{i=1}^{\infty} \left(\sum_{j=1}^{i-1} \sqrt{2} z_{\kappa(i,j)} B_F(e_i, e_j) + z_{\kappa(i,i)} B_F(e_i, e_i) \right) \tag{3.5}$$

for all z which yield a convergent series is linear and its domain is dense in $\ell^2(\mathbb{N})$.

Proof. Linearity is obvious and since B_F is bounded we have $\|B_F(e_i, e_j)\| \leq c$ for some constant c and all i and j. Thus, the dense subspace $\ell^1(\mathbb{N})$ of $\ell^2(\mathbb{N})$ belongs to the domain of A. □

Now the proof of the main theorem is quite simple.

Proof of Theorem 3.1. With Q from (3.3) and A from (3.5) we have $F(x) = AQ(x)$ for all x in X. Since $F(x)$ is defined for each x, in particular we see that the range of Q belongs to the domain of A. □

The constructed decomposition in the proof of the main theorem is not the only one. Choosing another quadratic isometry Q one can improve the properties of A. If, for instance, A in the proof is injective and bounded, we can write it as $A = \tilde{A}U$ with selfadjoint $\tilde{A} : Y \to Y$ and a linear isometry $U : \ell^2(\mathbb{N}) \to Y$. This follows from the polar decomposition of the adjoint A^*. Then $\tilde{Q} := UQ$ is again a strong quadratic isometry and $F = \tilde{A}\tilde{Q}$.

With the decomposition (3.1) at hand regularization of a quadratic mapping F reduces to regularization of one possibly unbounded linear operator. At least if A is bounded, which always is the case after discretization, this can be done by standard techniques. Some care has to be taken if the chosen regularization method uses the norm in the space of preimages of A, since this is the $\ell^2(\mathbb{N})$-norm and not the norm in X. Using Tikhonov regularization

$$\|Az - y^\delta\|^2 + \alpha\|z\|^2 \to \min_{z\in\ell^2(\mathbb{N})}$$

for example, the standard norm penalty corresponds to $\|x\|^4$ in X because $\|Q(x)\|^2 = \|x\|^4$. Techniques that directly influence certain properties of the searched for function should also be used with care. For instance, total variation regularization for A would penalize the variation of $Q(x)$ and not the variation of x. Other regularization methods like truncated singular value decomposition or landweber iteration work without any difficulties.

The interested reader finds information about regularization of unbounded linear operators in [12]

After inverting A by some regularization method we have to invert the strong quadratic isometry Q. As shown in Proposition 2.8 such mappings are continuously invertible and therefore no regularization is required. Only the fact that the regularized solution z^δ, which results from invertion of A, typically lies in the orthogonal complement of the nullspace of A and possibly not in the range of Q has to be handled somehow. This can be done by projecting z^δ onto the range of Q, that is, by solving

$$\frac{1}{2}\|Q(x) - z^\delta\|^2 \to \min_{x \in X}. \tag{3.6}$$

Since the range of Q in general is not convex, existence of minimizers is not obvious. The following theorem shows that there are minimizers and how to calculate them. The adjoint B_Q^* of B_Q has been defined in Definition 2.9.

Theorem 3.4. *If Q is a weak quadratic isometry and if the bounded linear operator $x \mapsto B_Q^*(x, z^\delta)$ is compact, then the minimization problem* (3.6) *has a solution. If this operator has positive eigenvalues all global minimizer attain the form $\sqrt{\lambda}x$ where λ is the largest eigenvalue and x is a corresponding normed eigenvector. If there are no positive eigenvalues, then the minimizer is the null element.*

Proof. At first we simplify the minimization problem (3.6) by setting $x := tu$ where $t \geq 0$ and $\|u\| = 1$. In a first step we can miminize over t for each u and in a second step we can minimize over u with constraint $\|u\| = 1$.

The minimum with respect to t of

$$h_u(t) := \frac{1}{2}\|Q(tu) - z^\delta\|^2 = \frac{1}{2}t^4 - t^2\langle Q(u), z^\delta\rangle + \frac{1}{2}\|z\|^2$$

is at

$$t = \begin{cases} 0, & \langle Q(u), z^\delta\rangle \leq 0, \\ \sqrt{\langle Q(u), z^\delta\rangle}, & \langle Q(u), z^\delta\rangle > 0. \end{cases} \tag{3.7}$$

Thus, for each u with $\|u\| = 1$ we have

$$\min_{t \geq 0} h_u(t) = \begin{cases} \frac{1}{2}\|z^\delta\|^2, & \langle Q(u), z^\delta\rangle \leq 0, \\ \frac{1}{2}\|z^\delta\|^2 - \frac{1}{2}\langle Q(u), z^\delta\rangle^2, & \langle Q(u), z^\delta\rangle > 0. \end{cases}$$

If $\langle Q(u), z^\delta\rangle \leq 0$ for all u in X the solution to (3.6) is $x = 0$. If there is some u with $\langle Q(u), z^\delta\rangle > 0$ the minimization problem (3.6) is equivalent to

$$\langle Q(u), z^\delta\rangle^2 \to \max_{\substack{\|u\|=1 \\ \langle Q(u), z^\delta\rangle > 0}}$$

and thus to

$$\langle Q(u), z^\delta \rangle \to \max_{\|u\|=1} . \tag{3.8}$$

Now let $\bar\lambda$ be the largest eigenvalue of $u \mapsto B_Q^*(u, z^\delta)$ and denote by $\bar u$ a corresponding normed eigenvector, that is,

$$B_Q^*(\bar u, z^\delta) = \bar\lambda \bar u. \tag{3.9}$$

Then

$$\langle Q(\bar u), z^\delta \rangle = \langle \bar u, B_Q^*(\bar u, z^\delta)\rangle = \bar\lambda \langle \bar u, \bar u\rangle = \bar\lambda$$

and for all other normed elements u in X we obtain

$$\langle Q(u), z^\delta \rangle = \langle u, B_Q^*(u, z^\delta)\rangle \le \|B_Q^*(u, z^\delta)\| \le \bar\lambda \|u\| = \bar\lambda.$$

Therefore, $\bar u$ solves (3.8) and in view of (3.7) the solution of (3.6) is given by

$$x = \sqrt{\langle Q(\bar u), z^\delta\rangle}\,\bar u = \sqrt{\bar\lambda}\,\bar u.$$

Remember that $\langle Q(u), z^\delta \rangle > 0$ for some u and thus $\bar\lambda > 0$. □

The strong isometry Q from Lemma 3.2 satisfies the assumptions of Theorem 3.4 because the adjoint $x \mapsto B_Q^*(x, z^\delta)$ is compact, as the following proposition shows.

Proposition 3.5. *Let Q be defined by* (3.3) *and let $(e_i)_{i\in\mathbb{N}}$ be the corresponding orthonormal basis of X. Then $x \mapsto B_Q^*(x, z^\delta)$ has a symmetric matrix representation C_{z^δ} in $\mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ given by*

$$(C_{z^\delta})_{i,j} = \begin{cases} \frac{1}{\sqrt{2}} z^\delta_{\kappa(i,j)}, & j < i, \\ z^\delta_{\kappa(i,j)}, & j = i. \end{cases} \tag{3.10}$$

Here, κ is the index map defined in (3.2)*. The mapping $x \mapsto B_Q^*(x, z^\delta)$ is a Hilbert-Schmidt operator and thus compact.*

Proof. From

$$\begin{aligned}\langle e_j, B_Q^*(x, z^\delta)\rangle = \langle B_Q(x, e_j), z^\delta\rangle &= \left\langle B_Q\left(\sum_{i=1}^\infty \langle x, e_i\rangle e_i, e_j\right), z^\delta \right\rangle \\ &= \sum_{i=1}^\infty \langle x, e_i\rangle \langle B_Q(e_i, e_j), z^\delta\rangle\end{aligned}$$

we see

$$\begin{aligned}(C_{z^\delta})_{i,j} &= \langle B_Q(e_i,e_j), z^\delta\rangle \\ &= \sum_{k=1}^{\infty}\left(\sum_{l=1}^{k-1}\big(B_Q(e_i,e_j)\big)_{\kappa(k,l)} z^\delta_{\kappa(k,l)} + \big(B_Q(e_i,e_j)\big)_{\kappa(k,k)} z^\delta_{\kappa(k,k)}\right)\end{aligned}$$

and by the definition of B_Q, see (3.4), for $j \leq i$ we have

$$\big(B_Q(e_i,e_j)\big)_{\kappa(k,l)} = \begin{cases} \frac{1}{\sqrt{2}}, & j < i \text{ and } k = i \text{ and } l = j, \\ 1, & j = i \text{ and } k = l = i, \\ 0, & \text{else.} \end{cases}$$

To show the Hilbert-Schmidt property we estimate

$$\sum_{i=1}^{\infty} \|B_Q^*(e_i, z^\delta)\|^2 = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \langle e_j, B_Q^*(e_i, z^\delta)\rangle^2 = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} (C_{z^\delta})_{i,j}^2 \leq 2\|z^\delta\|^2,$$

that is, the series at the left-hand side converges. □

At the end of this section we discuss an issue that only becomes visible at the second sight. As mentioned before, classical regularization techniques for the linear part A in our decomposition yield solutions z^δ in the orthogonal complement $\mathcal{N}(A)^\perp$ of the nullspace of A. If A is not injective it might happen that $\mathcal{N}(A)^\perp$ is not a subset of the range $\mathcal{R}(Q)$ of Q. Then the projection (3.6) onto $\mathcal{R}(Q)$ could lead to a solution x^δ for which $Q(x^\delta)$ is not of the form $z^\delta + z_0$ with $z_0 \in \mathcal{N}(A)$. In other words, the projection onto the range of Q perhaps corrupts our solution.

If A is injective or if $\mathcal{N}(A)^\perp$ is a subset of $\mathcal{R}(Q)$ the mentioned problem does not occur. Thus, we should try to use only injective linear parts A, which can be achieved, for example, by choosing a suitable discretization. This idea will be put in concrete terms in the next section. Another approach would be to modify classical regularization techniques in a way which yields solutions not lying in $\mathcal{N}(A)^\perp$. In case of Tikhonov regularization this can be achieved by choosing a suitable penalty. But further research on such approaches and on the structure of $\mathcal{R}(Q)$ is necessary.

4 Influence of the discretization

In this section we first discuss a simple discretization technique for quadratic mappings and then apply it to an autoconvolution equation.

Choosing an orthonormal basis $(e_i)_{i\in\mathbb{N}}$ in X and defining X_n to be the linear hull of the first n basis elements, we approximate elements from X by elements from the finite-dimensional subspace X_n. The image of X_n under a quadratic mapping $F : X \to Y$ is a manifold of dimension at most n and the linear hull of this manifold has dimension of at most $\frac{n(n+1)}{2}$. This can be seen from

$$\begin{aligned} F(x) = B_F\left(\sum_{i=1}^n \langle x, e_i\rangle e_i, \sum_{j=1}^n \langle x, e_j\rangle e_j\right) &= \sum_{i=1}^n \sum_{j=1}^n \langle x, e_i\rangle \langle x, e_j\rangle B_F(e_i, e_j) \\ &= \sum_{i=1}^n \left(\sum_{j=1}^{i-1} 2\langle x, e_i\rangle \langle x, e_j\rangle B_F(e_i, e_j) + \langle x, e_i\rangle^2 B_F(e_i, e_i)\right) \end{aligned}$$

for x in X_n, because the set $\{B_F(e_i, e_j) : i \in \mathbb{N},\, j \leq i\}$ contains at most $\frac{n(n+1)}{2}$ elements.

With such a discretization the quadratic isometry Q in Lemma 3.2 maps an element x to some element $Q(x)$ in $\ell^2(\mathbb{N})$ which has zeros at all but the first $\frac{n(n+1)}{2}$ components. Due to Theorem 3.4 the inversion of Q requires the calculation of the largest eigenvalue and a corresponding eigenvector of a symmetric $(n \times n)$-matrix. There are several methods for doing this numerically, the power method, the inverse power method, or the Rayleigh quotient iteration, to name a few.

Discretization in the data space Y can be realized by a linear discretization operator $P_m : Y \to \mathbb{R}^m$. The linear part in our decomposition approach then can be seen as a matrix with m rows and $\frac{n(n+1)}{2}$ columns. The columns are exactly the discretized versions $P_m B_F(e_i, e_j)$ of $B_F(e_i, e_j)$.

As discussed at the end of the previous section the linear part in the decomposition should be injective, that is, the corresponding matrix should have full rank. The rank of this matrix can be influenced by the choice of the orthonormal basis $(e_i)_{i\in\mathbb{N}}$. We demonstrate this fact for the autoconvolution equation (1.3).

Let $X = L^2(0, 1)$, $Y = L^2(0, 2)$, and

$$\big(B_F(x, u)\big)(s) := \int_{\mathbb{R}} x(t)u(s - t)\,\mathrm{d}t, \tag{4.1}$$

where x and u are set to zero outside $(0, 1)$. For given grid points $t_0, t_1, \ldots, t_n$ in $[0, 1]$ with $0 = t_0 < t_1 < \cdots < t_n = 1$ we set

$$e_i(t) := \begin{cases} (t_i - t_{i-1})^{-\frac{1}{2}}, & t_{i-1} \leq t < t_i, \\ 0, & \text{else.} \end{cases}$$

Then $F(x)$ for x from the linear hull X_n of $\{e_1, \ldots, e_n\}$ is a piecewise linear function with grid points $s_{i,j} := t_i + t_j$ and $(F(x))(0) = 0 = (F(x))(2)$. Thus, the number of grid points $\frac{(n+1)(n+2)}{2} - 2 = \frac{n(n+1)}{2} + n - 1$ is larger than the maximal dimension of the image set's linear hull, which is $\frac{n(n+1)}{2}$.

The functions $B_F(e_i, e_j)$ are supported on $[s_{i-1,j-1}, s_{i,j}]$ and are one between $s_{i-1,j}$ and $s_{i,j-1}$. Besides the four mentioned points there are no other grid points where $B_F(e_i, e_j)$ is not differentiable.

The canonical choice $t_i = \frac{i}{n}$ for the grid points leads to grid points $s_{i,j} = \frac{i+j}{n}$ in the image space and the dimension of the image set's linear hull can be at most $2n - 1$, which is less than $\frac{n(n+1)}{2}$. In other words, the discretized linear part in our decomposition is not injective.

But if we choose non-equispaced grid points $t_0, t_1, \ldots, t_n$ such that all $s_{i,j}$ are different we can be sure that the image set's linear hull has maximal dimension $\frac{n(n+1)}{2}$ and the linear part is injective. The simplest way for constructing non-equispaced grid points is a small random shift of equispaced ones. First numerical experiments have shown that this idea works. However, systematic non-equispaced choices such that all $s_{i,j}$ differ are also possible.

5 Conclusions

Paying attension to the quadratic structure of autoconvolution and other ill-posed problems allows to develop new specialized regularization methods that exploit well-known techniques from the world of ill-posed linear equations.

The proposed method replaces the ill-posed nonlinear quadratic problem by one ill-posed linear equation and an eigenvalue problem. The linear equation can be regularized using classical techniques and the eigenvalue problem can be solved numerically by standard algorithms. Most notable only the easier to find largest eigenvalue is of interest, together with a corresponding eigenvector.

First numerical experiments have shown that the method works as expected from the theoretical results in this article. These numerical results and a comparision of our method to other ones [3, 17] are in preparation.

The developed notions of quadratic isometries and adjoint bilinear mappings give rise to further investigations from a functional analytic view point up to questions related to spectras of nonlinear mappings and their value for solving ill-posed inverse problems.

Another important issue for research in the near future is the adaption of the proposed method to complexvalued functions, which appear in the driving application, the SD SPIDER method mentioned in the introduction.

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