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# Regularization in Hilbert space under unbounded operators and general source conditions*

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**Abstract**

The authors study ill-posed equations with unbounded operators in Hilbert space. This setup has important applications, but only a few theoretical studies are available. First, the question is addressed and answered whether every element satisfies some general source condition with respect to a given self-adjoint unbounded operator. This generalizes a previous result from Mathé and Hofmann (2008 *Inverse Problems* **24** 015009). The analysis then proceeds to error bounds for regularization, emphasizing some specific points for regularization under unbounded operators. The study finally reviews two examples within the light of the present study, as these are fractional differentiation and some Cauchy problems for the Helmholtz equation, both studied previously and in more detail by U Tautenhahn and co-authors.

## 1. Introduction and main result

There are many papers that outline the regularization theory of ill-posed operator equations (with given right-hand side $y$ and unknown solution $x$)

$$A\,x = y, \tag{1}$$

where the forward operator $A$ is an injective linear and *bounded* mapping between two *Hilbert spaces* $X$ and $Y$ with inner products $\langle\cdot,\cdot\rangle$ and associated norms $\|\cdot\|$. The ill-posedness of the problem comes from the fact that the range $\mathcal{R}(A)$ is a non-closed subset of $Y$, i.e.,

$$\mathcal{R}(A) \neq \overline{\mathcal{R}(A)}. \tag{2}$$

* To Eberhard Schock on the occasion of his 70th birthday.

Following Nashed (see [8]) we have to distinguish *ill-posedness of type I*, where condition (2) is complemented by the fact that the range $\mathcal{R}(A)$ contains a closed infinite dimensional subspace, and *ill-posedness of type II*, where $A$ with infinite-dimensional range is compact.

In this paper, we focus on more general ill-posed equations (1) with a *closed, densely defined linear operator*

$$A : \mathcal{D}(A) \subseteq X \longrightarrow Y,$$

with domain $\mathcal{D}(A)$, i.e., $\mathcal{D}(A)$ is a dense linear subspace of $X$, and convergences $x_n \to x_0$ in $X$ with $x_n \in \mathcal{D}(A)$ $(n \in \mathbb{N})$ and $Ax_n \to y$ in $Y$ imply $x_0 \in \mathcal{D}(A)$ and $Ax_0 = y_0$. This includes the cases that either the operator $A$ is bounded (with $\mathcal{D}(A) = X$) but not necessarily compact (see also [4]) or that $A$ with a proper subset $\mathcal{D}(A)$ of $X$ is *unbounded*.

One can find in the literature only a few studies which treat regularization theory under unbounded operators. The monograph [15] discusses inverse problems under unbounded operators, but only marginally and without details. We also mention [10], where Tikhonov regularization is considered. Therefore, we feel that it makes sense to highlight some important features of this more general setup. Special care is required to apply the usual *functional calculus* from regularization theory, see [1, 7].

There are interesting applications which lead to the problem of regularization under unbounded operators, and we review some of them in section 4. Specifically, inverse problems under non-compact and unbounded operators cover cases beyond integral equations as shown in section 4.1. However, also for initial and boundary value problems of partial differential equations such situations may occur. An example is the Cauchy problem for the Helmholtz equation, which is presented in section 4.2.

Smoothness is often measured in terms of source conditions. The most general setup assumes that a solution to (1) belongs to the range of some index function applied to the operator $A^*A$, and we refer to section 2 for a formal introduction. The main objective of this study is to prove the existence of such source condition whenever the underlying operator $A$ from (1) is a closed, densely defined and injective operator. In that case the operators $A^*A$ and $AA^*$ are non-negative self-adjoint operators.

**Theorem 1.** *Let $X$ be a Hilbert space and let $H : \mathcal{D}(H) \subseteq X \to X$ be an injective, non-negative self-adjoint linear operator. Then the following results hold true.*

*(a) For every element $x \in X$ and $\varepsilon > 0$ there is a bounded index function $\psi : (0, \infty) \to (0, \infty)$ such that the general source condition*

$$x = \psi(H)w \qquad \text{with} \quad w \in X \qquad \text{and} \quad \|w\| \leqslant (1+\varepsilon)\|x\|$$

*is satisfied, and hence $x \in \mathcal{R}(\psi(H))$.*

*(b) If $x \in \mathcal{R}(\psi(H))$ for some unbounded index function $\psi$, then $x \in \mathcal{R}(\psi_0(H))$ for every bounded index function $\psi_0$ which coincides with $\psi$ on $(0, t_0]$ for some $t_0 > 0$.*

Below, we shall agree to call a self-adjoint operator *positive* if it is injective and non-negative.

We shall first develop some calculus for domains and ranges for positive self-adjoint operators, and then prove with theorem 1 the main result. In particular, we highlight that smoothness given in terms of Sobolev–Hilbert spaces can be translated to some source condition, a fact which is easy to establish but may not be of common knowledge. After that we are going to discuss several topics of regularization theory under unbounded operators, which differ in some aspects from the classical one. We conclude with two examples.

## 2. Calculus for general source conditions under unbounded operators

Source conditions use index functions and these must be defined on the spectrum $\sigma(H)$ of some positive self-adjoint operator $H$. If this operator is unbounded, then it is reasonable to assume that the corresponding function is defined on $\mathbb{R}^+$. Hence, we call a positive function $\psi : (0, \infty) \to (0, \infty)$ an *index function* if it is an increasing (non-decreasing) and continuous function with $\lim_{t\to 0} \psi(t) = 0$. We consider both *bounded* and *unbounded* index functions, i.e. $\lim_{t\to\infty} \psi(t)$ can be finite or $+\infty$.

In [6] the authors have shown the existence of an index function $\psi$ for every element $x^\dagger \in X$ in the Hilbert space $X$ such that a general source condition $x^\dagger = \psi(H)$ with source element $w \in X$ holds whenever $H$ is a positive and *compact* self-adjoint linear operator, which implies that $H$ is bounded. Theorem 1 extends the theorem from [6] to the missing cases that either $H$ is bounded but *non-compact* or that $H$ is *unbounded*.

When measuring smoothness with respect to an operator equation (1), we let $H := A^*A$. Under the assumed closedness of $A$ the adjoint operator $A^*$ is densely defined, and both operators $A^*A$ and $AA^*$ are *self-adjoint*, see [11, chapter VIII]. We agree to say that the solution $x^\dagger$ satisfies a source condition if there are a (bounded or unbounded) index function $\psi$ and a source element $w \in \mathcal{D}(\psi(A^*A))$ such that

$$x^\dagger = \psi(A^*A)w. \tag{3}$$

Plainly this is a reformulation of $x^\dagger \in \mathcal{R}(\psi(A^*A))$. In case of doubt we say in extenso that $x^\dagger$ satisfies a source condition given by the index function $\psi$ with source element $w$.

### *2.1. Domain and range relations for index functions*

Here we develop some calculus for operators given by index functions of some positive self-adjoint operator $H$. The following lemma establishes that given such an operator $H$, and some (bounded or unbounded) index function $\psi$, the range of $\psi(H)$ is determined by the behavior of $\psi$ near zero.

We shall use the following calculus for ranges and domains of operator functions. For this purpose we restrict our attention to unbounded index functions when studying domains of operators. If $H$ is some unbounded positive self-adjoint operator $H$ mapping in $X$, and if the index function $\psi$ is unbounded, then the associated operator $\psi(H)$ is unbounded, positive and self-adjoint. Furthermore, the mapping $\tilde{}: \psi \mapsto \tilde{\psi}$, given by $\tilde{\psi}(t) := 1/\psi(1/t),\ t > 0$,[4] maps unbounded index functions to unbounded index functions, and it holds true that

$$\mathcal{D}(\psi(H)) = \mathcal{R}((1/\psi)(H)) = \mathcal{R}(\tilde{\psi}(H^{-1})). \tag{4}$$

We shall also use the following simple facts.

**Lemma 1.** *Let $H$ be a positive self-adjoint operator. If two index functions $\psi$ and $\psi'$ obey $0 < \psi'(t) \leqslant \psi(t) < \infty$ for $t > 0$, then we have*

$$\mathcal{R}(\psi'(H)) \subseteq \mathcal{R}(\psi(H)) \qquad \textit{and} \quad \mathcal{D}(\psi(H)) \subseteq \mathcal{D}(\psi'(H)).$$

**Proof.** The range inclusion is easy to see. By the assumed ordering the operator $\left(\frac{\psi'}{\psi}\right)(H)$ is bounded. If for some $v \in X$ we have that $x^\dagger = \psi'(H)v$ then $w := \left(\frac{\psi'}{\psi}\right)(H)v$ is in the domain of $\psi(H)$, and we have the identity $x^\dagger = \psi'(H)v = \psi(H)\left(\frac{\psi'}{\psi}\right)(H)v = \psi(H)w$. The domain inclusion is trivial if $\psi'$ is bounded. Otherwise both index functions are unbounded, and the

[4] For monomials $\psi_p(t) := t^p$ this mapping is identical as we have that $\tilde{\psi}_p = \psi_p$.

domain inclusion is a consequence of the range inclusion. Indeed, if $\psi' \leqslant \psi$ then this yields that $\tilde{\psi} \leqslant \tilde{\psi}'$, and thus

$$\mathcal{D}(\psi(H)) = \mathcal{R}(\tilde{\psi}(H^{-1})) \subseteq \mathcal{R}(\tilde{\psi}'(H^{-1})) = \mathcal{D}(\psi'(H)),$$

which completes the proof. □

**Lemma 2.** *Let $H : \mathcal{D}(H) \subseteq X \to X$ be a positive self-adjoint operator and suppose that $\psi_1, \psi_2$ are two index functions. If both index functions coincide on $(0, t_0]$, for some $t_0 > 0$, then $\mathcal{R}(\psi_1(H)) = \mathcal{R}(\psi_2(H))$.*

**Proof.** Given $t_0 > 0$, we can assign to any index function $\psi$ the bounded index function

$$\psi_0(t) := \begin{cases} \psi(t), & 0 < t \leqslant t_0 \\ \psi(t_0), & t > t_0. \end{cases} \tag{5}$$

We claim that $\mathcal{R}(\psi(H)) = \mathcal{R}(\psi_0(H))$, from which the assertion of the lemma is a consequence. The inclusion $\mathcal{R}(\psi_0(H)) \subseteq \mathcal{R}(\psi(H))$ follows from lemma 1 since $\psi_0 \leqslant \psi$. The other inclusion is a consequence of the decomposition

$$\frac{\psi}{\psi_0}(t) = \frac{\psi}{\psi_0}(t)\chi_{0<t\leqslant t_0} + \frac{\psi}{\psi_0}(t)\chi_{t_0<t<\infty}, \qquad t \in \mathbb{R}^+,$$

and hence

$$\left(\frac{\psi}{\psi_0}\right)(t) \leqslant 1 + \left(\frac{1}{\psi(t_0)}\right)\psi(t), \qquad t \in \mathbb{R}^+.$$

Consequently, if $x^\dagger = \psi(H)v$ for $v \in \mathcal{D}(\psi(H)) \subseteq \mathcal{D}\big(\frac{\psi}{\psi_0}(H)\big)$, then $w := \big(\frac{\psi}{\psi_0}\big)(H)v$ is well defined and $x^\dagger = \psi_0(H)w$, and the proof is complete. □

By using the identification (4) the result from lemma 2 turns to the following lemma.

**Lemma 3.** *Let $H : \mathcal{D}(H) \subseteq X \to X$ be a positive self-adjoint operator and suppose that $\psi_1, \psi_2$ are two index functions. If both index functions coincide with $[t_0, \infty)$, for some $t_0 > 0$, then $\mathcal{D}(\psi_1(H)) = \mathcal{D}(\psi_2(H))$.*

**Remark 1.** In the case that the unbounded operator $H$ is $m$-accretive (semi-bounded), i.e., when $\langle Hx, x\rangle \geqslant m\|x\|^2,\ x \in \mathcal{D}(H)$, for some $m > 0$, then the inverse $H^{-1}$ is bounded. In this case, the domain restriction translates into a source condition with respect to a bounded operator $H^{-1}$ as

$$x^\dagger \in \mathcal{D}(\psi(H)) \qquad \text{if and only if} \quad x^\dagger = \tilde{\psi}(H^{-1})v \qquad \text{for some} \quad v \in X.$$

In many applications, for instance in example 1 below, the operator $H$ is such a semi-bounded differential operator, and then source conditions with respect to $H^{-1}$ and domain restrictions with respect to $H$ are equivalent.

### *2.2. Source conditions under unbounded operators*

We now proceed to prove theorem 1, and we start with an auxiliary result.

**Lemma 4.** *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, and let $f$ be a real measurable function, taking finite positive values $\mu$-almost everywhere. For each $g \in L^2(\Omega, \mathcal{A}, \mu)$ there is a bounded index function $\psi$ such that*

$$\tilde{g} := g/\psi(f) \in L^2(\Omega, \mathcal{A}, \mu) \qquad \text{and} \quad \|\tilde{g}\| \leqslant (1+\varepsilon)\|g\|. \tag{6}$$

**Proof.** Given $f$ as above, we introduce the following ($\mu$-a.e.) partition of the space $\Omega$. We set $\Omega_0 = \{t \in \Omega : 1 \leqslant f(t)\}$ and let

$$\Omega_k = \{t \in \Omega : 2^{-k} \leqslant f(t) < 2^{-k+1}\}, \qquad k = 1, 2, \ldots.$$

Then we have $\mu\big(\Omega \setminus \bigcup_{k=0}^{\infty} \Omega_k\big) = 0$. From the construction we obtain that

$$\|g\|^2 = \sum_{j=0}^{\infty} \|g\chi_{\Omega_k}\|^2,$$

for $\mu$-a.e. disjoint characteristic functions $\chi_{\Omega_k}$, $k = 0, 1, \ldots$. Then for every $\varepsilon > 0$ there is a sequence $\sigma_0 = 1 \geqslant \sigma_1 \geqslant \sigma_2 \geqslant \cdots > 0$ such that $\lim_{k\to\infty} \sigma_k = 0$, and

$$\sum_{k=0}^{\infty} \frac{\|g\chi_{\Omega_k}\|^2}{\sigma_k^2} \leqslant (1+\varepsilon)^2 \|g\|^2.$$

This follows, for example as in the proof of the theorem in [6], from [9, section 8.6.4]. On that basis we define a continuous piecewise linear function $\psi$ which takes the values $\sigma_k$ at the grid point $t = 2^{-k}$ and $\psi(t) = 1$ for $t \geqslant 1$. Then $\psi$ is an index function in our sense. We let $\tilde{g} := g/\psi(f)$ and claim that this function belongs to $L^2(\Omega, \mathcal{A}, \mu)$ and satisfies the norm bound. By construction we have that

$$\|\tilde{g}\|^2 = \int_\Omega \frac{|g(t)|^2}{\psi(f(t))^2}\, \mathrm{d}\mu \leqslant \int_\Omega \frac{|g(t)|^2}{\tilde{\psi}(f(t))^2}\, \mathrm{d}\mu = \sum_{k=0}^{\infty} \frac{\|g\chi_{\Omega_k}\|^2}{\sigma_k^2} \leqslant (1+\varepsilon)^2 \|g\|^2,$$

and the proof is complete. □

**Remark 2.** We should mention the following consequence. Suppose that $\Omega := (0, \infty)$, equipped with the Borel $\sigma$-algebra, and some measure $\mu$, and let $f(s) = s$, $s > 0$. Then the assertion of lemma 4 may be rephrased as follows: for each square integrable function $g$ there is an index function $\psi$ such that $g/\psi$ is still square integrable.

To each real-valued measurable function $f$ on a measure space $(\Omega, \mathcal{A}, \mu)$, we may assign a *multiplication operator* $M_f$ in $L^2(\Omega, \mathcal{A}, \mu)$, given by

$$M_f h := f \cdot h. \tag{7}$$

Its domain of definition is given as

$$\mathcal{D}(M_f) = \{h \in L^2(\Omega, \mathcal{A}, \mu) : f \cdot h \in L^2(\Omega, \mathcal{A}, \mu)\}. \tag{8}$$

In complex Hilbert space $X$, each self-adjoint operator, say $H : \mathcal{D}(H) \subseteq X \to X$ is unitarily invariant to a multiplication operator, i.e., there are a measure space $(\Omega, \mathcal{A}, \mu)$, a measurable function $f$ and a unitary operator $U : X \to L^2(\Omega, \mathcal{A}, \mu)$ such that

$$x \in \mathcal{D}(H) \iff f \cdot Ux \in L^2(\Omega, \mathcal{A}, \mu) \tag{9}$$

and

$$UHU^*h = f \cdot h \qquad \text{whenever} \quad h \in \mathcal{D}(M_f). \tag{10}$$

Moreover, for any bounded measurable function $\psi$ it holds that

$$\psi(H) = U^*\psi(M_f)U = U^*M_{\psi(f)}U. \tag{11}$$

We refer to [11, chapter VIII] for details.

**Remark 3.** Actually, in [11], the proof of the spectral theorem is presented for separable Hilbert spaces, and it is shown that in this case the representing measure $\mu$ can be chosen finite. However, the theorem also holds for non-separable Hilbert spaces, but the structure of the measure space $(\Omega, \mathcal{A}, \mu)$ may be more complicated.

Most proofs of the spectral theorem for unbounded operators use the Caley transform, and are thus formulated in complex Hilbert space. Other proofs reduce the spectral theorem for unbounded operators to the bounded case by using a suitable transformation, see [2, section 30]. In this case the argument works for both, real and complex Hilbert spaces.

**Remark 4.** The relation between positive self-adjoint operators and multiplication operators, as presented above sheds some new light on the assertions of lemmas 1–3. Indeed, it holds $x \in \mathcal{D}(\psi(M_f))$ if $x \cdot \psi(f) \in L^2(\Omega, \mathcal{A}, \mu)$, see (8), and consequently that $x \in \mathcal{R}(\psi(M_f))$ if $x/\psi(f) \in L^2(\Omega, \mathcal{A}, \mu)$, giving rise to alternative proofs of the lemmas.

As the above discussion reveals, multiplication operators are representative for (unbounded) self-adjoint operators, and properties of the operator $H$ can be seen as properties of the representing function $f$ in $M_f$. Specifically we call $\mathcal{R}(f) := \sigma(M_f) \setminus \{0\} = \sigma(H) \setminus \{0\}$ the essential range of the function $f$. We refer to [11, chapter VII.2] for more details.

Ill-posed problems are characterized by operators $H$ with a non-closed range $\mathcal{R}(H)$, where zero is an accumulation point of the set $\mathcal{R}(f)$. Also, a positive operator $H$ is unbounded if and only if $+\infty$ is an accumulation point of $\mathcal{R}(f)$.

**Example 1** (Sobolev–Hilbert spaces)**.** The above approach is taken in the definition of Sobolev–Hilbert spaces on $\mathbb{R}$. We let $f(s) := (1+s^2)^{1/2}$, $s \in \mathbb{R}$. For any $p > 0$ we assign the positive self-adjoint multiplication operator $M_{f^p}$ with corresponding domain. The Fourier transformation $\mathcal{F}$ constitutes an isometry on $L^2(\mathbb{R})$, and we then let $G_p := \mathcal{F} M_{f^p} \mathcal{F}^{-1}$ in $L^2(\mathbb{R})$ with domain

$$H^p(\mathbb{R}) := \{x \in L^2(\mathbb{R}) : \quad \|x\|_p := \|\mathcal{F} M_{f^p} \mathcal{F}^{-1} x\|_{L^2(\mathbb{R})} < \infty\}.$$

We refer to [14, 2.3.3] for a general outline.

The operator $G_p$ is clearly semi-bounded, and hence we may apply the reasoning from remark 1: an element $x \in H^p(\mathbb{R})$ belongs to the Sobolev space exactly if it satisfies a source condition $x \in \mathcal{R}\big(G_p^{-1}\big) = \mathcal{R}\big(\big(G_1^{-1}\big)^p\big)$. This will be relevant for the example of fractional differentiation, studied in section 4.1.

Note that the function $f$ representing the operator $G_p$ has the essential range $\mathcal{R}(f) = [1, \infty)$, indicating that $G_p$ possesses a closed range.

In the following lemma, we shall assume that the measure $\mu$ is finite, which is typical for separable Hilbert spaces $X$, see remark 3. The result extends to the non-separable case with a technical modification of the proof.

**Lemma 5.** *If the non-negative self-adjoint operator $H : \mathcal{D}(H) \subseteq X \to X$ is injective, then the measurable function $f$ which represents the corresponding multiplication operator can be chosen to be strictly positive.*

**Proof.** This follows from the following identity for functions $0 \neq h \in \mathcal{D}(M_f)$:

$$\int_\Omega f|h|^2 \, \mathrm{d}\mu = \langle M_f h, h\rangle = \langle UHU^*h, h\rangle = \langle HU^*h, U^*h\rangle > 0.$$

Assume now that $f \leqslant 0$ on a set $S$ of positive measure. Making $S$ smaller if necessary we may also assume that $f$ is bounded on $S$. Since $\mu$ is finite the function $h = \chi_S$ then is in $\mathcal{D}(M_f)\setminus\{0\}$ which yields a contradiction to the estimate above. Hence $f > 0$ $\mu$-a.e. and $U^*h \neq 0$. Hence

$f$ would be a function with $\int_\Omega f|h|^2\,\mathrm{d}\mu = 0$. Therefore $f > 0$ $\mu$-a.e. and we can change $f$ to be positive on the remaining null set. □

We are now in a position to prove the main result.

**Proof of theorem 1.** Under the assumptions on $H$, by using lemma 5, the representing multiplication operator $M_f$ possesses a positive function $f$. If $x \in X$ then $g := Ux \in L^2(\Omega, \mathcal{A}, \mu)$, and we can apply lemma 4 to find a bounded index function $\psi$ such that $\tilde{g} := g/\psi(f) \in L^2(\Omega, \mathcal{A}, \mu)$. By using (11) this means that

$$x = U^* g = U^* M_{\psi(f)} \tilde{g} = U^* \psi(M_f) U U^* \tilde{g} = \psi(H) w,$$

where we defined $w := U^* \tilde{g} \in X$. This proves assertion (a). On the other hand, for proving (b) let $x \in \mathcal{R}(\psi(H))$ for some unbounded index function. Then we can replace this by any bounded one as done in lemma 2, and the proof is complete. □

We draw some consequences of theorem 1 in the following.

**Corollary 1.** *Under the assumptions of theorem 1, the following holds true.*

*(a) If $x \in \mathcal{R}(\psi(H))$ for some index function $\psi$, then there is an index function $\psi_1$ such that $x \in \mathcal{R}([\psi \cdot \psi_1](H))$.*

*(b) If $x \in \mathcal{D}(\psi(H))$ for some unbounded index function $\psi$, then there is an unbounded index function $\psi_1$ such that $x \in \mathcal{D}([\psi \cdot \psi_1](H))$.*

**Proof.** To prove (a) we apply theorem 1 once more to $w \in X$ in order to find an index function $\psi_1$ with $w = \psi_1(H)v$, which proves the result. To prove (b) we assume without loss of generality that the operator $H$ is unbounded. Then we use the identity (4) to infer that $x \in \mathcal{R}(\tilde{\psi}(H^{-1}))$. By item (a) there is an index function $\psi_1$ such that $x \in \mathcal{R}([\tilde{\psi} \cdot \psi_1](H^{-1}))$. We note that the function $\tilde{\psi} \cdot \psi_1$ constitutes an unbounded index function, and applying the identity (4) once more we obtain that $x \in \mathcal{D}([\psi \cdot \tilde{\psi}_1](H))$. The function $\tilde{\psi}_1$ may not be an index function as it may be bounded away from zero. However, in the light of lemma 3 we may replace $\tilde{\psi}_1$ by any unbounded index function sharing the same behavior near infinity, which concludes the proof of the corollary. □

**Remark 5.** Assertion (a) of the corollary strengthens the assertion of theorem 1, and it emphasizes that there is no *maximal* smoothness, a fact which has various consequences in regularization theory, a topic which will not be discussed here.

## 3. Regularization under unbounded operators

As mentioned in section 1, there are only a few papers which study linear equations (1) for unbounded operator $A : \mathcal{D}(A) \subseteq X \to Y$ with $x \in \mathcal{D}(A)$, and we recall the study [10] in which Tikhonov regularization is considered. There, the solution representation for Tikhonov regularization is derived from minimizing the functional

$$J_\alpha(x, y^\delta) := \|Ax - y^\delta\|^2 + \alpha\|x\|^2, \qquad \alpha > 0.$$

Theorem 2 in [10] establishes that there is a unique solution, say $x_\alpha^\delta$, to this problem, and this solution has the representation

$$x_\alpha^\delta := A^*(AA^* + \alpha I)^{-1} y^\delta \tag{12}$$

for any data $y^\delta \in Y$, where we assume $\|y^\delta - y\| \leqslant \delta$. We shall extend this to more general regularization, and we recall the notion of linear regularization in Hilbert space, see e.g. [5, definition 2.2].

**Definition 1.** *A family of functions* $g_\alpha : (0, \infty) \mapsto \mathbb{R}$, $0 < \alpha \leqslant \bar{\alpha}$, *of bounded measurable functions is called regularization if they are piecewise continuous in* $\alpha$ *and the following properties hold, where* $r_\alpha(t) := 1 - t g_\alpha(t)$, $t > 0$, *denotes the residual function.*

*(a) For each* $t > 0$ *there is convergence* $|r_\alpha(t)| \to 0$ *as* $\alpha \to 0$.
*(b) There is a constant* $\gamma_1$ *such that* $|r_\alpha(t)| \leqslant \gamma_1$ *for all* $0 < \alpha \leqslant \bar{\alpha}$.
*(c) There is a constant* $\gamma_*$ *such that* $\sup_{t>0} \sqrt{t}|g_\alpha(t)| \leqslant \gamma_*/\sqrt{\alpha}$, *for* $0 < \alpha \leqslant \bar{\alpha}$.

**Remark 6.** For Tikhonov regularization as in (12), the family is given by $g_\alpha(t) := 1/(t+\alpha)$, and it is easily seen to satisfy the properties of a regularization.

In analogy to (12), given some general regularization, and data $y^\delta \in Y$ we let

$$x_\alpha^\delta := A^* g_\alpha(AA^*) y^\delta \tag{13}$$

be the approximate solution to (1) by using the parameter $\alpha$. Our goal is first to see why (13) is correctly defined in the case of an unbounded operator $A$, and then to derive error bounds under smoothness in terms of general source conditions $x^\dagger = \psi(H)w$, with $H = A^*A$, for the solution $x^\dagger$.

### *3.1. Auxiliary analysis*

In the light of definition 1(c) we introduce the following space $\mathcal{W}$ of bounded (Borel-measurable) functions on $\mathbb{R}^+$ as

$$\mathcal{W} := \left\{ g : \mathbb{R}^+ \to \mathbb{R}^+ \text{ bounded and } \quad \|g\|_{\mathcal{W}} := \sup_{s \in \mathbb{R}^+} \sqrt{s}|g(s)| < \infty \right\}. \tag{14}$$

The calculus for general regularization schemes is based on the following.

**Proposition 1.** *Let* $A : \mathcal{D}(A) \subset X \to Y$ *be a densely defined and closed operator, and let* $g$ *be any bounded measurable function in* $\mathbb{R}^+$. *If* $y \in \mathcal{D}(A^*)$ *then also* $g(AA^*)y \in \mathcal{D}(A^*)$, *and it holds that*

$$A^* g(AA^*) y = g(A^*A) A^* y. \tag{15}$$

*In addition, for all functions in* $\mathcal{W}$ *the operator defined in (15) extends to a bounded operator from* $Y \to X$, *with norm bounded by* $\|g\|_{\mathcal{W}}$. *(Analogous statements hold for the equation* $Ag(A^*A) = g(AA^*)A$.*)*

**Proof.** We shall use the polarization identity for closed densely defined operators, see e.g. [11, theorem VIII.32], i.e., there is a partial isometry $U$ (on $\ker(A)^\perp$ onto the closure of $\mathcal{R}(A)$) such that $A = U|A|$, where $|A| := (A^*A)^{1/2}$. Therefore, it holds that $AA^* = U|A|^2U^*$, and for any $y \in \mathcal{D}(A^*)$ we have that

$$\begin{aligned} g(A^*A)A^*y &= g(|A|^2)|A|U^*y = |A|g(|A|^2)U^*y \\ &= A^*Ug(|A|^2)U^*y = A^*g(AA^*)y. \end{aligned}$$

Thus, we have that $g(AA^*)y \in \mathcal{D}(A^*)$ and the identity (15) holds.

Finally, if $g \in \mathcal{W}$ then we assign to a given $y \in \mathcal{D}(A^*)$ the element $z := g(A^*A)y \in \mathcal{D}(A^*)$. We have that

$$\|A^*z\|^2 = \langle A^*z, A^*z\rangle = \langle AA^*z, z\rangle = \|(AA^*)^{1/2}z\|^2 \leqslant \|g\|_{\mathcal{W}}^2 \|y\|^2, \tag{16}$$

and the operator from $A^*g(A^*A)$ extends to a bounded operator on all of $Y$ with norm bound $\|g\|_{\mathcal{W}}$. The proof is complete. □

This allows for the following consequences with regard to error bounds of regularization.

**Corollary 2.** *Let $A : \mathcal{D}(A) \subset X \to Y$ be a closed, densely defined operator, and let $g_\alpha$, $0 < \alpha \leqslant \bar{\alpha}$, constitute a regularization. Then the following assertions hold true:*

*(a) The identity $A^* g_\alpha(AA^*)A = g_\alpha(A^*A)A^*A$ holds, and the self-adjoint operator $r_\alpha(A^*A)$ extends to a bounded operator in $X$ with norm less than or equal to $\gamma_1$.*

*(b) For each $y \in Y$ the element $g_\alpha(AA^*)y$ belongs to the domain of $A^*$ and*

$$\|A^* g_\alpha(AA^*)y\| \leqslant \gamma_* \frac{\|y\|}{\sqrt{\alpha}}.$$

Both assertions are consequences of proposition 1 and we omit the proof. The norm bound in (b) follows as in the proof of (16), and spectral calculus yields that in this case $\|g_\alpha\|_{\mathcal{W}} \leqslant \gamma_*/\sqrt{\alpha}$.

Assertion (b) confirms that $x_\alpha^\delta$ from (13) is well defined for any data $y^\delta \in Y$. Correspondingly to (13), we let

$$x_\alpha := A^* g_\alpha(AA^*)Ax^\dagger, \tag{17}$$

for the solution $x^\dagger \in D(A)$, which is also well defined by corollary 2. Error bounds are obtained from

$$\left\|x^\dagger - x_\alpha^\delta\right\| \leqslant \|x^\dagger - x_\alpha\| + \left\|x_\alpha - x_\alpha^\delta\right\|, \tag{18}$$

and we treat both summands, separately. An estimate for the second term in the right-hand side of (18) expresses the noise propagation, and it holds

**Lemma 6.** *For each $\alpha > 0$ we have that*

$$\left\|x_\alpha - x_\alpha^\delta\right\| \leqslant \gamma_* \frac{\delta}{\sqrt{\alpha}}.$$

**Proof.** We rewrite

$$\left\|x_\alpha - x_\alpha^\delta\right\| = \|A^* g_\alpha(AA^*)(Ax^\dagger - y^\delta)\| \leqslant \delta \|A^* g_\alpha(AA^*)\|.$$

Corollary 2(b) allows us to complete the proof. □

The first term $\|x^\dagger - x_\alpha\|$, which is called *bias*, requires smoothness of the solution $x^\dagger$, and by theorem 1 we know that there exists a bounded index function $\psi$ such that $x^\dagger$ satisfies a source condition (3).

**Lemma 7.** *Suppose that the solution satisfies a source condition (3) given by the index function $\psi$ and with a source element $w \in \mathcal{D}(\psi(H))$. Then for each $0 < \alpha \leqslant \bar{\alpha}$ we have that*

$$\|x^\dagger - x_\alpha\| \leqslant \|w\| \sup_{s \in \sigma(A^*A)} |r_\alpha(s)|\psi(s).$$

**Proof.** We rewrite

$$\|x^\dagger - x_\alpha\| = \|x^\dagger - A^* g_\alpha(A^*A)Ax^\dagger\|.$$

Using corollary 2(a) we can continue with

$$\begin{aligned}\|x^\dagger - x_\alpha\| &= \|(I - A^* g_\alpha(AA^*)A)x^\dagger\| \\ &= \|r_\alpha(A^*A)\psi(A^*A)w\| \leqslant \|w\| \|r_\alpha(A^*A)\psi(A^*A)\|.\end{aligned}$$

We exploit functional calculus from (11) for the self-adjoint operator $A^*A$ to conclude that

$$\|r_\alpha(A^*A)\psi(A^*A)\| \leqslant \sup_{t \in \Omega} |r_\alpha(f(t))|\psi(f(t)) \leqslant \sup_{s \in \sigma(A^*A)} |r_\alpha(s)|\psi(s), \tag{19}$$

by substituting $s := f(t) > 0$. □

*3.2. Qualification of regularization under unbounded operators*

Bounds for the bias are important for understanding regularization, and the related concept is called *qualification*. For the convenience of the reader we recall this notion, see [5, definition 2.6]. The goal is to bound the bias at the solution $x^\dagger = \psi(A^*A)w$ in terms of the involved index function $\psi$, as this was done in (19), at least for $\alpha > 0$ small. This leads to the following definition.

**Definition 2.** *An index function $\psi$ is a qualification of the regularization $g_\alpha$, if there are a constant $\gamma := \gamma_\psi < \infty$ and a value $0 < \bar{\alpha} < \infty$ such that*

$$\sup_{s\in\sigma(A^*A)} |r_\alpha(s)|\psi(s) \leqslant \gamma\psi(\alpha), \qquad 0 < \alpha \leqslant \bar{\alpha}. \tag{20}$$

In the case that the spectrum $\sigma(A^*A)$ is bounded, i.e., for bounded operators $A$, it is well known that the qualification is determined in a neighborhood of zero. Moreover, if one index function is a qualification then this is also valid for all index functions which coincide with it in a neighborhood of zero. However, such assertion does not hold under unbounded operators. It extends only if we impose some growth condition (GC). The following result is a generalization of [5, proposition 2.7] to general regularization and under unbounded operators.

**Proposition 2.** *Let $g_\alpha$ be a regularization with some known qualification $\varphi$. An index function $\psi$ is a qualification of $g_\alpha$ if there is $s_0 > 0$ for which the function $s \mapsto \psi(s)/\varphi(s)$, $0 < s \leqslant s_0$, is non-increasing and moreover the growth condition*

$$C := \sup_{s>s_0,\ s\in\sigma(A^*A)} \frac{\psi(s)}{\varphi(s)} < \infty \tag{GC}$$

*holds.*

**Proof.** Suppose that the assumption of the proposition is fulfilled. We shall prove (20) for $0 < \alpha \leqslant \bar{s} := \min\{\bar{\alpha}, s_0\}$. To do so we distinguish the three cases that $0 < s \leqslant \alpha$, $\alpha < s \leqslant \bar{s}$, and $s > \bar{s}$ with $s \in \sigma(A^*A)$. In the first case by using the monotonicity we have that $|r_\alpha(s)|\psi(s) \leqslant \gamma_1\psi(\alpha)$, with $\gamma_1$ from definition 1(b). For $\alpha < s \leqslant \bar{s}$ using the assumed monotonicity of the quotient we have

$$|r_\alpha(s)|\psi(s) = |r_\alpha(s)|\varphi(s)\frac{\psi(s)}{\varphi(s)} \leqslant \gamma_\varphi\varphi(\alpha)\frac{\psi(\alpha)}{\varphi(\alpha)} \leqslant \gamma_\varphi\psi(\alpha).$$

For the remaining case $\alpha < \bar{s} < s$ we reduce the problem to the known qualification of the function $\varphi$. We bound

$$\begin{aligned}|r_\alpha(s)|\psi(s) = |r_\alpha(s)|\varphi(s)\frac{\psi(s)}{\varphi(s)} &\leqslant \gamma_\varphi\psi(\alpha)\frac{\varphi(\alpha)}{\psi(\alpha)}\frac{\psi(s)}{\varphi(s)}\\ &\leqslant \gamma_\varphi\psi(\alpha)\frac{\varphi(s_0)}{\psi(s_0)}\frac{\psi(s)}{\varphi(s)} \leqslant \gamma_\varphi C\frac{\varphi(s_0)}{\psi(s_0)}\psi(\alpha),\end{aligned}$$

yielding the qualification bound with $\gamma_\psi := \max\left\{\gamma_\varphi, \gamma_1, \gamma_\varphi C\frac{\varphi(s_0)}{\psi(s_0)}\right\}$. □

**Remark 7.** Condition (GC) is a growth constraint at the infinite part of the spectrum. If this holds for some $s_0 > 0$ then it holds for every $s_0 > 0$.

There are two important cases where the constant $C$ in (GC) is finite. First, if the operator $A$ is bounded (in which case $C = \psi(\|A^*A\|)/\varphi(s_0)$), or in the case that the index function $\psi$ is uniformly bounded, say by $C \cdot \psi(s_0)$. Therefore, this additional assumption (GC) cannot be seen when treating qualification under bounded operators.

If the monotonicity assumption on $\psi(s)/\varphi(s)$ in proposition 2 holds globally, then (GC) is automatically fulfilled with $C = \psi(s_0)/\varphi(s_0)$.

The monotonicity assumption of the above proposition represents low smoothness. We accompany the above result with the case of high smoothness. It yields the best to expect even under highest possible smoothness. The proof is similar to the above one and we omit it.

**Proposition 3.** *Let $g_\alpha$ be a regularization with known qualification $\varphi$. Suppose that the index function $\psi$ obeys (GC). If there is $s_0 > 0$ such that the function $\psi(s)/\varphi(s)$ is non-decreasing for all $0 < s \leqslant s_0$, then there is a constant $1 \leqslant C < \infty$ such that*

$$\sup_{s\in\sigma(A^*A)} |r_\alpha(s)|\psi(s) \leqslant C\varphi(\alpha) \qquad \text{for all} \quad 0 < \alpha \leqslant \bar{\alpha}.$$

The notion of qualification uses global properties of the index functions on all of $\sigma(A^*A)$. It thus may happen that for two functions which coincide on some interval $(0, s_0]$ one is a qualification of the regularization and the other is not. This is a paradox which cannot be observed for bounded operators and it is artificial. In fact, by using theorem 1(b) based on lemma 2 we can replace any unbounded index function $\psi$ by a bounded one $\psi_0$, which equals the original one on some initial segment $(0, s_0]$. For the latter $\psi_0$, the proposition applies and thus yields a convergence rate for the bias. Taking this into account we summarize the above analysis as

**Proposition 4.** *Let $g_\alpha$ be a regularization with known qualification $\varphi$. Suppose that the solution satisfies a source condition (3) given by the index function $\psi$ with source element $w \in X$.*

*(a) If there is $s_0 > 0$ such that the function $\psi(s)/\varphi(s)$, $0 < s \leqslant s_0$, is non-increasing then there are a constant $1 \leqslant C_1 < \infty$ and some $\bar{\alpha} > 0$ such that*

$$\|x^\dagger - x_\alpha\| \leqslant C_1\, \psi(\alpha)\|w\|, \qquad 0 < \alpha \leqslant \bar{\alpha}. \tag{21}$$

*(b) If there is $s_0 > 0$ such that the function $\psi(s)/\varphi(s)$, $0 < s \leqslant s_0$, is non-decreasing then there are a constant $1 \leqslant C_2 < \infty$ and some $\bar{\alpha} > 0$ such that*

$$\|x^\dagger - x_\alpha\| \leqslant C_2\, \varphi(\alpha)\|w\|, \qquad 0 < \alpha \leqslant \bar{\alpha}. \tag{22}$$

We postpone further discussion, in particular for Tikhonov regularization, to the end of section 3.3.

### *3.3. Convergence rates*

Having established both, bounds for the bias and for the noise propagation term, we obtain explicit error bounds (as functions of the noise level $\delta > 0$) for the cases covered by proposition 4. As in usual regularization theory for general smoothness, see [7], we assign the strictly increasing (from zero to $\infty$) index function

$$\Theta_\psi(t) := \sqrt{t}\psi(t), \qquad 0 < t < \infty, \tag{23}$$

to any index function $\psi$. The convergence rates in the following result are the usual ones in regularization theory under bounded operators, and under the smoothness assumption $x^\dagger = \psi(A^*A)w$. However, as was highlighted in the above discussion, the proof is not a straightforward consequence of the qualification, but needs to take into account the fact that we can replace unbounded index functions by appropriately chosen bounded ones.

**Theorem 2.** *Let $g_\alpha$ be a regularization with known qualification $\varphi$. Suppose that the solution satisfies a source condition (3) given by the index function $\psi$ with source element $w \in \mathcal{D}(\psi(H))$.*

*(a) If some estimate of the form (21) holds true, then we have the convergence rate*

$$\left\|x^\dagger - x^\delta_{\alpha(\delta)}\right\| = \mathcal{O}\big(\psi\big(\Theta_\psi^{-1}(\delta)\big)\big) \qquad \textit{as} \quad \delta \to 0,$$

*for an a priori parameter choice $\Theta_\psi(\alpha(\delta)) = \delta$.*

*(b) If even some estimate of the form (22) is valid, then we have*

$$\left\|x^\dagger - x^\delta_{\alpha(\delta)}\right\| = \mathcal{O}\big(\varphi\big(\Theta_\varphi^{-1}(\delta)\big)\big) \qquad \textit{as} \quad \delta \to 0,$$

*for an a priori parameter choice $\Theta_\varphi(\alpha(\delta)) = \delta$.*

**Proof.** We first mention that by the properties of the functions $\Theta_\psi, \Theta_\varphi$, the parameters $\alpha(\delta)$ exist.

Now, the proof follows from the error decomposition (18), lemma 6 and proposition 4. In case (a) we have that

$$\left\|x^\dagger - x^\delta_\alpha\right\| \leqslant C_1\psi(\alpha)\|w\| + \gamma_*\frac{\delta}{\sqrt{\alpha}} = (C_1\|w\| + \gamma_*)\max\left\{\psi(\alpha), \frac{\delta}{\sqrt{\alpha}}\right\},$$

from which the first assertion follows. The proof of the second one is similar and we omit it. □

**Example 2** (Tikhonov regularization)**.** We shall review the outline in this section for Tikhonov regularization, i.e., when the solution is given by (12). This fits in the general approach by letting $g_\alpha(t) := 1/(t+\alpha),\ t > 0, \alpha > 0$, see remark 6. It is easily verified that the index function $\varphi(s) = s,\ s > 0$, is a qualification (with constant $\gamma = 1$) for Tikhonov regularization, both under bounded or unbounded operator $A$.

Let us now discuss propositions 2 and 3 for Tikhonov regularization, and smoothness given in terms of monomials $\psi_p(s) = s^p,\ s > 0$, for $p > 0$. Since the index function $\varphi(s) = s,\ s > 0$, is a qualification, the monomials $\psi_p(s)$ satisfy the assumptions of proposition 2 whenever $0 < p \leqslant 1$, where the monotonicity of $\psi(s)/s$ holds globally.

If $p > 1$ then the quotient $s^p/s = s^{p-1},\ s > 0$, is globally increasing and tends to infinity as $s \to \infty$. If the operator $A$ is unbounded then (GC) fails to hold for $p > 1$. This means that we cannot apply proposition 3, and we cannot deduce a decay rate for the bias, even if the solution smoothness is high. However, we can replace the global monomial $\psi_p$ by the function $\psi_0$ which coincides with $\psi_p$ on (0, 1], and is equal to 1 for $s > 1$. This modified index function is bounded and thus allows to apply proposition 4 and also theorem 2. We then obtain convergence rates of the order $\delta^{2p/(2p+1)}$ for $0 < p \leqslant 1$, and $\delta^{2/3}$ for $p > 1$. This expresses the well-known effect of *saturation* for the method of Tikhonov regularization.

More generally, suppose that $g_\alpha$ is some regularization with qualification, say $\varphi(s) := s^{p_0},\ s > 0$ for some exponent $p_0 > 0$. If the index function $\psi$ is of power-type smoothness in a right neighborhood of $s = 0$, i.e., $\psi(s) := s^p,\ 0 < s \leqslant \bar{s}$, for some $0 < p < \infty$, then the corresponding bounds from theorem 2 are of the orders $\delta^{\frac{2p}{2p+1}}$ for $0 < p \leqslant p_0$. If $p > p_0$ then we obtain the rate $\delta^{\frac{2p_0}{2p_0+1}}$ as $\delta \to 0$.

## 4. Two examples

In the literature on linear ill-posed operator equations there are numerous examples of problems (1), where die operator $A : \mathcal{D}(A) \subset X \to Y$ mapping between Hilbert spaces $X$ and $Y$ has zero and infinity as accumulation points of the spectrum. Frequently, for $X$ and $Y$ the $L^2$-spaces of functions on unbounded domains $\mathbb{R}^k$ $(k = 1, 2, \ldots)$ are under consideration, which sometimes allows us to identify the corresponding multiplier function $f$ according to lemmas 4 and 5 in an explicit manner. We mention two of such examples, analyzed in detail by U Tautenhahn and co-authors in [12, 13].

### *4.1. Fractional differentiation*

In [13] the problem of fractional differentiation, i.e., of finding the derivative $x = D_\beta y$ of order $0 < \beta < 1$ of a function $y$ is analyzed. For applications of that problem in natural sciences and engineering we refer, e.g., to the monograph [3]. The problem can be written as an operator (fractional integration operator) equation (1) with $X = Y = L^2(\mathbb{R})$,

$$[A_\beta x](s) = \frac{1}{\Gamma(\beta)} \int_{-\infty}^{s} \frac{x(t)}{(s-t)^{1-\beta}}\,\mathrm{d}t, \qquad s \in \mathbb{R}.$$

As mentioned by the authors of [13] the operator $A_\beta$ is densely defined, injective, normal and closed on its domain

$$\mathcal{D}(A_\beta) = \{x \in L^2(\mathbb{R}) : \ |s|^{-\beta}[\mathcal{F}x](s) \in L^2(\mathbb{R})\},$$

where $\mathcal{F}$ is the Fourier transform operator. It can then be seen that the corresponding self-adjoint operator $A_\beta^* A_\beta$ has the domain

$$\mathcal{D}(A_\beta^* A_\beta) = \{x \in L^2(\mathbb{R}) : \ |s|^{-2\beta}[\mathcal{F}x](s) \in L^2(\mathbb{R})\}.$$

Evidently, the corresponding multiplier function is $f_\beta(s) = |s|^{-2\beta}$ with $\mathcal{R}(f_\beta) = (0, \infty)$ which indicates that $H = A_\beta^* A_\beta$ has continuous spectrum. Hence (1) is ill-posed of type I ($H$ non-compact).

We briefly show how Sobolev smoothness of the solution $x^\dagger$, as discussed in example 1, translates to a source condition with respect to $A_\beta^* A_\beta$. Since both classes are related to multiplication by the Fourier transform, we see that we need to find a function $\psi : (0, \infty) \mapsto (0, \infty)$ with

$$\psi(|s|^{-2\beta}) = (1 + |s|^2)^{-p/2}, \qquad s \in \mathbb{R}. \tag{24}$$

An easy calculation shows that this is achieved by

$$\psi_{\beta,p}(t) := \left(\frac{t^{1/\beta}}{1 + t^{1/\beta}}\right)^{p/2}, \qquad t > 0. \tag{25}$$

The following result holds true, see also [13, proposition 3.1].

**Lemma 8.** *The function $\psi_{\beta,p}$ from (25) is a bounded index function for which $\psi_{\beta,p}(A_\beta^* A_\beta) = \left(G_1^{-1}\right)^p$. It is concave exactly if $0 < p \leqslant 2\beta$.*

This result allows us to apply theorem 2. Note that $\psi_{\beta,p}(t) \leqslant t^{p/(2\beta)}$ for all $t > 0$. If $p \leqslant 2\beta$ then this is concave, and theorem 2(a) applies and provides us with the convergence rate $\delta^{p/(p+\beta)}$ if $\alpha$ is chosen as $\alpha \sim \delta^{2\beta/(p+\beta)}$. Tikhonov regularization, when written in spectral domain is given by the Wiener filter, damping high frequency signals, as

$$[\mathcal{F}y^\delta](s) \mapsto \frac{s^\beta}{1 + \alpha s^{2\beta}}[\mathcal{F}y^\delta](s), \qquad s \in \mathbb{R}.$$

We have thus reproved the result for Tikhonov regularization in [13, theorem 4.1].

*4.2. Cauchy problem for the Helmholtz equation*

The second example belongs to the field of inverse problems in partial differential equations. In [12] the authors study Cauchy problems for the Helmholtz equation

$$\Delta u + k^2 u = 0, \qquad (r, z) \in \mathbb{R}^2 \times (0, d),$$

in a cylindrical domain with solution $u(r,z)$ and wave number $k > 0$. Precisely, for fixed $d$ and $z \in [0, d)$ the function $x = u(r, z)$ is to be determined from $y = u(r, d)$, where $u(r,z)$ satisfies the Helmholtz equation under homogeneous Neumann boundary conditions. We consider $X = Y = L^2(\mathbb{R}^2)$ for fixed $z \in (0, d)$. The inverse problem under consideration has applications in optoelectronics and in specific laser beam models and can be written as a linear operator equation

$$A_z u(r, z) = u(r, d), \tag{26}$$

where the operator $A_z$ is bounded for small wave numbers $k(d - z) < \frac{\pi}{2}$ and unbounded for large wave numbers $k(d - z) > \frac{\pi}{2}$. Fourier transform again yields the corresponding multiplier function, here

$$f(t) = \begin{cases} [\cosh((d-z)\sqrt{|t|^2 - k^2})]^{-2} & \text{if} \quad |t| \geqslant k \\ [\cos((d-z)\sqrt{k^2 - |t|^2})]^{-2} & \text{if} \quad |t| \leqslant k \end{cases}, \qquad t = (t_1, t_2).$$

The authors of [12] demonstrate that the range $\mathcal{R}(f)$ of $f$ has an accumulation point zero in any case, and equation (26) is ill-posed. Specifically, the following results are proved.

(a) In the case of large wave numbers they find that $\mathcal{R}(f) = (0, \infty)$.

(b) For small wave numbers $k(d - z) < \pi/2$, the operator $A_z$ is bounded and we have $\mathcal{R}(f) = (0, [\cos k(d - z)]^{-2}]$.

(c) The initial condition $\{u \in D,\ \|u(\cdot, 0)\| \leqslant 1\}$ translates to a source condition (3) given by the index function

$$\psi_z(t) := \frac{1}{\sqrt{t}} \left[ \cosh\left( \frac{d}{d-z} \operatorname{acosh} \frac{1}{\sqrt{t}} \right) \right]^{-1}, \qquad t > 0,$$

see [12, proposition 3.1], i.e.,

$$x^\dagger \in H_{\psi_z} := \{x \in L^2(\mathbb{R}^2) : \ x = \psi_z(A_z^* A_z)v, \ \|v\| \leqslant 1\}. \tag{27}$$

A local analysis reveals that

$$\psi_z(t) \sim \tilde{\psi}_z(t) := \left( \frac{\sqrt{t}}{2} \right)^{z/(d-z)} \qquad \text{as} \quad t \to 0.$$

This shows two items. First, for each $\varepsilon$ there is $\bar{t}$ such that

$$\psi_z(t) \leqslant (1 + \varepsilon)\bar{\psi}_z(t), \ 0 < t \leqslant \bar{t}. \tag{28}$$

Therefore, we may use the function $(1 + \varepsilon)\bar{\psi}_z(\alpha)$ to bound the bias. Second, this bound is concave in the range $0 < z \leqslant \frac{2}{3}d$, i.e., when $z$ is not close to the level $d$, and the degree of ill-posedness depends on the relative distance of $z$ to the observation location $d$, see [12, section 3] for more details.

(d) It follows from the bias bound related to (28) that the best possible accuracy expressed by the corresponding modulus of continuity on the set (27) for $\delta \to 0$ is given by

$$\sup_{x^\dagger \in H_{\psi_z}} \inf_R \sup_{\|Ax^\dagger - y^\delta\| \leqslant \delta} \|x^\dagger - R(y^\delta)\| = \left( \frac{\delta}{2} \right)^{z/d} (1 + \mathcal{O}(1)),$$

where $R : Y \to X$ is any linear or nonlinear method of reconstruction. This slightly improves the bound given in [12, theorem 4.2].

The results of the previous section show that within the range $0 < z \leqslant \frac{2}{3}d$, Tikhonov regularization is appropriate for regularization, and it yields the error bound

$$\|x^\dagger - x_{\alpha_*}\| \leqslant \frac{3}{2}\left(\frac{\delta}{2}\right)^{z/d}, \qquad 0 < \delta \leqslant \bar{\delta},$$

with parameter choice $\alpha_* := 4(\delta^2/4)^{(d-z)/d}$. This is close to the best possible accuracy, as mentioned in (d).

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