On the analysis of distance functions for linear ill-posed problems with an application to the integration operator in $\mathbf{L}^2$

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Abstract

The paper is devoted to the analysis of linear ill-posed operator equations $Ax = y$ with solution x_0 in a Hilbert space setting. In an introductory part, we recall assertions on convergence rates based on general source conditions for wide classes of linear regularization methods. The source conditions are formulated by using index functions. Error estimates for the regularization methods are developed by exploiting the concept of Mathé and Pereverzev that assumes the qualification of such a method to be an index function. In the main part of the paper we show that convergence rates can also be obtained based on distance functions $d(R)$ depending on radius $R > 0$ and expressing for x_0 the violation of a benchmark source condition. This paper is focused on the moderate source condition $x_0 = A^*v$. The case of distance functions with power type decay rates $d(R) = \mathcal{O}\left(R^{-\frac{\eta}{1-\eta}}\right)$ as $R \to \infty$ for exponents $0 < \eta < 1$ is especially discussed. For the integration operator in $L^2(0,1)$ aimed at finding the antiderivative of a square-integrable function the distance function can be verified in a rather explicit way by using the Lagrange multiplier method and by solving the occurring Fredholm integral equations of the second kind. The developed theory is illustrated by an example, where the optimal decay order of $d(R) \to 0$ for some specific solution x_0 can be derived directly from explicit solutions of associated integral equations.

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1 Introduction

In recent papers the first named author and coauthors have introduced the concept of *approximate source conditions* for obtaining convergence rates in regularization of *linear ill-posed* operator equations

$$A\,x \,=\, y \qquad (x \in X,\;\; y \in Y)\,. \tag{1.1}$$

Here, X and Y are infinite dimensional separable Hilbert spaces, where the symbol $\|\cdot\|$ denotes the generic norms in both spaces as well as associated operator norms, and $\langle\cdot,\cdot\rangle$ designates the inner product. Moreover, the bounded linear operator $A : X \to Y$ is assumed to be *injective* with *nonclosed range* $\mathcal{R}(A)$ and hence the inverse operator $A^{-1} : \mathcal{R}(A) \subset Y \to X$ is unbounded. Then finding the uniquely determined solution $x_0 \in X$ of (1.1) for $y \in \mathcal{R}(A)$ in a stable manner requires *regularization methods*, for example on the basis of linear regularization schemes (see, e.g., [24], [5, Chapters 3 and 4], [1, Chapter 2]), whenever only noisy data $y^\delta \in Y$ with $\|y^\delta - y\| \le \delta$ are given instead of y.

The analysis of convergence and *convergence rates* for regularized solutions (see, e.g., [5], [14], [16], [17], [19], [20] and [23]) gives some essential insight into the interplay of smoothing properties of the forward operator A characterized by its degree of ill-posedness (cf., e.g., [11], [12] and [6]) and the relative smoothness of the solution x_0 with respect to A expressed by appropriate source conditions. The initial version of the concept of approximate source conditions (see [8] and [13]) was formulated for the Tikhonov regularization and motivated by Baumeister's theorem (see [2, Theorem 6.8]) based on the *distance function*

$$d(R) \,:=\, \inf\,\{\|x_0 - A^*\,v\| \,:\, v \in Y,\; \|v\| \le R\} \qquad (R > 0) \tag{1.2}$$

depending on a radius R. This distance function measures for the solution element x_0 the violation of the *moderate source condition*

$$x_0 \,=\, A^*\,v \qquad (v \in Y) \tag{1.3}$$

In subsequent papers the concept was extended to general linear regularization methods (see [10]) and to distance functions

$$d_\psi(R) \,:=\, \inf\,\{\|x_0 - \psi(A^*A)\,w\| \,:\, w \in X,\; \|w\| \le R\} \qquad (R > 0) \tag{1.4}$$

(see [9], also [4]) with general benchmark functions $\psi : (0, \|A^*A\|] \to (0,\infty)$.

Such an approach, however, is only applicable to practical situations if the occurring distance functions or at least appropriate majorants can be verified explicitly. Explicit majorant functions could be constructed in [3], [10] and [13] whenever *range inclusions* $\mathcal{R}(\varrho(G)) \subset \mathcal{R}((A^*A)^{\frac{1}{2}})$ or equivalent link conditions between A and an operator $G : X \to X$ are supposed, where $x_0 = G\,w$ with some element $w \in X$ expresses the solution smoothness. Moreover, for specific solutions x_0 and for *noncompact* multiplication operators $A := M$ defined as

$$[Mx](t) := m(t)\,x(t) \qquad (0 \le t \le 1)$$

mapping in $X = Y = L^2(0,1)$ close upper bounds for the distance function $d(R)$ were derived in [8, §3]. The corresponding operator equations (1.1) for those multiplication operators are ill-posed of type I in the sense of Nashed (see [21]).

The present paper, however, is going to complement the analysis with an example where distance functions (1.2) can be verified explicitly and A is a *compact* linear integral operator, hence (1.1) is ill-posed of type II in Nashed's sense. More precisely, we will consider $A := J$, where J mapping in $X = Y = L^2(0,1)$ is the simple *integration operator*

$$[J\,x](s) := \int\limits_0^s x(t)\,\mathrm{d}t \qquad (0 \le s \le 1)\,. \tag{1.5}$$

with a well-known singular system $\{\sigma_n; u_n; v_n\}_{n=1}^\infty$, where the decreasing sequence

$$\sigma_n \;=\; \frac{1}{\pi\,\left(n-\frac{1}{2}\right)} \;\sim\; \frac{1}{\pi\,n} \qquad (n=1,2,...) \tag{1.6}$$

describes the singular values and

$$u_n(t) = \sqrt{2}\,\cos\left(\left(n-\frac{1}{2}\right)\pi t\right), \quad v_n(t) = \sqrt{2}\,\sin\left(\left(n-\frac{1}{2}\right)\pi t\right) \qquad (0 \le t \le 1) \tag{1.7}$$

the corresponding eigenfunctions satisfying the equations $J\,u_n = \sigma_n\,v_n,\;\; J^*\,v_n = \sigma_n\,u_n$ $(n=1,2,...)$.

On the one hand, we derive a close majorant for the distance function $d(R)$ in case of the *Volterra integral equation of the first kind* $Ax = y$ with $A := J$, but on the other hand we also show that finding such a function can be realized by verifying families of solutions to *Fredholm integral equations of the second kind* with one scalar parameter and solving associated eigenvalue problems. By the authors' opinion the formulated cross-connections between first and second kind integral equations can be interesting for the understanding of the inner structure of such problems.

The paper is organized as follows. In Section 2 we recall for linear ill-posed operator equations (1.1) with solution x_0 in the Hilbert space setting assertions on convergence rates based on general source conditions for wide classes of linear regularization methods. The source conditions are formulated by using index functions. Error estimates for the regularization methods are developed by exploiting the concept of Mathé and Pereverzev that assumes the qualification of such a method to be an index function. In Section 3 we show that convergence rates can also be obtained based on distance functions $d(R)$ depending on radius $R > 0$ and expressing for x_0 the violation of a benchmark source condition. This paper is focused on the moderate source condition $x_0 = A^*v$. Section 4 discusses the case of distance functions with power type decay rates $d(R) = \mathcal{O}\left(R^{-\frac{\eta}{1-\eta}}\right)$ as $R \to \infty$ for exponents $0 < \eta < 1$. For the integration operator J mapping in $L^2(0,1)$ the distance function can be verified in a rather explicit way by using the Lagrange multiplier method and by solving the occurring Fredholm integral equations of the second kind. This is presented in detail in Section 5. Section 6 completes the paper with an example, where the optimal decay order of $d(R) \to 0$ for some specific solution x_0 can be derived directly from the theory presented in Section 5 and from an explicit solution of the associated integral equation.

2 Source conditions and convergence rates for general linear regularization methods

Throughout this paper, we will focus on general linear regularization methods for the stable approximate solution of equation (1.1). Every such method is generated by a piecewise continuous function

$$g_\alpha(t) \quad (0 < t \le a := \|A^*A\|, \ \ 0 < \alpha \le \overline{\alpha} \le a)\,.$$

In this context, we distinguish regularized solutions

$$x_\alpha \ = \ g_\alpha\,(A^*A)\,A^*\,y$$

in the case of noise-free data and

$$x_\alpha^\delta \ = \ g_\alpha\,(A^*A)\,A^*y^\delta$$

in the case of noisy data. For fixed A and x_0 the regularization error of the noise-free case as a function $f(\alpha)$ of the regularization parameter $\alpha > 0$ can be written as

$$f(\alpha) := \|x_\alpha - x_0\| \ = \ \|(g_\alpha(A^*A)A^*A - I)\,x_0\| \ = \ \|r_\alpha(A^*A)\,x_0\|\,, \tag{2.1}$$

where $r_\alpha(t) := 1 - t\,g_\alpha(t)$ $(0 < t \le a)$ is the residual function of the regularization method. As obvious in regularization theory (cf. [10]) we pose the following *standing assumption*:

Assumption 2.1 *There exist two constants $C_1, C_2 > 0$ such that for all $0 < t \le a$*

$$\begin{array}{lll} (i) & \lim\limits_{\alpha\to 0} r_\alpha(t) \ = 0\,, & \\ (ii) & |r_\alpha(t)| \ \le \ C_1 & (0 < \alpha \le \overline{\alpha})\,; \\ (iii) & \sqrt{t}\,|g_\alpha(t)| \ \le \ \dfrac{C_2}{\sqrt{\alpha}} & (0 < \alpha \le \overline{\alpha})\,. \end{array}$$

Example 2.2 The most prominent regularization method is classical Tikhonov regularization with generator function $g_\alpha(t) = 1/(t+\alpha)$ and residual function $r_\alpha(t) = \alpha/(t+\alpha)$. This method satisfies Assumption 2.1 with $C_1 = 1$ and $C_2 = 1/2$.

The requirements (i) and (ii) of Assumption 2.1 ensure based on the noise-free error formula (2.1) the convergence $f(\alpha) \to 0$ as $\alpha \to 0$, but this convergence depends on 'smoothness' properties of x_0 and can be arbitrarily slow. Taking into account the noise level $\delta > 0$ the total error of regularization can be estimated by the triangle inequality in the form

$$\|x_\alpha^\delta - x_0\| \le \|x_\alpha - x_0\| + \|x_\alpha^\delta - x_\alpha\|$$

and by the requirement (iii) of Assumption 2.1 as

$$\|x_\alpha^\delta - x_0\| \ \le f(\alpha) + \frac{C_2\,\delta}{\sqrt{\alpha}} \qquad (0 < \alpha \le \overline{\alpha})\,. \tag{2.2}$$

Index functions (cf. [7], [16]) play an important role in our theory.

Definition 2.3 *We call $\psi(t)$ $(0 < t \leq \bar{t})$ an index function if this function is continuous and strictly increasing with limit condition $\lim_{t\to 0} \psi(t) = 0$.*

To obtain *convergence rates* for the regularization method g_α *source conditions*

$$x_0 = \varphi(A^*A)\, w \qquad (w \in X) \tag{2.3}$$

with index functions $\varphi(t)$ $(0 \leq t \leq a)$ have to be used. Based on (2.3) we then have from spectral theory

$$f(\alpha) = \|r_\alpha(A^*A)\, \varphi(A^*A)\, w\| \leq \left(\sup_{0<t\leq a} |r_\alpha(t)|\, \varphi(t) \right) \|w\|\,. \tag{2.4}$$

This can be estimated further from above if we follow the ideas of Mathé and Pereverzev (see [15] and [16]) to consider the *qualification* of a regularization method to be an index function.

Definition 2.4 *An index function $\psi(t)$ $(0 < t \leq a)$ is called a qualification with constant $C_0 \in [1, \infty)$ of the regularization method g_α if*

$$\sup_{0<t\leq a} |r_\alpha(t)|\, \psi(t) \;\leq\; C_0\, \psi(\alpha) \qquad (0 < \alpha \leq \bar{\alpha})\,.$$

Then from formula (2.4) we immediately obtain the following proposition.

Proposition 2.5 *Let x_0 satisfy the source condition (2.3) and let the index function φ be a qualification with constant $C_0 \in [1, \infty)$ of the regularization method g_α. Then*

$$f(\alpha) \leq C_0\, \varphi(\alpha)\, \|w\| \qquad (0 < \alpha \leq \bar{\alpha}) \tag{2.5}$$

and hence

$$\|x_\alpha^\delta - x_0\| \;\leq\; C_0\, \varphi(\alpha)\, \|w\| + \frac{C_2\, \delta}{\sqrt{\alpha}} \qquad (0 < \alpha \leq \bar{\alpha})\,. \tag{2.6}$$

As is well-known (see [16]) by balancing the two terms in the bound of (2.6) for sufficiently small $\bar{\delta} > 0$ we find a constant $K > 0$ such that

$$\|x_{\alpha(\delta)}^\delta - x_0\| \;\leq\; K\, \varphi(\Theta^{-1}(\delta)) \qquad (0 < \delta \leq \bar{\delta})\,, \tag{2.7}$$

where with φ also

$$\Theta(\alpha) := \sqrt{\alpha}\, \varphi(\alpha) \qquad (0 < \alpha \leq \bar{\alpha})$$

is an index function and the regularization parameter is chosen a priori as $\alpha(\delta) := \Theta^{-1}(\delta)$.

In particular for the Tikhonov regularization from the literature (see, e.g., [18] and [3]) we get a variety of sufficient conditions that characterize qualifications and therefore ensure estimates (2.6) and (2.7).

Proposition 2.6 *Let $\psi(t)$ $(0 < t \leq a)$ be an index function. If (a) $\psi(t)/t$ is monotonically decreasing on $(0, a]$, or (b) $\psi(t)$ is concave on $[0, a]$, then φ is a qualification with constant $C_0 = 1$ of Tikhonov regularization. If there exists a real number $\hat{t} \in (0, a)$ such that (c) $\psi(t)/t$ is monotonically decreasing on $(0, \hat{t}\,]$ or (d) $\psi(t)$ is concave on $[0, \hat{t}\,]$, then the same is true, but with the constant $C_0 = \psi(a)/\psi(\hat{t}\,)$.*

Note that any function $\psi(t) = t^{\frac{\nu}{2}}$ with exponent $0 < \nu \le 2$ is concave and hence a qualification of Tikhonov's method (see Example 2.2) with constant $C_0 = 1$.

Remark 2.7 In case that a source condition (2.3) is valid with

$$\varphi(t) = \sqrt{t} \qquad (0 < t \le a)\,, \tag{2.8}$$

we can rewrite it as (1.3), because of $\mathcal{R}(A^*) = \mathcal{R}((A^*A)^{\frac{1}{2}})$. Provided that (2.8) is a qualification with constant C_0 for the regularization method g_α the error estimates (2.5), (2.6) and (2.7) in that case attain the form

$$f(\alpha) \le C_0 \sqrt{\alpha}\, \|v\| \qquad (0 < \alpha \le \overline{\alpha})\,, \tag{2.9}$$

$$\|x_\alpha^\delta - x_0\| \le C_0 \sqrt{\alpha}\, \|v\| + \frac{C_2\, \delta}{\sqrt{\alpha}} \qquad (0 < \alpha \le \overline{\alpha})\,,$$

and

$$\|x_{\alpha(\delta)}^\delta - x_0\| \le K \sqrt{\delta} \qquad (0 < \delta \le \overline{\delta})$$

for the a priori choice $\alpha(\delta) \sim \delta$.

3 Distance functions yielding convergence rates

If the solution $x_0 \in X$ of the ill-posed operator equation (1.1) is not smooth enough to satisfy the moderate source condition (1.3) (or in more generality a source condition $x_0 = \psi(A^*A)\, w$ $(w \in X)$ with some benchmark index function ψ), we can suggest an alternative approach for finding convergence rates for the regularization method generated by g_α. This approach exploits the fact that x_0 satisfies the considered source condition at least in an approximate manner and uses the distance function $d(R)$ from (1.2) (or more generally $d_\psi(R)$ from (1.4)) for constructing convergence rates $f(\alpha) = \mathcal{O}(\varphi(\alpha))$ as $\alpha \to 0$, where the rate function φ is determined by properties of the associated distance function. In the remaining part of this section we formulate such rate results for (1.3) and $d(R)$, because this situation will be studied later for equation (1.1) with the integration operator $A := J$. For an extension of the assertions given below to the case d_ψ with general benchmark function ψ we refer to [10, Theorem 5.5].

Evidently, for every $x_0 \in X$ the nonnegative distance function (1.2) is well-defined and nonincreasing for all radii $R > 0$ and satisfies the limit condition $\lim\limits_{R \to \infty} d(R) = 0$ as a consequence of the injectivity of A implying $\overline{\mathcal{R}(A^*))} = X$. There are two cases: Case (a) with $x_0 \notin \mathcal{R}(A^*)$, where $d(R) > 0$ for all $R > 0$, as well as case (b) with $x_0 \in \mathcal{R}(A^*)$, where we have for some $R_0 > 0$ the situation $d(R) > 0$ $(0 < R < R_0)$ and $d(R) = 0$ $(R \ge R_0)$. Only the case (a) is of interest here. For that case one can show using the Lagrange multiplier method (cf. [8, Proof of Lemma 2.5]) that $d(R)$ is a strictly decreasing function for $R \in (0, \infty)$ and consequently that $d(1/t)$ is an index function for $t > 0$. Hence

$$\theta(t) := t\, d(1/t) \qquad (t > 0) \tag{3.1}$$

is an index function on every interval $[0, \overline{t}\,]$. We also use the notation $\theta(t)$ if $d(R)$ in (3.1) is replaced by a *strictly decreasing majorant* function $\widetilde{d}(R)$ such that

$$d(R) \le \widetilde{d}(R) \quad (0 < \underline{R} \le R < \infty), \qquad \lim_{R \to \infty} \widetilde{d}(R) = 0.$$

Lemma 3.1 *Let $\sqrt{t}$ $(0 \le t \le a)$ be a qualification with constant $1 \le C_0 < \infty$ for the regularization method generated by g_α. Then with $d(R)$ from (1.2) we obtain for that method the error estimate*

$$f(\alpha) \,=\, \|x_\alpha - x_0\| \,\le\, C_1\, d(R) + C_0 \sqrt{\alpha}\, R \,\le\, \max\,(C_0, C_1)\, \left(\widetilde{d}(R) + \sqrt{\alpha}\, R\right) \tag{3.2}$$

for all $0 < \underline{R} \le R < \infty$ and $0 < \alpha \le \overline{\alpha}$.

Proof: Taking into account the fact that the square-root function is a qualification for g_α, for any $v \in X$ with $\|v\| \le R$ we can estimate by the triangle inequality as follows:

$$\begin{aligned}
\|x_\alpha - x_0\| &= \|r_\alpha(A^*A)\, x_0\| \\
&= \|r_\alpha(A^*A)\, x_0 - r_\alpha(A^*A)\, A^*\, v + r_\alpha(A^*A)\, A^*\, v\| \\
&\le \|r_\alpha(A^*A)\, (x_0 - A^*\, v)\| \,+\, \|r_\alpha(A^*A)\, A^*\, v\| \\
&\le C_1\, \|x_0 - A^*\, v\| \,+\, \left(\sup_{0<t\le a} |r_\alpha(t)| \sqrt{t}\right) \|v\| \\
&\le C_1\, \|x_0 - A^*\, v\| \,+\, C_0 \sqrt{\alpha}\, R\,.
\end{aligned}$$

Since this estimate remains true when $\|x_0 - A^*\, v\|$ is substituted by its infimum over all v from the centered ball of X with radius $R > 0$, we immediately obtain the required inequality (3.2). This proves the lemma. □

We recall that the assumption of Lemma 3.1 is satisfied for the Tikhonov regularization with $C_0 = 1$.

Theorem 3.2 *Let the assumptions of Lemma 3.1 hold. Moreover let*

$$x_0 \notin \mathcal{R}(A^*)\,. \tag{3.3}$$

Then with $\widetilde{\alpha} \in (0, \overline{\alpha}]$ sufficiently small we have an error estimate

$$f(\alpha) = \|x_\alpha - x_0\| \,\le\, 2\, \max\,(C_0, C_1)\, \frac{\sqrt{\alpha}}{\theta^{-1}(\sqrt{\alpha})} \qquad (0 < \alpha \le \widetilde{\alpha}) \tag{3.4}$$

for the regularization method generated by g_α.

Proof: We use the estimate (3.2), which is valid for sufficiently large $R > 0$, and equate the terms $\widetilde{d}(R)$ and $\sqrt{\alpha}\, R$. By setting $t := 1/R$ this is equivalent to the equation $\theta(t) = \sqrt{\alpha}$ for $\theta(t) = t\, \widetilde{d}(1/t)$. Having $\alpha > 0$ small enough there is some $t = t(\alpha) = \theta^{-1}(\sqrt{\alpha})$ such that this equation is fulfilled and we find (3.4) from (3.2) taking into account that all the function $\sqrt{t}$, $\theta(t)$, and $\theta^{-1}(t)$ are index functions for sufficiently small $t > 0$. This proves the theorem. □

Note that the estimate (3.4) is of the form (2.5), $f(\alpha) = \mathcal{O}(\varphi(\alpha))$ as $\alpha \to 0$, with rate $\varphi(\alpha) = \frac{\sqrt{\alpha}}{\theta^{-1}(\sqrt{\alpha})}$ implying a corresponding estimate (2.6) in the noisy data case. It is important to mention that this rate $\varphi(\alpha)$ is slower than the rate $\sqrt{\alpha}$ provided by the moderate source condition (1.3), since $\lim\limits_{t\to 0} 1/\theta^{-1}(t) = \infty$. This is a natural consequence of the assumption (3.3) expressing the missing smoothness of x_0.

4 Power-type decay of distance functions

Now let $x_0 \notin \mathcal{R}(A^*)$ be a solution of (1.1) such that for some constants $c > 0$ and $0 < \eta < 1$

$$d(R) \le \frac{c}{R^{\frac{\eta}{1-\eta}}} \qquad (0 < \underline{R} \le R < \infty)\,. \tag{4.1}$$

Note that $\frac{\eta}{1-\eta}$ attains all positive real numbers if η varies through the open interval $(0,1)$. Whenever $\sqrt{t}$ $(0 < t \le a)$ is a qualification with constant C_0 of the regularization method generated by g_α, we obtain from (3.4) and (4.1) with $\widetilde{d}(R) = c\,R^{-\frac{\eta}{1-\eta}}$ and $\theta(t) = c\,t^{\frac{1}{1-\eta}}$ the estimate

$$f(\alpha) = \|x_\alpha - x_0\| \le \widetilde{c}\,\alpha^{\frac{\eta}{2}} \qquad (0 < \alpha \le \widetilde{\alpha})\,, \tag{4.2}$$

where the constant can be made explicit as $\widetilde{c} = 2\,\max\,(C_0, C_1)\;c^{1-\eta}$ and $\widetilde{\alpha} > 0$ is sufficiently small.

In the case of *compact* operators A there has been formulated a *converse result* on the distance function $d(R)$ in [9, Theorem 1] (for an extension to $d_\psi(R)$ with monomials ψ as benchmark see also [4]). Using Young's inequality and the equivalence

$$x_0 \in \mathcal{R}\left((A^*A)^{\frac{\eta}{2}}\right) \quad\Longleftrightarrow\quad \sum_{i=1}^{\infty} \frac{\langle x_0, u_i\rangle^2}{\sigma_i^{2\eta}} < \infty \tag{4.3}$$

(see [5, Proposition 3.13]) for the singular system $\{\sigma_n; u_n; v_n\}_{n=1}^{\infty}$ of A it could be proven that, for all $0 < \eta < 1$,

$$x_0 = (A^*A)^{\frac{\eta}{2}}\,w \qquad (w \in X) \tag{4.4}$$

implies an inequality of form (4.1). As is well-known (cf. [22]) the rate $f(\alpha) = \mathcal{O}(\alpha^{\frac{\mu}{2}})$ as $\alpha \to 0$ yields a source condition (4.4) for all $0 < \eta < \mu$. If then the supremum μ_{sup} of all such μ is positive, due to $x_0 \notin \mathcal{R}(A^*)$ we have $\mu_{sup} \in (0,1]$ and μ_{sup} equals the supremum of all η satisfying the source condition (4.4). Moreover, for compact A and if $0 < \mu_{sup} < 1$, this value is also the maximum η_{max} of all η satisfying an inequality (4.1).

Now we focus on the integration operator $A := J$ in $X = Y = L^2(0,1)$ introduced in formula (1.5). Since the adjoint operator J^* of J is explicitly given by

$$[J^*\,y](t) := \int_t^1 y(s)\,\mathrm{d}s \qquad (0 \le t \le 1)\,,$$

the moderate source condition (1.3) is equivalent to

$$x_0 \in H^1[0,1]\,, \qquad x_0(1) = 0\,. \tag{4.5}$$

If, for example, the function $x_0(t)$ $(0 \le t \le 1)$ is continuously differentiable, but fails to satisfy the boundary condition $x_0(1) = 0$, then (1.3) cannot hold, but a weaker source condition of power type (4.4) with $0 < \eta < 1$ may be valid. For studying such a situation we consider for simplicity in the sequel the constant function

$$x_0(t) = 1 \qquad (0 \le t \le 1) \tag{4.6}$$

and the integration operator $A := J$ (see also [9, §5]). Then using the explicit structure of the singular system (1.6) and (1.7) we find for the singular values $\sigma_n \sim n^{-1}$ and for the inner products in $L^2(0,1)$ occurring in (4.3) $\langle x_0, u_n \rangle \sim \sigma_n \sim n^{-1}$ as $n \to \infty$. Consequently,

$$\sum_{i=1}^{\infty} \frac{\langle x_0, u_i \rangle^2}{\sigma_i^{2\eta}} < \infty \qquad \text{if and only if} \qquad \sum_{i=1}^{\infty} i^{2\eta-2} < \infty \,. \tag{4.7}$$

Therefore, we can state that (4.4) is satisfied for all $0 < \eta < 1/2$ and hence we have $\eta_{max} = \frac{1}{2}$ characterizing the maximum of all η satisfying an estimate of the form (4.1) for the distance function $d(R)$ and x_0 from (4.6). So the limit rate in the right-hand side of (4.1) for that x_0 is

$$d(R) \,\leq\, \frac{c}{R} \qquad (0 < \underline{R} \leq R < \infty)\,. \tag{4.8}$$

In the next section we will prove this estimate directly by analyzing linear integral equations of the second kind.

5 Distance functions and Fredholm integral equations of the second kind in $\mathbf{L^2(0,1)}$

In the general Hilbert space setting of the linear operator equation (1.1) the distance function $d(R)$ from (1.2) can be verified for given linear operator $A: X \to Y$ and solution $x_0 \in X$ by exploiting the Lagrange multiplier method. Precisely, for all $\lambda > 0$ the uniquely determined solution $v = v_\lambda$ of the extremal problem

$$\|A^* v - x_0\|^2 + \lambda \left(\|v\|^2 - R^2\right) \to \min, \quad \text{subject to} \quad v \in Y,$$

can be found by solving the normal equation

$$(AA^* + \lambda\, I)\, v_\lambda \;=\; A\, x_0\,.$$

Because of the equivalence

$$(AA^* + \lambda\, I)^{-1} A = A\,(A^*A + \lambda\, I)^{-1}$$

we can write

$$d(R) \;=\; \|A^* v_\lambda - x_0\| \;=\; \lambda\, \|(A^*A + \lambda I)^{-1} x_0\|\,, \tag{5.1}$$

where for all $R > 0$ the Lagrange multiplier $\lambda = \lambda(R) > 0$ is the uniquely determined solution of the equation

$$\|v_\lambda\|^2 \;=\; \|A(A^*A + \lambda I)^{-1} x_0\|^2 \;=\; R^2\,. \tag{5.2}$$

In order to use the interrelations (5.1) and (5.2) for verifying $d(R)$ in case of the integration operator $A := J$ mapping in $X = Y = L^2(0,1)$, we can search for families $w_\lambda := (J^*J + \lambda I)^{-1} x_0 \in X$ with

$$d(R) \;=\; \lambda\, \|w_\lambda\|\,, \tag{5.3}$$

where for all $R > 0$ the corresponding parameter $\lambda = \lambda(R) > 0$ is determined as the uniquely determined solution of the equation $\|J\,w_\lambda\|^2 = R^2$. We note that $v_\lambda = J\,w_\lambda$ is the antiderivative of the function w_λ. It can easily be shown that the functions w_λ are just the solutions of the family of *Fredholm integral equations of the second kind*

$$\int_0^1 (1 - \max(s,t))\, w_\lambda(t)\,\mathrm{d}t + \lambda\, w_\lambda(s) = x_0(s) \qquad (0 < s < 1) \tag{5.4}$$

with family parameter $\lambda > 0$. For all $\lambda > 0$ we will present an explicit solution to equation (5.4) in the subsequent theorem.

Theorem 5.1 *For all $x_0 \in L^2(0,1)$ and all parameters $\lambda > 0$ the integral equation (5.4) has a uniquely determined solution $w_\lambda \in L^2(0,1)$ of the form*

$$w_\lambda(s) = \frac{x_0(s)}{\lambda} + \frac{1}{\lambda^{3/2}} \left[\int_0^s x_0(t)\,\sinh\left(\frac{s-t}{\sqrt{\lambda}}\right) dt - \frac{\cosh\left(\frac{s}{\sqrt{\lambda}}\right)}{\cosh\left(\frac{1}{\sqrt{\lambda}}\right)} \int_0^1 x_0(t)\,\sinh\left(\frac{1-t}{\sqrt{\lambda}}\right) dt \right] \tag{5.5}$$

$$(0 < s < 1)\,.$$

Proof: The equation (5.4) is for all $\lambda > 0$ equivalent to

$$\lambda w_\lambda(s) + \int_0^s (1-s)\, w_\lambda(t)\,\mathrm{d}t + \int_s^1 (1-t)\, w_\lambda(t)\,\mathrm{d}t = x_0(s) \quad (0 < s < 1)\,. \tag{5.6}$$

This is a Fredholm integral equation of the second kind with a bounded measurable kernel. Further, for all $\lambda > 0$ the corresponding homogeneous equation has the trivial solution $w_{\lambda,hom} = 0$ only since $w_{\lambda,hom} \in C^1[0,1]$ and

$$\int_0^1 [w_{\lambda,hom}(s)]^2\,ds + \lambda \int_0^1 [w'_{\lambda,hom}(s)]^2\,ds = 0$$

(cf. (5.8) below with $x_0 = 0$). Hence, the non-homogeneous equation (5.6) is uniquely solvable in any space $L^p(0,1)$ with $1 \le p \le \infty$, in particular in $L^2(0,1)$. Of course, the unique solvability of (5.6) in $L^2(0,1)$ is also a direct consequence of the positive definitness of the operator $J^*J + \lambda I$ for all $\lambda > 0$.

To show that (5.5) is the corresponding resolvent representation of the solution to (5.6) we reduce equation (5.6) in usual way to an explicitly solvable boundary value problem for a second order differential equation supposing

$$x_0, w_\lambda \in E := C[0,1] \cap C^1[0,1) \cap C^2(0,1)\,.$$

The validity for any $x_0, w_\lambda \in L^2(0,1)$ then follows via approximation by C^2-functions or by inserting of (5.5) directly into (5.6).

Now let be $x_0, w_\lambda \in E$. At the right boundary $s = 1$ we then have from (5.6)

$$\lambda\, w_\lambda(1) = x_0(1)\,. \tag{5.7}$$

Moreover, by differentiation the integral equation (5.6) can be reformulated as integro-differential equation

$$\lambda\, w_\lambda'(s) - \int\limits_0^s w_\lambda(t)\,\mathrm{d}t = x_0'(s) \qquad (0 < s < 1)\,, \tag{5.8}$$

with boundary condition

$$\lambda\, w_\lambda'(0) = x_0'(0) \tag{5.9}$$

at the left side $s = 0$. Further differentiation of (5.8) yields

$$\lambda\, w_\lambda''(s) - w_\lambda(s) = x_0''(s) \qquad (0 < s < 1)\,. \tag{5.10}$$

It can be seen that the integral equation (5.4) is equivalent to a boundary value problem of the differential equation (5.10) with boundary conditions (5.7) and (5.9). The general solution of (5.10) has the explicit form

$$w_\lambda(s) = \widetilde{K}_1 \cosh\left(\frac{s}{\sqrt{\lambda}}\right) + \widetilde{K}_2 \sinh\left(\frac{s}{\sqrt{\lambda}}\right) + \frac{1}{\sqrt{\lambda}} \int\limits_0^s x_0''(t) \sinh\left(\frac{s-t}{\sqrt{\lambda}}\right)\,\mathrm{d}t\,. \tag{5.11}$$

Integration by parts of the integral in (5.11) yields

$$\begin{aligned}
&\int\limits_0^s x_0''(t) \sinh\left(\frac{s-t}{\sqrt{\lambda}}\right)\,\mathrm{d}t \\
&= x_0'(t) \sinh\left(\frac{s-t}{\sqrt{\lambda}}\right)\Bigg|_0^s + \frac{1}{\sqrt{\lambda}} \int\limits_0^s x_0'(t) \cosh\left(\frac{s-t}{\sqrt{\lambda}}\right)\,\mathrm{d}t \\
&= -x_0'(0) \sinh\left(\frac{s}{\sqrt{\lambda}}\right) + \frac{1}{\sqrt{\lambda}} \int\limits_0^s x_0'(t) \cosh\left(\frac{s-t}{\sqrt{\lambda}}\right)\,\mathrm{d}t \\
&= -x_0'(0) \sinh\left(\frac{s}{\sqrt{\lambda}}\right) + \frac{1}{\sqrt{\lambda}} x_0(t) \cosh\left(\frac{s-t}{\sqrt{\lambda}}\right)\Bigg|_0^s + \frac{1}{\lambda} \int\limits_0^s x_0(t) \sinh\left(\frac{s-t}{\sqrt{\lambda}}\right)\,\mathrm{d}t \\
&= -x_0'(0) \sinh\left(\frac{s}{\sqrt{\lambda}}\right) + \frac{1}{\sqrt{\lambda}} x_0(s) - \frac{1}{\sqrt{\lambda}} x_0(0) \cosh\left(\frac{s}{\sqrt{\lambda}}\right) \\
&\quad + \frac{1}{\lambda} \int\limits_0^s x_0(t) \sinh\left(\frac{s-t}{\sqrt{\lambda}}\right)\,\mathrm{d}t\,.
\end{aligned}$$

Hence we have

$$\begin{aligned}
w_\lambda(s) \;=\; & \left(\widetilde{K}_1 - \frac{1}{\sqrt{\lambda}} x_0(0)\right) \cosh\left(\frac{s}{\sqrt{\lambda}}\right) + \left(\widetilde{K}_2 - \frac{1}{\sqrt{\lambda}} x_0'(0)\right) \sinh\left(\frac{s}{\sqrt{\lambda}}\right) \\
& + \frac{1}{\lambda} x_0(s) + \frac{1}{\sqrt{\lambda}} \frac{1}{\lambda} \int\limits_0^s x_0(t) \sinh\left(\frac{s-t}{\sqrt{\lambda}}\right)\,\mathrm{d}t\,.
\end{aligned} \tag{5.12}$$

Finally, the coefficients $K_1 := \left(\widetilde{K}_1 - \frac{1}{\sqrt{\lambda}} x_0(0)\right)$ and $K_2 := \left(\widetilde{K}_2 - \frac{1}{\sqrt{\lambda}} x_0'(0)\right)$ are to be determined from the boundary conditions (5.7) and (5.9). Differentiation of (5.12) gives

$$\begin{aligned} w_\lambda'(s) &= \frac{K_1}{\sqrt{\lambda}} \sinh\left(\frac{s}{\sqrt{\lambda}}\right) + \frac{K_2}{\sqrt{\lambda}} \cosh\left(\frac{s}{\sqrt{\lambda}}\right) + \frac{1}{\lambda} x_0'(s) \\ &\quad + \frac{1}{\lambda^2} \int_0^s x_0(t) \cosh\left(\frac{s-t}{\sqrt{\lambda}}\right) \mathrm{d}t \end{aligned}$$

with boundary condition

$$w_\lambda'(0) = \frac{K_2}{\sqrt{\lambda}} + \frac{x_0'(0)}{\lambda}\,. \tag{5.13}$$

From (5.13) we obtain by (5.9) that $K_2 = 0$. Moreover, (5.12) yields for $s = 1$

$$w_\lambda(1) = K_1 \cosh\left(\frac{1}{\sqrt{\lambda}}\right) + \frac{1}{\lambda} x_0(1) + \frac{1}{\sqrt{\lambda}} \frac{1}{\lambda} \int_0^1 x_0(t) \sinh\left(\frac{1-t}{\sqrt{\lambda}}\right) \mathrm{d}t \tag{5.14}$$

and together with (5.7) the factor

$$K_1 = -\frac{1}{\cosh\left(\frac{1}{\sqrt{\lambda}}\right)} \cdot \frac{1}{\lambda^{3/2}} \int_0^1 x_0(t) \sinh\left(\frac{1-t}{\sqrt{\lambda}}\right) \mathrm{d}t\,. \tag{5.15}$$

This, however, provides us with the explicit solution formula (5.5) coming from (5.12). $\square$

6 An example of explicit verification

We consider now for $A := J$ mapping in $L^2(0,1)$ the *special case* of a constant solution $x_0 \equiv 1$ (see the discussion around formula (4.6) in Section 4). Then Theorem 5.1 yields the family

$$\begin{aligned} w_\lambda(s) &= \frac{1}{\lambda} + \frac{1}{\lambda^{3/2}} \left[\int_0^s \sinh\left(\frac{s-t}{\sqrt{\lambda}}\right) \mathrm{d}t - \frac{\cosh\left(\frac{s}{\sqrt{\lambda}}\right)}{\cosh\left(\frac{1}{\sqrt{\lambda}}\right)} \int_0^1 \sinh\left(\frac{1-t}{\sqrt{\lambda}}\right) \mathrm{d}t \right] \\ &= \frac{1}{\lambda} + \frac{\cosh\left(\frac{s}{\sqrt{\lambda}}\right) - 1}{\lambda} - \frac{\cosh\left(\frac{s}{\sqrt{\lambda}}\right)}{\cosh\left(\frac{1}{\sqrt{\lambda}}\right)} \cdot \frac{\cosh\left(\frac{1}{\sqrt{\lambda}}\right) - 1}{\lambda} \\ &= \frac{1}{\lambda} \frac{\cosh\left(\frac{s}{\sqrt{\lambda}}\right)}{\cosh\left(\frac{1}{\sqrt{\lambda}}\right)} \end{aligned}$$

of solutions to (5.4) for that x_0. From (5.3) we get for these solutions an explicit expression for the distance function

$$d(R) = \sqrt{\int_0^1 \frac{\cosh^2\left(\frac{s}{\sqrt{\lambda(R)}}\right)}{\cosh^2\left(\frac{1}{\sqrt{\lambda(R)}}\right)}\, \mathrm{d}s} = \sqrt{\frac{\sqrt{\lambda(R)}\sinh\left(\frac{2}{\sqrt{\lambda(R)}}\right)+2}{4\cosh^2\left(\frac{1}{\sqrt{\lambda(R)}}\right)}}\,, \tag{6.1}$$

where $\lambda(R)$ denotes the uniquely determined positive number λ satisfying the equation $\|J\,w_\lambda\|^2 = R^2$. Also the antiderivatives $J\,w_\lambda$ of the functions w_λ can be made explicit for all $\lambda > 0$ as

$$[J\,w_\lambda](t) = \frac{1}{\lambda}\int_0^t \frac{\cosh\left(\frac{s}{\sqrt{\lambda}}\right)}{\cosh\left(\frac{1}{\sqrt{\lambda}}\right)}\,\mathrm{d}s = \frac{1}{\sqrt{\lambda}}\,\frac{\sinh\left(\frac{t}{\sqrt{\lambda}}\right)}{\cosh\left(\frac{1}{\sqrt{\lambda}}\right)} \qquad (0 \le t \le 1)\,.$$

Then we derive by some algebra

$$\begin{aligned}
\|J\,w_\lambda\|^2 &= \int_0^1 \frac{1}{\lambda}\,\frac{\sinh^2\left(\frac{t}{\sqrt{\lambda}}\right)}{\cosh^2\left(\frac{1}{\sqrt{\lambda}}\right)}\,\mathrm{d}t = \frac{\exp\left(-\frac{2}{\sqrt{\lambda}}\right)\left[\lambda\exp\left(\frac{4}{\sqrt{\lambda}}\right) - 4\sqrt{\lambda}\exp\left(\frac{2}{\sqrt{\lambda}}\right) - \lambda\right]}{8\lambda^{3/2}\cosh^2\left(\frac{1}{\sqrt{\lambda}}\right)} \\
&= \frac{2\lambda\sinh\left(\frac{2}{\sqrt{\lambda}}\right) - 4\sqrt{\lambda}}{8\lambda^{3/2}\cosh^2\left(\frac{1}{\sqrt{\lambda}}\right)} = \frac{\sqrt{\lambda}\sinh\left(\frac{2}{\sqrt{\lambda}}\right) - 2}{4\lambda\cosh^2\left(\frac{1}{\sqrt{\lambda}}\right)}\,.
\end{aligned}$$

Using the inequalities

$$\sinh t < \frac{1}{2}\exp(t) < \cosh t\,,$$

which are valid for all real numbers t, we can further estimate

$$\|J\,w_\lambda\|^2 \le \frac{\frac{1}{2}\sqrt{\lambda}\exp\left(\frac{2}{\sqrt{\lambda}}\right) - 2}{4\lambda\left[\frac{1}{2}\exp\left(\frac{1}{\sqrt{\lambda}}\right)\right]^2} = \frac{\frac{1}{2}\sqrt{\lambda}\exp\left(\frac{2}{\sqrt{\lambda}}\right) - 2}{\lambda\exp\left(\frac{2}{\sqrt{\lambda}}\right)} = \frac{1}{2\sqrt{\lambda}} - \frac{2}{\lambda\exp\left(\frac{2}{\sqrt{\lambda}}\right)} \le \frac{1}{2\sqrt{\lambda}}\,.$$

Hence, the positive value $\widetilde{\lambda}(R)$ satisfying the equation $\frac{1}{2\sqrt{\lambda}} = R^2$ is not less than $\lambda(R)$ satisfying the equation $\|J\,w_\lambda\|^2 = R^2$, i.e., $\lambda(R) \le \widetilde{\lambda}(R)$. Note that $\lambda(R)$ as well as $\widetilde{\lambda}(R)$ are well-determined positive numbers for all $R > 0$.

Finally, for evaluating the distance function (6.1) we note that for all $R > 0$

$$d(R) \le \sqrt{\frac{\frac{1}{2}\sqrt{\lambda(R)}\exp\left(\frac{2}{\sqrt{\lambda(R)}}\right)+2}{4\left[\frac{1}{2}\exp\left(\frac{1}{\sqrt{\lambda(R)}}\right)\right]^2}} = \sqrt{\frac{\sqrt{\lambda(R)}}{2} + \frac{2}{\exp\left(\frac{2}{\sqrt{\lambda(R)}}\right)}}\,. \tag{6.2}$$

For sufficiently large R, however, say $0 < \underline{R} \le R < \infty$, we have $\lambda(R) > 0$ small enough such that $\frac{2}{\exp\left(\frac{2}{\sqrt{\lambda(R)}}\right)} \le \frac{\sqrt{\lambda(R)}}{2}$ holds. Then by (6.2) we have

$$d(R) \;\le\; \sqrt[4]{\lambda(R)} \;\le\; \sqrt[4]{\widetilde{\lambda}(R)} \;=\; \frac{1}{\sqrt{2}\,R} \qquad (0 < \underline{R} \le R < \infty)\,. \tag{6.3}$$

Summarizing the last two sections we can state that our explicit approach via solving second kind integral equations leads to an upper estimate (4.8) with constant $c_0 = 1/\sqrt{2}$ of the distance function $d(R)$ if the solution (4.6) of equation (1.1) and the integration operator $A := J$ in $L^2(0,1)$ is under consideration. In view of (4.7) this estimate is *order optimal* for the limiting process $R \to \infty$.

By private communication we learned from P. Mathé that *Fenchel-Moreau duality* (cf. [25]) allows us to rewrite the distance function (1.2) as

$$d(R) \;=\; \sup\left\{\langle x_0, w\rangle - R\,\|A\,w\| \;:\; w \in X,\; \|w\| \le 1\right\} \qquad (R > 0) \tag{6.4}$$

yielding *lower bounds* for appropriate elements w. Precisely, for $X = Y = L^2(0,1)$, $A := J$ and x_0 from (4.6) we have for $\xi \in (0,1)$ and

$$w(t) := \frac{\chi_{[\xi,1]}(t)}{\sqrt{1-\xi}} \qquad (0 \le t \le 1)$$

with the characteristic function χ the properties $\|w\| = 1$ and $\langle x_0, w\rangle - R\,\|J\,w\| = \sqrt{1-\xi} - \frac{R(1-\xi)}{\sqrt{3}} \le \frac{\sqrt{3}}{4R}$. This inequality holds as an equation if we set $\xi := 1 - \frac{3}{4R^2}$. Then together with the upper bound (6.3) we obtain in that case the two-sided estimate

$$\frac{\sqrt{3}}{4\,R} \le d(R) \le \frac{1}{\sqrt{2}\,R}$$

for sufficiently large $R > 0$.

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