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On the nature of ill-posedness of an inverse problem arising in option pricing

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Abstract
Inverse problems in option pricing are frequently regarded as simple and resolved if a formula of Black–Scholes type defines the forward operator. However, precisely because the structure of such problems is straightforward, they may serve as benchmark problems for studying the nature of ill-posedness and the impact of data smoothness and no arbitrage on solution properties. In this paper, we analyse the inverse problem (IP) of calibrating a purely time-dependent volatility function from a term-structure of option prices by solving an ill-posed nonlinear operator equation in spaces of continuous and power-integrable functions over a finite interval. The forward operator of the IP under consideration is decomposed into an inner linear convolution operator and an outer nonlinear Nemytskii operator given by a Black–Scholes function. The inversion of the outer operator leads to an ill-posedness effect localized at small times, whereas the inner differentiation problem is ill posed in a global manner. Several aspects of regularization and their properties are discussed. In particular, a detailed analysis of local ill-posedness and Tikhonov regularization of the complete IP including convergence rates is given in a Hilbert space setting. A brief numerical case study on synthetic data illustrates and completes the paper.

1. Introduction

The past ten years can be considered as a very active period in developing the practice of pricing structured financial instruments in the context of modern risk management. This was also the reason for a dramatically growing interest in derivative pricing theory as an actual part of financial mathematics. Proceeding from the basic papers of Black, Scholes and Merton [6, 31] stochastic calculus combined with advanced numerical techniques could be applied successfully for the fair price calculation of options and other financial derivatives written on an underlying asset in arbitrage-free markets (see, for example, [5, 26, 29] and [33]).

There also occur inverse option pricing problems aimed at *calibrating* (identifying) not directly observable *volatilities* σ in general as functions depending on time τ and current asset price X from option prices u observed at the financial market. In particular, the mathematical background of the so-called volatility smile phenomenon of strike-dependent implied volatilities is under consideration. Research results concerning inverse problems (IPs) of option pricing have been intensively published in recent years (see, e.g., [3, 7, 8, 10–13] and [30]). Most of the papers remark on and motivate the fact that the IPs under consideration are *ill posed* in Hadamard's sense. Frequently they discuss *regularization* approaches for stable solutions of the IPs without analysing the *ill-posedness phenomena* of such problems in detail.

Inverse problems in option pricing are frequently regarded as simple and resolved if a formula of Black–Scholes type defines the forward operator, as in the case of a constant volatility, where the classical Black–Scholes formula holds. Also purely time-dependent volatility functions in combination with families of maturity-dependent option prices do not seem to be of much interest, since the model is rather restricted. But precisely because the structure of such problems is straightforward, they may serve as *benchmark problems* for studying several ill-posedness phenomena occurring in inverse option pricing problems. In this paper, based on the preliminary studies in [16] and [21], we try to fill a gap in the literature by analysing ill-posed situations and additional conditions enforcing well-posed subproblems associated with time-dependent option price and volatility functions in spaces of continuous and power-integrable functions over a finite time interval. This also provides an insight into the impact of data smoothness and no arbitrage on solution properties and into the singular character of at-the-money options. Neither phenomenon becomes apparent if one considers asset price-dependent volatilities and strike-dependent option prices. We believe that the analysis of the purely time-dependent case is important as an intermediary step towards the more general problem of fitting the volatility smile as a whole.

The paper is organized as follows: in the remaining part of the introduction we formulate in the context of time-dependent functions the option price formula using the Black–Scholes function and define the specific IP under consideration. The IP consists of solving a nonlinear operator equation in Banach spaces of real functions defined on a finite interval. The solution process is decomposed into solving a nonlinear outer operator equation by inverting a Nemytskii operator and solving a linear inner operator equation by differentiation. Main properties of the used Black–Scholes function and Nemytskii operator are summarized in section 2. Based on those properties section 3 deals with the solution of the outer equation of the IP for smooth option data in spaces of continuous functions. Both ill-posed and, in the case of arbitrage-free data, well-posed situations occur for this outer equation, whereas the inner equation acting as numerical differentiation is always ill posed. In section 4 we consider quasisolutions of the outer equation of the IP and their properties in the case of noisy data in L^p-spaces. Section 5 is devoted to the study of local ill-posedness properties of the complete ill-posed nonlinear IP in a Hilbert space setting by considering the character of convergence conditions for the Tikhonov regularization. A brief case study discussion of a discrete approach in section 6 illustrates and completes the paper.

We consider in this paper a variant of the Black–Scholes model, which is focused on time-dependent functions over the interval $[0, T]$ using a generalized *geometric Brownian motion* as stochastic process for the price $X(\tau) > 0$ of an asset, on which options are written. With constant drift $\mu \in \mathbb{R}$, time-dependent volatilities $\sigma(\tau) > 0$ and a standard Wiener process $W(\tau)$, the stochastic differential equation

$$\frac{\mathrm{d}X(\tau)}{X(\tau)} = \mu\,\mathrm{d}\tau + \sigma(\tau)\,\mathrm{d}W(\tau) \qquad (0 \leqslant \tau \leqslant T)$$

is assumed to hold. At the initial time $\tau = 0$ let there exist an idealized family of European vanilla call options written on the asset with current asset price $X := X(0) > 0$, fixed strike $K > 0$, fixed risk-free interest rate $r \geqslant 0$ and remaining times to maturity t continuously varying between zero and the upper time limit T.

Neglecting the role of dividends and setting for simplicity

$$a(\tau) := \sigma^2(\tau) \qquad (0 \leqslant \tau \leqslant T) \qquad \text{and} \qquad S(t) := \int_0^t a(\tau)\,\mathrm{d}\tau \qquad (0 \leqslant t \leqslant T)$$

it follows from stochastic and analytic considerations (for details see, e.g., [29, p 71f.]) that fair option prices $u(t)$ on *arbitrage-free* markets are explicitly given by the *Black–Scholes-type formula*

$$u(t) = U_{\mathrm{BS}}(X, K, r, t, S(t)) \qquad (0 \leqslant t \leqslant T). \tag{1}$$

This formula is based on the *Black–Scholes function* U_{BS}, which we can define for the variables $X > 0$, $K > 0$, $r \geqslant 0$, $\tau \geqslant 0$ and $s \geqslant 0$ as

$$U_{\mathrm{BS}}(X, K, r, \tau, s) := \begin{cases} X\Phi(d_1) - K\mathrm{e}^{-r\tau}\Phi(d_2) & (s > 0) \\ \max(X - K\mathrm{e}^{-r\tau}, 0) & (s = 0) \end{cases} \tag{2}$$

with

$$d_1 := \frac{\ln(\frac{X}{K}) + r\tau + \frac{s}{2}}{\sqrt{s}}, \qquad d_2 := d_1 - \sqrt{s} \tag{3}$$

and the cumulative density function of the standard normal distribution

$$\Phi(z) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \mathrm{e}^{-\frac{x^2}{2}}\,\mathrm{d}x. \tag{4}$$

In the following we always express the volatility term structure of the underlying asset by the not directly observable *volatility function* a. Although the formulae (1)–(4) were originally derived only for positive and Hölder continuous functions a, these formulae also yield well defined non-negative functions $u(t)$ $(0 \leqslant t \leqslant T)$ in the case of not necessarily continuous but Lebesgue-integrable and almost everywhere finite and non-negative functions $a(\tau)$ $(0 \leqslant \tau \leqslant T)$. Namely, such functions a have non-negative and absolutely continuous primitives $S(t)$ $(0 \leqslant t \leqslant T)$, which imply non-negative functions u as a consequence of the properties of the function U_{BS} listed in lemma 2.1 below.

Now let there be given at time $\tau = 0$ a *data function* $u^\delta(t)$ $(0 \leqslant t \leqslant T)$ of observed call option prices as noisy data of the *fair price function* $u(t)$ $(0 \leqslant t \leqslant T)$ according to formula (1) with a noise level $\delta \geqslant 0$. Then we can formulate the IP under consideration aimed at calibrating the volatility function a as follows.

Definition 1.1 (Inverse problem—IP). *Under the assumptions stated above find at time $\tau = 0$ the time-dependent volatility function $a(\tau)$ $(0 \leqslant \tau \leqslant T)$ from noisy observations $u^\delta(t)$ $(0 \leqslant t \leqslant T)$ of the maturity-dependent fair price function $u(t)$ $(0 \leqslant t \leqslant T)$.*

2. Black–Scholes function and Nemytskii operators

We first summarize the main properties of the *Black–Scholes function* U_{BS} defined by the formulae (2)–(4). The results of the following lemma can be proven straightforwardly by elementary calculations.

Lemma 2.1. *Let the parameters* $X > 0$, $K > 0$ *and* $r \geqslant 0$ *be fixed. Then the non-negative function* $U_{\mathrm{BS}}(X, K, r, \tau, s)$ *is continuous for* $(\tau, s) \in [0, \infty) \times [0, \infty)$. *Moreover, for* $(\tau, s) \in [0, \infty) \times (0, \infty)$*, this function is continuously differentiable with respect to* τ*, where we have*

$$\frac{\partial U_{\mathrm{BS}}(X, K, r, \tau, s)}{\partial \tau} = r K \mathrm{e}^{-r\tau} \Phi(d_2) \geqslant 0, \tag{5}$$

and twice continuously differentiable with respect to s, where we have with $\nu := \ln(\frac{X}{K})$

$$\begin{aligned} \frac{\partial U_{\mathrm{BS}}(X, K, r, \tau, s)}{\partial s} &= \Phi'(d_1) X \frac{1}{2\sqrt{s}} \\ &= \frac{X}{2\sqrt{2\pi s}} \exp\left(-\frac{[\nu + r\tau]^2}{2s} - \frac{[\nu + r\tau]}{2} - \frac{s}{8}\right) > 0 \end{aligned} \tag{6}$$

and

$$\begin{aligned} \frac{\partial^2 U_{\mathrm{BS}}(X, K, r, \tau, s)}{\partial s^2} &= -\Phi'(d_1) X \frac{1}{4\sqrt{s}} \left(-\frac{[\nu + r\tau]^2}{s^2} + \frac{1}{4} + \frac{1}{s}\right) \\ &= -\frac{X}{4\sqrt{2\pi s}} \left(-\frac{[\nu + r\tau]^2}{s^2} + \frac{1}{4} + \frac{1}{s}\right) \exp\left(-\frac{[\nu + r\tau]^2}{2s} - \frac{[\nu + r\tau]}{2} - \frac{s}{8}\right). \end{aligned} \tag{7}$$

Furthermore, we find the limit conditions

$$\lim_{s \to 0} \frac{\partial U_{\mathrm{BS}}(X, K, r, \tau, s)}{\partial s} = \begin{cases} \infty & (X = K\mathrm{e}^{-r\tau}) \\ 0 & (X \neq K\mathrm{e}^{-r\tau}) \end{cases} \tag{8}$$

and

$$\lim_{s \to \infty} U_{\mathrm{BS}}(X, K, r, \tau, s) = X. \tag{9}$$

On the other hand, the partial derivative

$$\frac{\partial U_{\mathrm{BS}}(X, K, r, \tau, s)}{\partial K} = -\mathrm{e}^{-r\tau} \Phi(d_2) < 0 \tag{10}$$

exists and is continuous for $(\tau, s) \in [0, \infty) \times (0, \infty)$.

The Black–Scholes function U_{BS} allows us to define a *Nemytskii operator N* by the formula

$$[N(v)](t) := U_{\mathrm{BS}}(X, K, r, t, v(t)) \qquad (0 \leqslant t \leqslant T). \tag{11}$$

This operator

$$N : D_+ := \{v(t)(0 \leqslant t \leqslant T) : v(t) \geqslant 0\} \longrightarrow D_+ \tag{12}$$

mapping the set D_+ of non-negative functions over the interval $[0, T]$ into itself will help verify the nature of the IP below.

From formula (6) of lemma 2.1 we obtain $\frac{\partial U_{\mathrm{BS}}(X,K,r,\tau,s)}{\partial s} > 0$ for all $(\tau, s) \in [0, T] \times (0, \infty)$ and hence the following lemma.

Lemma 2.2. *The Nemytskii operator N defined by formulae (11) and (12) is injective on its domain* D_+.

In general (see, e.g., [1, p 15]), a Nemytskii operator $N : v(t) \mapsto k(t, v(t))$ applied to real-valued scalar functions $v(t)$ is defined by a kernel function $k(t, s)$, where t varies in a finite interval $I \subset \mathbb{R}$ and s varies in $\mathbb{R}$. If $s \mapsto k(t, s)$ is continuous for almost every $t \in I$ and $t \mapsto k(t, s)$ is measurable for all $s \in \mathbb{R}$, the Nemytskii operator satisfies the *Carathéodory condition*. If the Nemytskii operator moreover satisfies a *growth* *condition* $|k(t,s)| \leqslant c_1 + c_2|s|^{p/q}$ with positive constants c_1 and c_2, then it maps continuously from $L^p(I)$ to $L^q(I)$ for $1 \leqslant p, q < \infty$ (see, e,g, [1, theorem 2.2]). In the context of formula (11) we set $I := [0, T]$ and

$$k(t,s) := U_{\mathrm{BS}}(X, K, r, t, s) \qquad (s \geqslant 0), \qquad k(t,s) := U_{\mathrm{BS}}(X, K, r, t, 0) \qquad (s < 0).$$

From lemma 2.1 it follows that the function $U_{\mathrm{BS}}(X, K, r, t, s)$ generating the Nemytskii operator N is continuous and uniformly bounded with $|U_{\mathrm{BS}}(X, K, r, t, s)| < X$ due to the formulae (6) and (9) for all $(t, s) \in [0, T] \times [0, \infty)$. Then the Carathéodory condition and a growth condition are satisfied and we have continuity of N between spaces of power-integrable functions on the interval $[0, T]$ as the following lemma asserts.

Lemma 2.3. *The Nemytskii operator N defined by formula (11) with domain $D_+ \cap L^p(0, T)$ maps continuously from $L^p(0, T)$ to $L^q(0, T)$ for all $1 \leqslant p, q < \infty$.*

As obvious throughout this paper we denote by $L^p(a, b)$ $(1 \leqslant p < \infty)$ the Banach space of p-power integrable real functions $x(t)$ $(a \leqslant t \leqslant b)$ with the norm $\|x\|_{L^p(a,b)} := (\int_a^b |x(t)|^p\,\mathrm{d}t)^{1/p}$, by $L^\infty(a, b)$ the Banach space of essentially bounded real functions on the interval (a, b) with the norm $\|x\|_{L^\infty(a,b)} := \operatorname{ess\,sup}_{t\in(a,b)} |x(t)|$ and by $C[a, b]$ the Banach space of continuous real functions defined on $[a, b]$ with the norm $\|x\|_{C[a,b]} := \max_{t\in[a,b]} |x(t)|$.

If we restrict the domain of N to the set

$$D_0 := \{v \in C[0, T] : v(0) = 0, v(t) \geqslant 0 \ (0 < t \leqslant T)\},$$

then because of lemma 2.1 we have

$$N : D_0 \subset C[0, T] \longrightarrow D_+ \cap C[0, T].$$

Using the substitutions $w := \sqrt{\frac{v}{t}}$ as well as $\Omega(t, w) := U_{\mathrm{BS}}(X, K, r, t, v)$ we derive for all $t > 0$ and $w > 0$

$$0 < \frac{\partial \Omega(t, w)}{\partial w} = X\sqrt{t}\,\Phi'(\bar{d}_1) \leqslant \frac{X\sqrt{t}}{\sqrt{2\pi}}$$

with

$$\bar{d}_1 := \frac{\ln(\frac{X}{K}) + t(r + \frac{w^2}{2})}{\sqrt{t}w}.$$

Consequently, for functions $v_1, v_2, w_1, w_2 \in D_0$ with $v_i(t) = t w_i^2(t)$ $(i = 1, 2)$ there are pointwise estimations

$$|[N(v_1)](t) - [N(v_2)](t)| \leqslant \left|\frac{\partial \Omega(t, w_t)}{\partial w}\right| \frac{1}{\sqrt{t}} \left|\sqrt{v_1(t)} - \sqrt{v_2(t)}\right| \qquad (0 < t \leqslant T)$$

with an intermediate value w_t between $w_1(t)$ and $w_2(t)$ and

$$|[N(v_1)](t) - [N(v_2)](t)| \leqslant \frac{X}{\sqrt{2\pi}} \left|\sqrt{v_1(t)} - \sqrt{v_2(t)}\right| \qquad (0 \leqslant t \leqslant T). \tag{13}$$

From (13) we directly obtain the following.

Lemma 2.4. *The Nemytskii operator N defined by formula (11) with domain D_0 maps continuously from $C[0, T]$ to $C[0, T]$.*

If we denote by B_1, B_2 and B_3 Banach spaces of functions defined on the interval $[0, T]$, then we can write the IP as a *nonlinear operator equation*

$$F(a) = u \qquad (a \in D(F) \subset B_1, u \in D_+ \cap B_2), \tag{14}$$

where the nonlinear operator

$$F = N \circ J : D(F) \subset B_1 \longrightarrow B_2$$

with domain

$$D(F) := \{\tilde{a} \in L^1(0,T) \cap B_1 : \tilde{a}(t) \geqslant 0 \text{ a.e. in } [0,T]\}$$

is decomposed into the *inner* linear convolution operator $J : B_1 \longrightarrow B_3$ with

$$[J(h)](t) := \int_0^t h(\tau)\,\mathrm{d}\tau \qquad (0 \leqslant t \leqslant T) \tag{15}$$

and the *outer* nonlinear Nemytskii operator $N : D_+ \cap B_3 \subset B_3 \longrightarrow B_2$ defined by (11).

Consequently, the problem of solving the operator equation (14) can be decomposed into solving, successively, the nonlinear outer operator equation

$$N(S) = u \qquad (S \in D_+ \cap B_3, u \in D_+ \cap B_2) \tag{16}$$

and the linear inner operator equation

$$J(a) = S \qquad (a \in D(F) \subset B_1, S \in D_+ \cap B_3). \tag{17}$$

For our domain $D(F)$, all functions of the range $J(D(F))$ are absolutely continuous, non-negative and nondecreasing and belong to the set

$D_0^{\nearrow} := \{\tilde{S} \in C[0,T] : \tilde{S}(0) = 0, \tilde{S}(t_1) \leqslant \tilde{S}(t_2)\ (0 \leqslant t_1 < t_2 \leqslant T)\} \subset D_0 \subset D_+$.

Therefore the inner equation (17) is only solvable if the solution S of the outer equation (16) belongs to $D_0^{\nearrow}$.

Note that the composition $F = N \circ J$ under consideration in this paper is reverse to the situation discussed in [27, chapter 7.5], where as occurring in the case of Hammerstein integral equations nonlinear composite operators $\tilde{F} = A \circ N$ with an inner Nemytskii and an outer bounded linear operator A are analysed.

To solve forward problems of computing maturity-dependent price functions $\hat{u}(t) := U_{\mathrm{BS}}(\hat{X}, \hat{K}, \hat{r}, t, S(t))$ $(0 \leqslant t \leqslant T)$ of European vanilla call options with varying parameters $\hat{X}$, $\hat{K}$ and $\hat{r}$ based on the solution of the IP it is sufficient to determine the auxiliary function S from the outer equation (16). In view of the continuity of Nemytskii operators N under consideration here (see lemma 2.4), the problems of finding $\hat{u}$ from S are well posed if we measure the deviations of S and $\hat{u}$ in the maximum norm. On the other hand, the volatility function $a(t)$ $(0 \leqslant t \leqslant T)$ itself is not used explicitly for computing $\hat{u}$. As the subsequent section will show, this is an advantage. Namely, for arbitrage-free option data u^δ of the fair price function u the outer equation (16) is *well posed* in a C-space setting. However, the inner equation (17) aimed at finding the derivative $a(t) = S'(t)$ $(0 \leqslant t \leqslant T)$ of the function S is *ill posed* in usual Banach spaces B_1 and B_3 of integrable or continuous functions on the interval $[0, T]$ and leads to *ill-conditioned* problems after discretization (see, e.g., [18]). In the Hilbert space setting $B_1 = B_3 = L^2(0, T)$ the differentiation problem is weakly ill posed and has an ill-posedness degree of one (see, e.g., [28, p 235] and [22, p 33ff]).

Note that for the practitioners it is preferably of interest to solve the complete IP, since the calibration of volatility functions a is required for pricing American or exotic options by solving initial boundary value problems of parabolic PDEs, where the volatilities occur as parameters in the differential equation. The stable approximate solution of the overall IP is discussed in section 5 below.

Throughout this paper we only analyse the IP for calls. Since out-of-the-money option prices are more informative regarding the unknown volatilities than in-the-money option prices, it could be helpful to calibrate from real data of put options in the case $X > K$. Since for call prices u_{call} and associated put prices u_{put} with fixed parameters X, K, r and the same maturity t the usual put–call parity relation $u_{\mathrm{put}}(t) = u_{\mathrm{call}}(t) - X + K\mathrm{e}^{-rt}$ holds (see, e.g., [29, p 121]), the results of the call analysis can be easily transformed to the put case.

3. Solving the outer equation of the IP in C-spaces for smooth and arbitrage-free option data

In this section we are going to solve with $B_2 = B_3 = C[0, T]$ the *outer equation* (16) of the IP for a given function $u^\delta(t)$ $(0 \leqslant t \leqslant T)$ of observed option price data that approximate the fair price function $u = F(a) = N(S)$. Let the admissible volatility functions possess in the following a positive essential infimum, i.e., we assume $a \in D^*(F)$, where

$$D^*(F) := \{\tilde{a} \in L^1(0, T) : \operatorname*{ess\,inf}_{t\in(0,T)} \tilde{a}(t) > 0\}.$$

Moreover, let the data u^δ satisfy the following assumption, which is reasonable for data in an arbitrage-free market (see, e.g., [31]).

Assumption 3.1. *The data function $u^\delta(t)$ $(0 \leqslant t \leqslant T)$ is assumed to be continuous and strictly increasing with*

$$u^\delta(0) = \max(X - K, 0), \qquad \max(X - K\mathrm{e}^{-rt}, 0) < u^\delta(t) < X \qquad (0 < t \leqslant T). \tag{18}$$

Note that the assumption 3.1 is satisfied for all functions u^δ belonging to the range $F(D^*(F))$. Namely, the range $J(D^*(F))$ consists of strictly increasing functions $\tilde{S} = J(\tilde{a})$ with a minimum growth rate $\tilde{S}(t) \geqslant ct$ $(0 \leqslant t \leqslant T)$ and $c := \operatorname{ess\,inf}_{t\in(0,T)} \tilde{a}(t) > 0$. Consequently, due to lemma 2.1 the continuous functions $\tilde{u}$ of the range $F(D^*(F))$ are strictly increasing and fulfil a condition of type (18).

For noisy data u^δ the outer equation (16) can be rewritten as

$$U_{\mathrm{BS}}(X, K, r, t, S^\delta(t)) = u^\delta(t) \qquad (0 \leqslant t \leqslant T). \tag{19}$$

If there *exists* a solution $S^\delta \in D_+$ of equation (19) for given data u^δ, then from the *injectivity* of the Nemytskii operators N (see lemma 2.2) it follows that this solution is *unique*. Moreover, the following theorem shows that we can even find a uniquely determined function $S^\delta \in D_0 \subset C[0, T]$ satisfying (19).

Theorem 3.2. *Under the assumption 3.1 there exists a uniquely determined continuous function $S^\delta(t)$ $(0 \leqslant t \leqslant T)$ with $S^\delta(0) = 0$ and $0 < S^\delta(t) \leqslant \bar{S}$ $(0 < t \leqslant T)$ solving the equation (19), where $\bar{S}$ satisfies the equation $U_{\mathrm{BS}}(X, K, r, 0, \bar{S}) = u^\delta(T) = \|u^\delta\|_{C[0,T]}$.*

Proof. As a consequence of lemma 2.1 the function $k(t, s) := U_{\mathrm{BS}}(X, K, r, t, s)$ with

$$\frac{\partial k(t,s)}{\partial t} \geqslant 0 \qquad \text{and} \qquad \frac{\partial k(t,s)}{\partial s} > 0$$

is continuous in both variables t and s, nondecreasing with respect to t and strictly increasing with respect to s for $(t, s) \in [0, T] \times (0, \infty)$. Moreover, we have for all $t \in (0, T]$

$$\lim_{s\to 0} k(t, s) = k(t, 0) = \max(X - K\mathrm{e}^{-rt}, 0) < \lim_{s\to\infty} k(t, s) = X$$

(see the formulae (2) and (9)). Since the data u^δ with $u^\delta(t) \leqslant u^\delta(T)$ $(0 \leqslant t \leqslant T)$ satisfy the condition (18), from the family of equations

$$k(t, s) = u^\delta(t) \tag{20}$$

in s, where the parameter t varies in the interval $[0, T]$, we find in a unique manner values $s = S^\delta(t) > 0$ for all $t \in (0, T]$ and $s = S^\delta(0) = 0$ for $t = 0$ because of $k(0, 0) = u^\delta(0)$. The value $\bar{S}$ satisfying $k(0, \bar{S}) = u^\delta(T)$ is also uniquely determined. From the estimation $k(0, S^\delta(t)) \leqslant k(t, S^\delta(t)) = u^\delta(t) \leqslant u^\delta(T) = k(0, \bar{S})$ we get $S^\delta(t) \leqslant \bar{S}$. Finally, the continuity of the function $S^\delta(t)$ $(0 \leqslant t \leqslant T)$ follows from the *implicit function theorem* (see, e.g., [17, p 421]) considering that $k(t, s)$ is continuous in both variables and strictly monotonic with respect to s. This proves the theorem. □

Note that the functions S^δ provided by theorem 3.2 are not necessarily monotonic. We will evaluate pointwise for $0 < t \leqslant T$ the error $|S^\delta(t) - S(t)|$ of the continuous positive function $S^\delta(t)$ with $\lim_{t\to 0} S^\delta(t) = 0$ thus obtained, by using the formula

$$|S^\delta(t) - S(t)| = \left(\frac{\partial U_{\mathrm{BS}}(X, K, r, t, S_{\mathrm{im}}(t))}{\partial s}\right)^{-1} |u^\delta(t) - u(t)|, \tag{21}$$

where $S_{\mathrm{im}}(t) \in [\min(S^\delta(t), S(t)), \max(S^\delta(t), S(t))]$ is a positive intermediate function influencing the error amplification factor

$$\varphi(t) := \left(\frac{\partial U_{\mathrm{BS}}(X, K, r, t, S_{\mathrm{im}}(t))}{\partial s}\right)^{-1} > 0 \qquad (0 < t \leqslant T).$$

With $\lim_{t\to 0} S_{\mathrm{im}}(t) = 0$ we obtain from formula (6) in the case $X \neq K$ the limit conditions

$$\lim_{t\to 0} \frac{1}{\sqrt{S_{\mathrm{im}}(t)}} \exp\left(-\frac{[\ln(\frac{X}{K}) + rt]^2}{2S_{\mathrm{im}}(t)}\right) = 0$$

and consequently

$$\lim_{t\to 0} \varphi(t) = \infty \qquad (X \neq K) \tag{22}$$

for the error amplification factor. That means, in the case $X \neq K$, the problem of determining S^δ from data u^δ satisfying the assumption 3.1 is *ill posed* in a C-space setting. The ill-posedness is locally concentrated in a neighbourhood of $t = 0$. As a consequence of (22), for $X \neq K$ and sufficiently small t, the errors $|S^\delta(t) - S(t)|$ may remain large, although the data errors $\|u^\delta - u\|_{C[0,T]}$ get arbitrarily small. In practice the approximate solutions $S^\delta(t)$ tend to oscillate for small t in such a data situation (see also figures 3 and 4 in section 6).

On the other hand, the case $X = K$ is more ambiguous. Namely, in that case $\frac{1}{\sqrt{S_{\mathrm{im}}(t)}} \exp(-\frac{r^2t^2}{2S_{\mathrm{im}}(t)})$ tends to infinity as $t \to 0$ whenever we have an inequality of the form $S_{\mathrm{im}}(t) \geqslant Ct^2$ $(0 \leqslant t \leqslant T)$ with a constant $C > 0$ and we get from formula (6) the reverse limit condition

$$\lim_{t\to 0} \varphi(t) = 0 \qquad (X = K) \tag{23}$$

for the amplification factor. If however $\liminf_{t\to 0} \frac{1}{\sqrt{S_{\mathrm{im}}(t)}} \exp(-\frac{r^2t^2}{2S_{\mathrm{im}}(t)}) = 0$, then for $X = K$ we obtain $\limsup_{t\to 0} \varphi(t) = \infty$.

Closely connected with the limit jump in formula (8) we find a jump situation by comparing the formulae (22) and (23). *At-the-money options* with $X = K$ represent a *singular situation*, since the instability of the outer equation at $t = 0$ for in-the-money options and out-of-the-money options expressed by formula (22) disappears if formula (23) holds. Such a singular behaviour of at-the-money options seems to be well known in finance. Namely, for a constant volatility σ, the frequently used option measure *theta* written in our terms as

$$\Theta(t) := \frac{d}{d(-t)} U_{\mathrm{BS}}(X, K, r, t, S(t)) \qquad \text{with } S(t) = \sigma^2 t$$

explodes to $-\infty$ as the time to maturity t tends to zero if and only if $X = K$ (see figure 13.6 in [26, p 321]).

The ill-posedness effect just described in particular for $X \neq K$ as well as the missing monotonicity of S^δ can be overcome for the outer equation by posing a further assumption.

Assumption 3.3. *In addition to assumption 3.1 the data function $u^\delta(t)$ is assumed to be continuously differentiable for $0 < t \leqslant T$ with*

$$(u^\delta)'(t) - Kr\mathrm{e}^{-rt}\Phi\left(\frac{\ln(\frac{X}{K}) + rt - \frac{S^\delta(t)}{2}}{\sqrt{S^\delta(t)}}\right) \geqslant 0 \qquad (0 < t \leqslant T), \tag{24}$$

where u^δ implies the function $S^\delta \in D_0$ with $S^\delta(t) > 0$ for $t > 0$ via equation (19) in a unique manner.

The condition (24) is also a consequence of an arbitrage-free market. Namely, by comparing appropriate portfolios it can be shown that option prices $u(K, t)$ at time $\tau = 0$ considered as differentiable functions of strike price K and maturity t satisfy inequalities of the form (see [2, p 11])

$$\frac{\partial u(K,t)}{\partial t} + Kr\frac{\partial u(K,t)}{\partial K} \geqslant 0. \tag{25}$$

For the IP we have $u(K, t) = U_{\mathrm{BS}}(X, K, r, t, S(t))$, where $\frac{\partial u(K,t)}{\partial t} = u'(t)$ and with (10)

$$\frac{\partial u(K,t)}{\partial K} = \frac{\partial U_{\mathrm{BS}}(X, K, r, t, S(t))}{\partial K} = -\mathrm{e}^{-rt}\Phi\left(\frac{\ln(\frac{X}{K}) + rt - \frac{S(t)}{2}}{\sqrt{S(t)}}\right).$$

Consequently, the inequality (25) attains here the form (24).

Theorem 3.4. *Under the assumptions 3.1 and 3.3 the uniquely determined solution S^δ of equation (19) with $S^\delta(0) = 0$ and $S^\delta(t) > 0$ $(0 < t \leqslant T)$ is a nondecreasing and absolutely continuous function with a continuous and integrable derivative $(S^\delta)'(t) \geqslant 0$ $(0 < t \leqslant T)$, where $S^\delta(t) = \int_0^t (S^\delta)'(\tau)\,\mathrm{d}\tau$ $(0 < t \leqslant T)$ and*

$$(S^\delta)'(t) = \frac{2\sqrt{S^\delta(t)}[(u^\delta)'(t) - Kr\mathrm{e}^{-rt}\Phi(d_2^*)]}{\Phi'(d_1^*)X} \geqslant 0 \qquad (0 < t \leqslant T) \tag{26}$$

with

$$d_1^* := \frac{\ln(\frac{X}{K}) + rt + \frac{S^\delta(t)}{2}}{\sqrt{S^\delta(t)}}, \qquad d_2^* := d_1^* - \sqrt{S^\delta(t)}.$$

Proof. Considering the formulae (5), (6) and (24) for $0 < t \leqslant T$ from the implicit function theorem (see, e.g., [17, p 423ff]) we obtain continuous differentiability of S^δ with $(S^\delta)'(t) \geqslant 0$ and formula (26). Hence $S^\delta(t)$ $(0 \leqslant t \leqslant T)$ is nondecreasing and based on [32, theorems 4 and 5, p 236f] we have an integrable derivative $(S^\delta)' \in L^1(0, T)$ with $\int_0^t (S^\delta)'(\tau)\,\mathrm{d}\tau \leqslant S^\delta(t) - S^\delta(0) = S^\delta(t)$ $(0 \leqslant t \leqslant T)$. Choosing ε from the interval $0 < \varepsilon < t$ we get

$$\int_0^t (S^\delta)'(\tau)\,\mathrm{d}\tau = \int_0^\varepsilon (S^\delta)'(\tau)\,\mathrm{d}\tau + S^\delta(t) - S^\delta(\varepsilon) = S^\delta(t)$$

and absolute continuity of S^δ, since $\int_0^\varepsilon (S^\delta)'(\tau)\,\mathrm{d}\tau - S^\delta(\varepsilon)$ is a constant and tends to zero as $\varepsilon \to 0$. This proves the theorem. □

As a consequence of theorem 3.4 we obtain for arbitrage-free and sufficiently smooth option price data $u^\delta(t)$ $(0 \leqslant t \leqslant T)$ by solving equation (19) a function $S^\delta \in D_0^{\nearrow}$, which is continuously differentiable for positive t and provides an integrable volatility function $a^\delta(t) := (S^\delta)'(t) \geqslant 0$ $(0 < t \leqslant T)$. The function $a^\delta(t)$ is continuous for $t > 0$, but may tend to infinity as t tends to zero.

The next theorem will show that solving the equation (19) for smooth arbitrage-free data u^δ is a *well-posed* problem for the Banach spaces $B_2 = B_3 = C[0, T]$. That means that making option price data u^δ free of arbitrage acts as a specific variant of regularization for the outer equation (16).

Theorem 3.5. *Let $\{u_n = N(S_n)\}_{n=1}^{\infty}$ with N from formula (11) be a sequence of arbitrage-free noisy option price functions satisfying the assumptions 3.1 and 3.3 that converges in the Banach space $B_2 = C[0, T]$ to the fair option price function $u = N(S)$. Then the associated sequence of functions $\{S_n\}_{n=1}^{\infty}$ also converges to S in the Banach space $B_3 = C[0, T]$.*

Proof. In view of the positivity and continuity of the partial derivative $\dfrac{\partial U_{\mathrm{BS}}(X, K, r, t, s)}{\partial s}$ on the domain $(t, s) \in [0, T] \times (0, \infty)$ (see lemma 2.1) we have, for fixed $t \in (0, T]$,

$$|S_n(t) - S(t)| \leqslant \left(\frac{\partial U_{\mathrm{BS}}(X, K, r, t, S_{\mathrm{im}}(t))}{\partial s}\right)^{-1} |u_n(t) - u(t)|$$

with intermediate values $S_{\mathrm{im}}(t)$ between the positive values $S_n(t)$ and $S(t)$. Now, for given sufficiently small $\varepsilon > 0$ we choose $t_\varepsilon \in (0, T]$ such that $S(t_\varepsilon) = \frac{\varepsilon}{4}$. Since the function U_{BS} is increasing with respect to $s > 0$, the functions S_n and S are increasing for $t \in [t_\varepsilon, T]$ and there holds $\lim_{n\to\infty} u_n(t_\varepsilon) = u(t_\varepsilon) > \max(X - K\mathrm{e}^{rt_\varepsilon}, 0)$ as well as $\lim_{n\to\infty} u_n(T) = u(T) < X$, we find $0 < S_{\min} < S_{\max} < \infty$ and a positive integer n_1 depending on ε such that

$$S_{\min} \leqslant S_n(t) \leqslant S_{\max}, \qquad S_{\min} \leqslant S(t) \leqslant S_{\max} \quad (t_\varepsilon \leqslant t \leqslant T, n \geqslant n_1).$$

Then we obtain

$$\|S_n - S\|_{C[t_\varepsilon, T]} \leqslant C\|u_n - u\|_{C[t_\varepsilon, T]} \qquad (n \geqslant n_1(\varepsilon))$$

with the constant

$$C := \max_{(t,s)\in[t_\varepsilon,T]\times[S_{\min},S_{\max}]} \left(\frac{\partial U_{\mathrm{BS}}(X, K, r, t, s)}{\partial s}\right)^{-1}.$$

Moreover, there exists an integer n_2 depending on ε with

$$|u_n(t) - u(t)| \leqslant \frac{\varepsilon}{2C} \qquad (0 \leqslant t \leqslant T, n \geqslant n_2).$$

This provides

$$\|S_n - S\|_{C[t_\varepsilon, T]} \leqslant \frac{\varepsilon}{2} \qquad (n \geqslant \max(n_1, n_2)).$$

Using the growth of the functions S_n and S (see theorem 3.4) and the triangle inequality we get for $n \geqslant \max(n_1, n_2)$ the estimations

$$\|S_n - S\|_{C[0, t_\varepsilon]} \leqslant S_n(t_\varepsilon) + S(t_\varepsilon) \leqslant |S_n(t_\varepsilon) - S(t_\varepsilon)| + 2S(t_\varepsilon) \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and

$$\|S_n - S\|_{C[0,T]} \leqslant \varepsilon,$$

which prove the theorem. □

Note that for nil interest rates $r = 0$ and data functions $u^\delta(t)$ continuously differentiable for $0 < t \leqslant T$ the assumption 3.3 is reduced to a data monotonicity with respect to t. If, moreover, strictly increasing data functions are required as in assumption 3.1, then the function $S^\delta = J(a^\delta)$ of theorem 3.4 is also strictly increasing and corresponds to a strictly positive volatility function $a^\delta(t) > 0$ $(0 < t \leqslant T)$.

4. Solving the outer equation of the IP in L^p-spaces for noisy option data

In this section we measure deviations of the functions u^δ and S^δ from u and S on the interval $[0, T]$ by means of L^p-norms. We consider the Banach spaces $B_2 = L^q(0, T)$ and $B_3 = L^p(0, T)$ with $1 \leqslant p, q < \infty$ for the outer equation (16) of the IP.

The positive data function $u^\delta(t)(0 \leqslant t \leqslant T)$ of observed maturity-dependent option prices is not necessarily smooth and arbitrage free in the sense of assumptions 3.1 and 3.3, but it satisfies assumption 4.1.

Assumption 4.1. *The non-negative data function* $u^\delta \in L^q(0, T)$ $(1 \leqslant q < \infty)$ *is approximated by the estimate*

$$\|u^\delta - u\|_{L^q(0,T)} \leqslant \delta \tag{27}$$

the fair option price function $u = F(a) = N(S)$ *for a given noise level* $\delta > 0$. *Moreover, let* $a \in L^\infty(0, T)$ *hold for the volatility function, where we assume an upper bound* $\bar{c} \geqslant \|a\|_{L^\infty(0,T)}$ *implying* $0 \leqslant S(t) \leqslant \kappa$ $(0 \leqslant t \leqslant T)$ *with* $\kappa := \bar{c}T$.

We apply a variant of the *method of quasisolutions* exploiting the fact that

$$D_+^\kappa := \{\tilde{S} \in D_+ : 0 \leqslant \tilde{S}(t) \leqslant \kappa \ (0 \leqslant t \leqslant T), \tilde{S}(t_1) \leqslant \tilde{S}(t_2) \ (0 \leqslant t_1 < t_2 \leqslant T)\}$$

is a *compactum* in the Banach space $L^p(0, T)$ $(1 \leqslant p < \infty)$ (see, e.g., [4, example 3, p 26]). As an approximate solution of the outer equation (16) we use a quasisolution associated with the data u^δ, which is a minimizer $S^\delta \in D_+^\kappa$ of the extremal problem

$$\|N(\tilde{S}) - u^\delta\|_{L^q(0,T)} \longrightarrow \min, \qquad \text{subject to } \tilde{S} \in D_+^\kappa.$$

Then we can prove the following convergence assertion.

Theorem 4.2. *Let* $\{S^{\delta_n}\}_{n=1}^\infty$ *be a sequence of quasisolutions associated with a sequence of data* $\{u^{\delta_n}\}_{n=1}^\infty$ *satisfying the inequality (27), where* $\delta_n \to 0$ *as* $n \to \infty$. *Then the convergence properties*

$$\lim_{n\to\infty} \|S^{\delta_n} - S\|_{L^p(0,T)} = 0 \qquad (1 \leqslant p < \infty) \tag{28}$$

and

$$\lim_{n\to\infty} \|S^{\delta_n} - S\|_{L^\infty(0,\gamma)} = 0 \qquad \textit{for all } 0 < \gamma < T \tag{29}$$

hold.

Proof. Since the Nemytskii operator

$$N: \quad D_+^\kappa \subset L^p(0, T) \longrightarrow L^q(0, T)$$

is injective and continuous (see lemmas 2.2 and 2.3), we obtain the first limit condition (28) immediately from Tikhonov's lemma on the continuity of the inverse of an operator, which is injective, continuous and defined on a compactum (see, e.g., [4, lemma 2.2]). Moreover, from [4, theorem 2.8] based on the continuity of the function S we can formulate a further limit condition

$$\lim_{n\to\infty} \|S^{\delta_n} - S\|_{L^\infty(\beta,\gamma)} = 0 \qquad \text{for all } 0 < \beta < \gamma < T, \tag{30}$$

where the approximate solution $S^{\delta_n} \in D_+^\kappa$ may have discontinuities. Using the triangle inequality and the growth of the functions S and S^{δ_n} we find

$$\|S^{\delta_n} - S\|_{L^\infty(0,\beta)} \leqslant S^{\delta_n}(\beta) + S(\beta) \leqslant \|S^{\delta_n} - S\|_{L^\infty(\beta,\gamma)} + 2S(\beta)$$

for arbitrarily small values $\beta > 0$. For any given $\varepsilon > 0$ there is a value $\beta_0 > 0$ such that $S(\beta_0) < \frac{\varepsilon}{4}$, since $\lim_{\beta\to 0} S(\beta) = 0$. For sufficiently large n we moreover have with (30) $\|S^{\delta_n} - S\|_{L^\infty(\beta_0,\gamma)} < \frac{\varepsilon}{2}$ and hence $\|S^{\delta_n} - S\|_{L^\infty(0,\gamma)} < \varepsilon$. This implies the limit condition (29) and proves the theorem. □

Note that the set $D_+^\kappa \subset L^\infty(0, T)$ fails to be a compactum in the Banach space $L^\infty(0, T)$. Therefore the uniform convergence of approximate solutions S^δ to S cannot be shown on the whole interval $[0, T]$. Moreover, the L^q-data u^δ do not allow pointwise error estimations as given in formula (21).

The stabilization approach for the outer problem discussed in this section is a variant of *descriptive regularization* using the monotonicity of S as *a priori* information. On the other hand, constraints of the form $a(t) \geqslant 0$ or $0 \leqslant a(t) \leqslant \bar{c} < \infty$ a.e. on $[0, T]$ are not able to stabilize the solution process sufficiently. Therefore, the reconstruction of $a \in L^1(0, T)$ from data $S^\delta \in L^p(0, T)$ by solving the inner equation (17) is always an unstable component in solving the IP and requires an additional regularization.

5. On Tikhonov regularization in L^2

Now we are going to study ill-posedness properties of the complete ill-posed nonlinear IP written as operator equation (14) in the Hilbert space setting $B_1 = B_2 = L^2(0, T)$ by considering the behaviour of Tikhonov regularization with respect to convergence and convergence rates along the lines of the seminal paper [15] (see also [14]). In particular, we deal with the operator equation

$$F(a) = u \qquad (a \in D^\dagger(F) \subset L^2(0, T), u \in D_+ \cap L^2(0, T)), \tag{31}$$

where the domain of the nonlinear operator $F = N \circ J$ is restricted to

$$D^\dagger(F) := \{\tilde{a} \in L^2(0, T) : \operatorname*{ess\,inf}_{t\in(0,T)} \tilde{a}(t) \geqslant \underline{c} > 0\}$$

with a given uniform positive lower bound $\underline{c}$. Since $J : L^2(0, T) \to L^2(0, T)$ defined by formula (15) is a compact linear operator (see, e.g., [28, p 235]) and $N : D_+ \cap L^2(0, T) \subset L^2(0, T) \to L^2(0, T)$ defined by formula (11) is a continuous nonlinear operator as a consequence of lemma 2.3, the composite nonlinear operator $F = N \circ J : D^\dagger(F) \subset L^2(0, T) \to L^2(0, T)$ is also compact and continuous. Then based on results of section 2 we have the following lemma.

Lemma 5.1. *The nonlinear operator $F : D^\dagger(F) \subset L^2(0, T) \to L^2(0, T)$ possessing a convex and weakly closed domain $D^\dagger(F)$ is injective, compact, continuous, weakly continuous and consequently weakly closed, and the inverse operator F^{-1} defined on $F(D^\dagger(F))$ exists.*

Then proposition A.3 of [15] applies and we can formulate the following as a corollary of lemma 5.1.

Corollary 5.2. *For a given right-hand side $u \in F(D^\dagger(F))$ the operator equation (31) has a uniquely determined solution $a \in D^\dagger(F)$. For any ball $B_r(a)$ with centre a and radius $r > 0$ there exists a sequence $\{a_n\}_{n=1}^\infty \subset B_r(a) \cap D^\dagger(F)$ with*

$$a_n \rightharpoonup a \qquad \text{but } a_n \not\to a \qquad \text{and} \qquad F(a_n) \to u \qquad \text{in } L^2(0, T) \qquad \text{as } n \to \infty.$$

Thus, equation (31) is locally ill posed in the sense of [25, definition 2] and F^{-1} is not continuous in u.

Consequently, a regularization is required for the stable approximate solution of (31). We consider for data u^δ with

$$\|u^\delta - u\|_{L^2(0,T)} \leqslant \delta$$

and a fixed initial guess $a^* \in L^2(0,T)$ Tikhonov regularized solutions a_α^δ as minimizers of

$$\|F(\tilde{a}) - u^\delta\|^2_{L^2(0,T)} + \alpha\|\tilde{a} - a^*\|^2_{L^2(0,T)} \longrightarrow \min, \qquad \text{subject to } \tilde{a} \in D^\dagger(F),$$

which exist for all regularization parameters $\alpha > 0$ and stably depend on the data u^δ (see [15, theorem 2.1]). Moreover, for

$$\alpha_n = \alpha_n(\delta_n) \to 0 \qquad \text{and} \qquad \frac{\delta_n^2}{\alpha_n(\delta_n)} \to 0 \qquad \text{as } \delta_n \to 0 \qquad \text{for } n \to \infty$$

any sequence $\{a_{\alpha_n}^{\delta_n}\}_{n=1}^\infty$ converges to a in $L^2(0,T)$ (see [15, theorem 2.3]).

Now we analyse the usual sufficient conditions for obtaining a *convergence rate*

$$\|a_\alpha^\delta - a\|_{L^2(0,T)} = \mathrm{O}(\sqrt{\delta}). \tag{32}$$

Using a well known modification of theorem 2.4 in [15] we have the following proposition.

Proposition 5.3. *Under the conditions stated above we obtain for the parameter choice* $\alpha \sim \delta$ *a convergence rate (32) of the Tikhonov regularization if there exists a continuous linear operator*

$$G : L^2(0,T) \to L^2(0,T)$$

with adjoint G^* *and a positive constant L such that*

(i) $\|F(\tilde{a}) - F(a) - G(\tilde{a} - a)\|_{L^2(0,T)} \leqslant \frac{L}{2}\|\tilde{a} - a\|^2_{L^2(0,T)}$ *for all* $\tilde{a} \in D^\dagger(F)$*,*
(ii) there exists a function $w \in L^2(0,T)$ *satisfying* $a - a^* = G^*w$ *and*
(iii) $L\|w\|_{L^2(0,T)} < 1$.

If there exists a continuous linear operator G mapping in $L^2(0,T)$ and satisfying condition (i) in proposition 5.3, then it can be considered as the Fréchet derivative $\tilde{F}'(a)$ at the point a of an operator $\tilde{F}$, for which F is the restriction to the domain $D^\dagger(F)$ with an empty interior in the sense of [14, remark 10.30]. Following the ideas of [23], in particular the strength of requirements (ii) and (iii) yields information about the possibly locally varying ill-posedness character of the IP. If the derivative G at the point $a \in D^\dagger(F)$ exists in the case of equation (31), it is compact as a consequence of the compactness of F (cf [9, p 101]). Then the decay rate of the ordered singular values $\theta_i(G)$ of G to zero as $i \to \infty$ determines the *local degree of ill-posedness* (cf [25, section 3]) of (31) at the point a.

The operator G can be derived as a (formal) Gâteaux derivative by the limits

$$[G(h)](t) = \lim_{\varepsilon \to 0} \frac{[F(a+\varepsilon h)](t) - [F(a)](t)}{\varepsilon}$$

a.e. on $[0,T]$ for $\varepsilon > 0$ and admissible directions $h \in L^2(0,T)$. With $k(t,s) = U_{\mathrm{BS}}(X,K,r,t,s)$ we can write that limit for $0 < t \leqslant T$ as

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \frac{[F(a+\varepsilon h)](t) - [F(a)](t)}{\varepsilon} \\
&\quad = \lim_{\varepsilon \to 0} \frac{k(t, S(t) + \varepsilon[J(h)](t)) - k(t, S(t))}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\frac{\partial k(t, S^\varepsilon_{\mathrm{im}}(t))}{\partial s}\varepsilon[J(h)](t)}{\varepsilon} \\
&\quad = \left(\lim_{\varepsilon \to 0} \frac{\partial k(t, S^\varepsilon_{\mathrm{im}}(t))}{\partial s}\right)[J(h)](t) = \frac{\partial k(t, S(t))}{\partial s}[J(h)](t),
\end{aligned}$$

where $S^{\varepsilon}_{\mathrm{im}}$ is an intermediate function satisfying the inequalities

$$\min(S(t), S(t) + \varepsilon[J(h)](t)) \leqslant S^{\varepsilon}_{\mathrm{im}}(t) \leqslant \max(S(t), S(t) + \varepsilon[J(h)](t)).$$

This limiting process leads to a composition $G = M \circ J$ of the convolution operator J with a *multiplication operator* M described by a multiplier function m in the form

$$[G(h)](t) = m(t)[J(h)](t) \qquad (0 \leqslant t \leqslant T,\ h \in L^2(0, T)). \tag{33}$$

The multiplier function attains the form

$$m(0) = 0, \qquad m(t) = \frac{\partial U_{\mathrm{BS}}(X, K, r, t, S(t))}{\partial s} > 0 \qquad (0 < t \leqslant T) \tag{34}$$

with $S = J(a)$ and we can prove the following.

Theorem 5.4. *In the case $X \neq K$, the linear operator G defined by the formulae (33) and (34) maps continuously in $L^2(0, T)$ with $m \in L^\infty(0, T)$. Then condition (i) of proposition 5.3 is satisfied with a constant*

$$L = TC_2, \qquad \textit{where } C_2 := \sup_{(t,s)\in\mathcal{M}_{\underline{c}}} \left| \frac{\partial^2 U_{\mathrm{BS}}(X, K, r, t, s)}{\partial s^2} \right| < \infty$$

is determined from the set

$$\mathcal{M}_{\underline{c}} := \{(t, s) \in \mathbb{R}^2 : s \geqslant \underline{c}t,\ 0 < t \leqslant T\}.$$

Proof. To prove the continuity of $G = M \circ J$ in $L^2(0, T)$ with the continuous convolution operator J, it is sufficient to show $m \in L^\infty(0, T)$, since then the multiplication operator M is also continuous in $L^2(0, T)$. From formula (6) we obtain for $(t, s) \in [0, T] \times (0, \infty)$ in the case $X \neq K$ the estimate

$$\left| \frac{\partial U_{\mathrm{BS}}(X, K, r, t, s)}{\partial s} \right| \leqslant \sqrt{\frac{XK}{8\pi}} \frac{1}{\sqrt{s}} \exp\left(-\frac{[\ln(\frac{X}{K}) + rt]^2}{2s} \right).$$

This implies for $(t, s) \in \mathcal{M}_{\underline{c}}$

$$\left| \frac{\partial U_{\mathrm{BS}}(X, K, r, t, s)}{\partial s} \right| \leqslant \sqrt{\frac{XK}{8\pi}} \left(\frac{K}{X} \right)^{\frac{r}{\underline{c}}} \frac{1}{\sqrt{s}} \exp\left(-\frac{[\ln(\frac{X}{K})]^2}{2s} \right). \tag{35}$$

The right-hand expression in inequality (35) is continuous with respect to $s \in (0, \infty)$ and tends to zero as $s \to 0$ and as $s \to \infty$. With a finite constant $C_1 := \sup_{(t,s)\in\mathcal{M}_{\underline{c}}} |\frac{\partial U_{\mathrm{BS}}(X,K,r,t,s)}{\partial s}| < \infty$ we have $m \in L^\infty(0, T)$, where $\|m\|_{L^\infty(0,T)} \leqslant C_1$ comes from the inequality $S(t) \geqslant \underline{c}t$ $(0 \leqslant t \leqslant T)$, which is a consequence of $a \in D^\dagger(F)$. In order to prove condition (i) of proposition 5.3 we verify the structure of the second derivative $\frac{\partial^2 U_{\mathrm{BS}}(X,K,r,t,s)}{\partial s^2}$ from formula (7). Similar considerations as in the case of the first derivative also show the existence of a constant $C_2 := \sup_{(t,s)\in\mathcal{M}_{\underline{c}}} |\frac{\partial^2 U_{\mathrm{BS}}(X,K,r,t,s)}{\partial s^2}| < \infty$. Then we can estimate with $S = J(a)$, $\tilde{S} = J(\tilde{a})$ and $a, \tilde{a} \in D^\dagger(F)$ for all $t \in (0, T]$:

$$\begin{aligned} |[F(\tilde{a}) - F(a) - G(\tilde{a} - a)](t)| &= \left| U_{\mathrm{BS}}(X, K, r, t, \tilde{S}(t)) - U_{\mathrm{BS}}(X, K, r, t, S(t)) \right. \\ &\quad \left. - \frac{\partial U_{\mathrm{BS}}(X, K, r, t, S(t))}{\partial s} (\tilde{S}(t) - S(t)) \right| \\ &= \frac{1}{2} \left| \frac{\partial^2 U_{\mathrm{BS}}(X, K, r, t, S_{\mathrm{im}}(t))}{\partial s^2} (\tilde{S}(t) - S(t))^2 \right| \leqslant \frac{C_2}{2} \left(\int_0^t (\tilde{a}(\tau) - a(\tau))\, \mathrm{d}\tau \right)^2, \end{aligned}$$

where $S_{\rm im}$ with $\min(\tilde{S}(t), S(t)) \leqslant S_{\rm im}(t) \leqslant \max(\tilde{S}(t), S(t))$ for $0 < t \leqslant T$ is an intermediate function such that the pairs of real numbers $(t, \tilde{S}(t))$, $(t, S(t))$ and $(t, S_{\rm im}(t))$ all belong to the set $\mathcal{M}_{\underline{c}}$. By applying Schwarz's inequality this provides

$$\|F(\tilde{a}) - F(a) - G(\tilde{a} - a)\|_{L^2(0,T)} \leqslant \frac{TC_2}{2}\|\tilde{a} - a\|^2_{L^2(0,T)}$$

and hence the required condition (i), which proves the theorem. □

For $X \neq K$ the nature of local ill-posedness of (31) at a point $a \in D^\dagger(F)$ arises from two components, namely from the global decay rate of singular values $\theta_i(J) \sim 1/i$ of the linear integral operator J forming the compact part in G and from the local decay rate of $m(t) \to 0$ as t tends to zero of the multiplication operator M as the noncompact part in G. Both components will occur again in the following if we consider the *source condition* (ii) and the *closeness condition* (iii) of proposition 5.3.

In order to interpret the conditions (ii) and (iii) in the case $X \neq K$, we write (ii) as

$$(a - a^*)(t) = \int_t^T m(\tau)w(\tau)\,\mathrm{d}\tau \qquad (0 \leqslant t \leqslant T,\ w \in L^2(0,T)) \tag{36}$$

using the equations $G^* = J^* \circ M^* = J^* \circ M$ and $[J^*(h)](t) = \int_t^T h(\tau)\,\mathrm{d}\tau$ $(0 \leqslant t \leqslant T)$. Formula (36) directly implies

$$(a - a^*)(T) = 0 \qquad \text{and} \qquad \frac{(a - a^*)'}{m} \in L^2(0,T) \tag{37}$$

with a difference $a - a^* \in H^1(0,T)$, for which the generalized derivative belongs to a weighted L^2-space with a weight $\frac{1}{m} \notin L^\infty(0,T)$. The closeness condition (iii) then attains the form

$$\left\|\frac{(a - a^*)'}{m}\right\|_{L^2(0,T)} < \frac{1}{L}. \tag{38}$$

The right-hand condition in (37) and condition (38) express the character of ill-posedness of (31) at the point a as smoothness and smallness requirements on the difference $a - a^*$.

Following the concept of *ill-posedness rates* developed in [24, section 4] for IPs including multiplication operators it should be noted that we have an *exponential growth rate* of $\frac{1}{m(t)} \to \infty$ as $t \to 0$. Based on formula (6) we derive for $X \neq K$

$$\frac{1}{m(t)} = K\sqrt{S(t)}\exp(\psi(t)) \qquad (0 < t \leqslant T)$$

with a constant $K > 0$ and

$$\psi(t) = \frac{\nu^2}{2S(t)} + \frac{r^2t^2}{2S(t)} + \frac{\nu r t}{S(t)} + \frac{\nu}{2} + \frac{rt}{2} + \frac{S(t)}{8}, \qquad \nu := \ln\left(\frac{X}{K}\right) \neq 0.$$

For $S \in I(D^\dagger(F))$ we have $\underline{c}t \leqslant S(t) \leqslant \bar{c}\sqrt{t}$ $(0 \leqslant t \leqslant T)$ with $\bar{c} := \|a\|_{L^2(0,T)}$. This implies for positive constants $\underline{K}$ and $\bar{K}$ the estimates

$$\underline{K}\sqrt{t}\exp\left(\frac{\nu^2}{2\bar{c}\sqrt{t}}\right) \leqslant \frac{1}{m(t)} \leqslant \bar{K}\sqrt[4]{t}\exp\left(\frac{\nu^2}{2\underline{c}t}\right) \qquad (0 < t \leqslant T) \tag{39}$$

below and above. Since, for fixed $\nu \neq 0$, the function $\frac{1}{m(t)}$ exponentially grows to infinity as $t \to 0$, the condition (38) on the difference $a - a^*$ is very rigorous with respect to small t. Formula (39) also shows that for $X - K \to 0$ implying $\nu \to 0$ the norm $\|m\|_{L^\infty(0,T)}$ tends to infinity.

Here we also see that at-the-money options with $X = K$ represent a singular situation in our purely time-dependent model, since we derive from (6) and (7) for $\nu = 0$ the formulae

$\lim_{t\to 0} m(t) = \infty$, $\sup_{(t,s)\in\mathcal{M}_{\underline{c}}} \frac{\partial U_{\mathrm{BS}}(X,K,r,t,s)}{\partial s} = \infty$ and $\sup_{(t,s)\in\mathcal{M}_{\underline{c}}} |\frac{\partial^2 U_{\mathrm{BS}}(X,K,r,t,s)}{\partial s^2}| = \infty$. Hence, the multiplication operator M defined by the formulae (34) fails to be bounded in $L^2(0,T)$ in that case and condition (i) of proposition 5.3 cannot be verified along the lines of the proof of theorem 5.4. This singularity of $X = K$ disappears if a variety of strike prices K is used as done in the sophisticated paper [12] presenting a Tikhonov regularization analysis for the more general IP of option pricing that combines the time- and price-dependent case. We note, however, that the considerations of [12] with H^1-solutions a and data u from a non-Hilbertian Sobolev space do not implicate the L^2-results of this section.

6. The discrete approach and some case studies

Finally, we briefly address the situation where we have option data $u_j^\delta := u^\delta(t_j)$ approximating fair prices $u_j := u(t_j)$ only for a discrete set of maturities $t_0 = 0 < t_1 < t_2 < \cdots < t_k = T$ (for further studies see [20]). We assume according to formula (18)

$$u_0^\delta = \max(X-K,0), \qquad \max(X - K\mathrm{e}^{-rt_j}) < u_j^\delta < X \qquad (j = 1,2,\ldots,k). \tag{40}$$

Using the composition $F = N \circ J$ we will consider a discrete approach for solving the IP. In the first step we determine a vector $\underline{S}^\delta = (S_1^\delta,\ldots,S_k^\delta)^T \in \mathbb{R}_+^k$ of non-negative components by solving the nonlinear equations

$$U_{\mathrm{BS}}(X,K,r,t_j,S_j^\delta) = u_j^\delta \qquad (j = 1,2,\ldots,k). \tag{41}$$

Each of these k equations can be solved by a simple line search algorithm. Since $U_{\mathrm{BS}}(X,K,r,t_j,s)$ is strictly increasing with respect to $s > 0$, due to (6), (9) and (40) all values S_j^δ are uniquely determined from (41). The second step contains a numerical differentiation, which is regularized according to

$$\|\underline{J}\,\underline{a} - \underline{S}^\delta\|_2^2 + \alpha\|\underline{L}\,\underline{a}\|_2^2 \longrightarrow \min, \qquad \text{subject to } \underline{a} \in \mathbb{R}_+^k,$$

with a minimizing vector $\underline{a}^\alpha = (a_1^\alpha,\ldots,a_k^\alpha)^T \in \mathbb{R}_+^k$, where $\alpha > 0$ is the regularization parameter, $\|\cdot\|_2$ denotes the Euclidean norm, $\underline{J}$ is a discretization of the linear Volterra integral operator J and $\|\underline{L}\,\underline{a}\|_2^2$ expresses the usual discretization of the L^2-norm square $\|a''\|_{L^2(0,T)}^2$ of the second derivative of the function a.

For a case study with computer-generated option price data we use the values $X = 0.6$, $K = 0.5$, $r = 0.05$, $T = 1$, $t_j = \frac{j}{k}$ $(j = 1,\ldots,k = 20)$ and the convex function

$$\sigma(t) = (t-0.5)^2 + 0.1 \qquad (0 \leqslant t \leqslant 1).$$

The exact data $\underline{u} = (u_1,\ldots,u_k)^T$ are computed by using the generalized Black–Scholes formula (1)–(4). Perturbed with a random noise vector $\underline{\eta} = (\eta_1,\ldots,\eta_k)^T \in \mathbb{R}^k$ they yield noisy data in the form

$$u_j^\delta = u_j + \delta\frac{\|\underline{u}\|_2}{\|\underline{\eta}\|_2}\eta_j \qquad (j = 1,\ldots,k)$$

for a given relative error $\delta > 0$. Some results of the case study are presented by figures 1 and 2 showing on the one hand the exact solution as a solid curve and on the other hand the linearly interpolated approximate solution as a dashed curve. Figure 1 illustrates the oscillating character of the unregularized volatility reconstruction, although the data error is rather small ($\delta = 0.1\%$). For the same situation a quite good regularized solution is presented in figure 2, where the regularization parameter choice is based on Hansen's L-curve criterion (see [19]). As shown in section 3, arbitrage-free option data u^δ yield in a unique and stable manner nonincreasing functions S^δ. If, however, the noisy discrete option data u_j^δ are not necessarily arbitrage free, then for very small δ the monotonicity may also be lost for values S_j^δ obtained

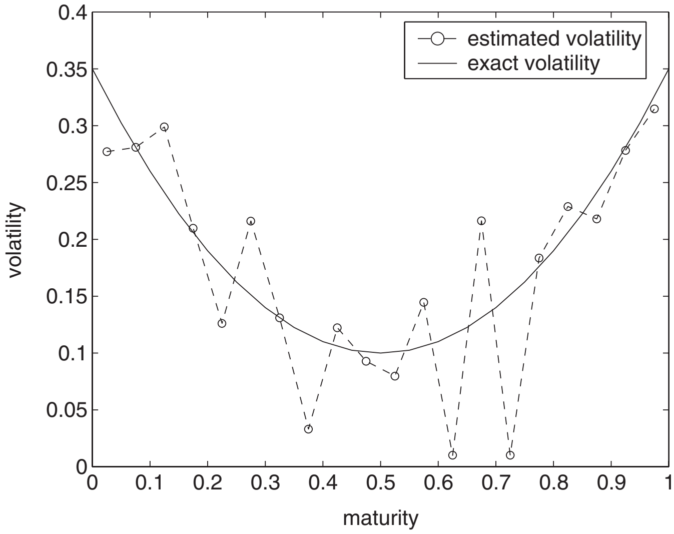

Figure 1. Unregularized solution ($\delta = 0.001$, $\alpha = 0$).

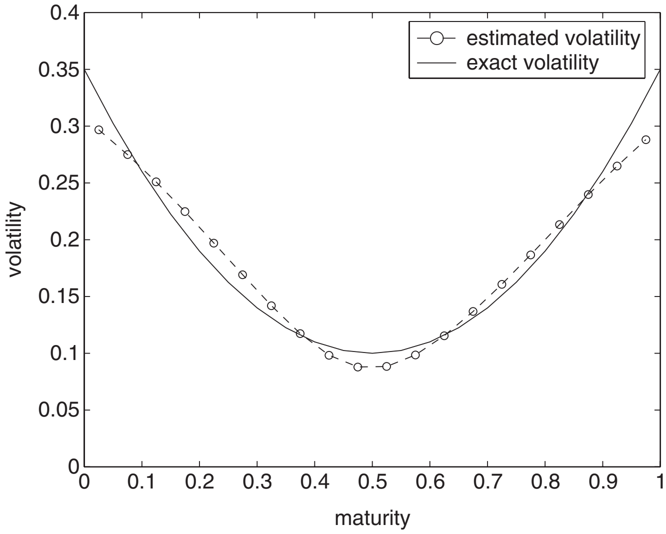

Figure 2. Regularized solution ($\delta = 0.001$, $\alpha = 7.1263 \times 10^{-7}$ from the L-curve method).

by a pointwise inversion of the Nemytskii operator N. In particular, if the remaining time to maturity t_j of the option is small, the corresponding values S_j^δ tend to oscillate (see figure 3). This phenomenon is a consequence of the fact that $S^\delta(t)$ tends to zero for small t. Namely, as shown in figure 4, the error amplification factor $\varphi(t)$ approximated by $\left(\frac{\partial U_{\mathrm{BS}}(X,K,r,t,S(t))}{\partial s}\right)^{-1}$ grows to infinity as t tends to zero.

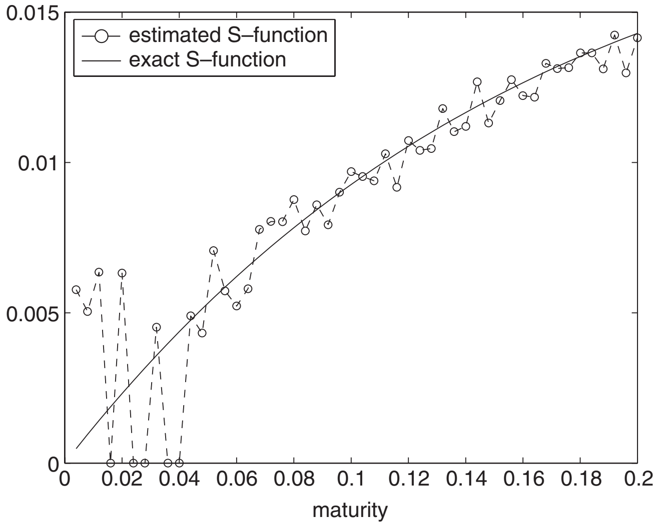

Figure 3. Pointwise reconstruction of $S^\delta(t)$ ($\delta = 0.001$, $k = 50$ grids on [0, 0.2]).

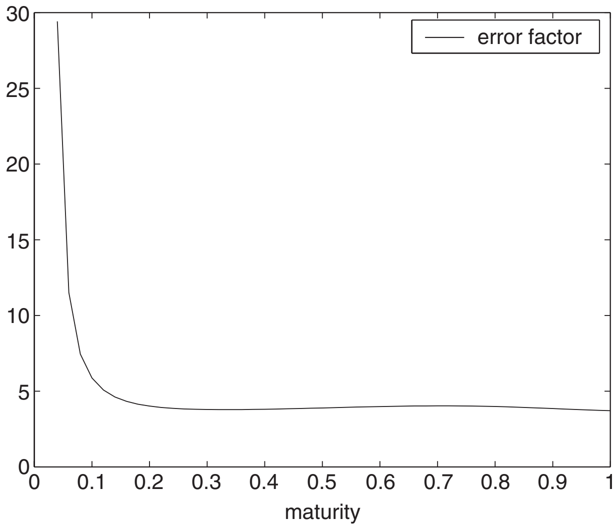

Figure 4. Behaviour of $\left(\frac{\partial U_{\mathrm{BS}}(X,K,r,t,S(t))}{\partial s}\right)^{-1}$ approximating the error factor $\varphi(t)$.

7. Conclusions

By studying the problem of calibrating a time-dependent volatility function from a term-structure of option prices and its ill-posedness phenomena the paper tries to fill a gap in the literature of IPs in option pricing. The explicitly available structure of the forward operator in the purely time-dependent case as a composition of an inner linear convolution operator and an outer nonlinear Nemytskii operator allows us to analyse in detail the occurring ill-posedness phenomena and ways of regularization. For the outer IP treated in a C-space setting the use of arbitrage-free data acts as a specific regularizer. In any case, however, the inner classic deconvolution (differentiation) problem requires an additional regularization. To overcome the local ill-posedness of the complete IP, Tikhonov regularization in L^2 is applicable, convergence rates can be proven and source conditions can be evaluated. It is pointed out that at-the-money options represent a singular situation, in which instability effects occurring for small times in the cases of in-the-money and out-of-the-money options may disappear and properties of the forward operator may degenerate. Although, due to the completely different problem structure, the mathematical analysis used in this paper cannot be generalized to the case of calibrating price-dependent volatility functions, the observed ill-posedness effects also influence the chances of the most important practical problem of fitting the volatility smile as a whole.

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