Indefinite linear matrix pencils and the multi-eigenvalue problem

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Outline

1. Pencils
2. A matrix pencil example
3. Multi-Parametric Eigenvalue Problem

Pencils

Self-adjoint pencils

Let

$$\mathcal{P} = \mathcal{P}(\lambda) := T_0 + \lambda T_1 + \lambda^2 T_2 + \ldots + \lambda^m T_m,$$

be a family of (bounded) operators in a Hilbert space $\mathcal{H}$, which depends on a spectral parameter $\lambda \in \mathbb{C}$, with self-adjoint operator coefficients

$$T_j = (T_j)^*, \qquad j = 1, \ldots, n.$$

Such a family is called a self-adjoint (polynomial) operator pencil.
I shall only deal with a linear self-adjoint operator pencil written as

$$\mathcal{P}(\lambda) = T - \lambda S.$$

Spectrum of a linear pencil

λ_0 is an *eigenvalue* of $\mathcal{P}$ if there exists $x \in \mathcal{H} \setminus \{0\}$ such that

$$\mathcal{P}(\lambda_0)\, x = 0,$$

i.e., if 0 is an eigenvalue of $\mathcal{P}(\lambda_0)$.
The spectrum is the set of values λ_0 for which there is no bounded inverse $\mathcal{P}(\lambda_0)^{-1}$, i.e. if $0 \in \mathrm{Spec}\mathcal{P}(\lambda_0)$.
For a linear pencil, the eigenvalue problem becomes

$$(T - \lambda S)\, x = 0,$$

and if S is invertible, then it reduces to the eigenvalue problem for a (non-self-adjoint) operator $S^{-1}T$;

$$S^{-1}Tx = \lambda x.$$

- If either T or S is sign-definite, then the problem may be reduced to the one for a self-adjoint operator

$$S^{-1/2} T S^{-1/2},$$

and the spectrum is real.

- If, however, both T, S are sign-indefinite, then the spectrum may be non-real.

A matrix pencil example

Example - A matrix pencil [DaLe]

- Fix an integer $n \in \mathbb{N}$, $N = 2n$, and define the $N \times N$ classes of matrices $H_c^{(N)}$ and S, where

$$H_c^{(N)} := \begin{pmatrix} c & 1 & & \\ 1 & c & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & c \end{pmatrix}, \qquad S := \begin{pmatrix} I_n & \\ & -I_n \end{pmatrix}$$

  where $c \in \mathbb{R}$ is a parameter and I_n is the identity matrix.
- The behavior of eigenvalues of the linear operator pencil

$$\mathcal{P} := \mathcal{P}(\lambda) = H_c^{(N)} - \lambda S$$

  as $N \to \infty$ was studied by Davies & Levitin(2014).

Example - A matrix pencil [DaLe]

- If $|c| \geq 2$, then $\mathrm{Spec}\,(\mathcal{P}) \subset \mathbb{R}$.
- $\mathrm{Spec}\,(\mathcal{P})$ is invariant under the symmetry $c \to -c$.
- $\mathrm{Spec}\,(\mathcal{P})$ is symmetric with respect to $\mathrm{Re}\,\lambda = 0$ and $\mathrm{Im}\,\lambda = 0$.
- Davies & Levitin studied the asymptotic behaviour of eigenvalues of $\mathcal{P}$ for large n;
  - For $c = 0$,

$$|\mathrm{Im}\,\lambda| \sim \frac{1}{n} Y_0\,(|\mathrm{Re}\,\lambda|)\,.$$

  - For $0 < c < 2$,

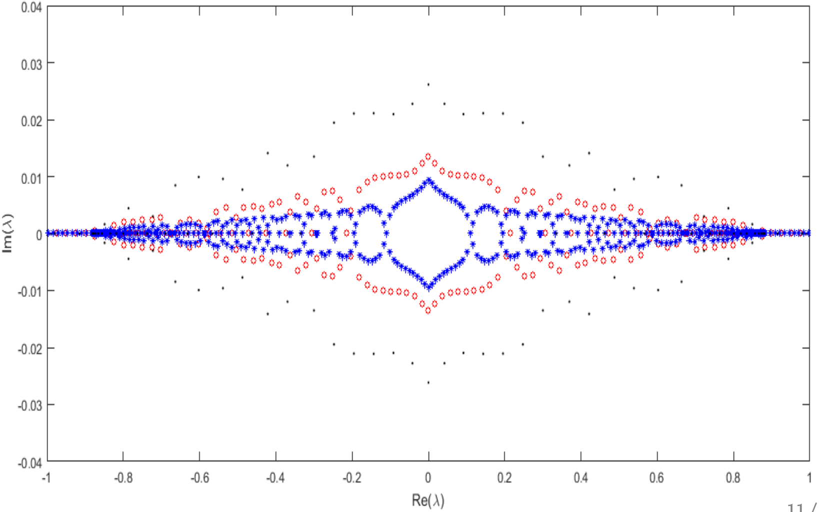
$$|\mathrm{Im}\,\lambda| \lesssim \frac{1}{n} Y_c\,(|\mathrm{Re}\,\lambda|)\,.$$

Functions Y_0 and Y_c are explicit (though rather complicated), and have logarithmic singularities at $\mathrm{Re}\,\lambda = 0$.

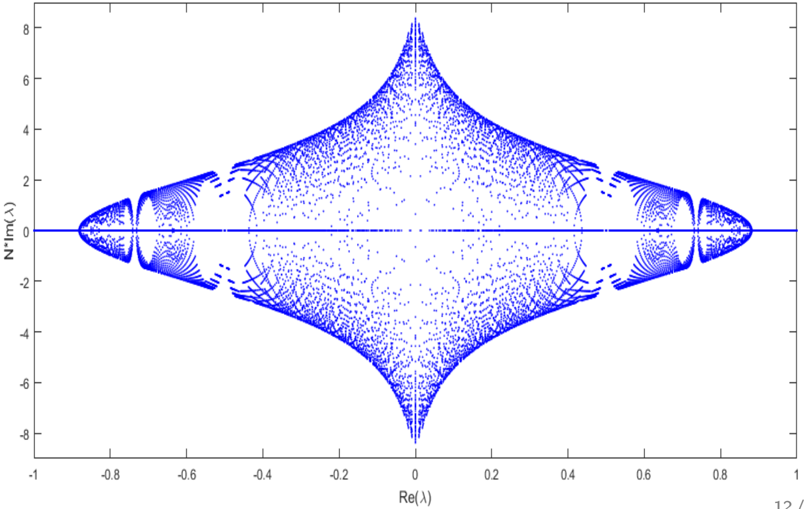
Example - A matrix pencil [DaLe]

- Video time!

$\mathrm{Spec}\,(\mathcal{P})$ for $c = \sqrt{5}/2$ and $n = 500, 250, \mathbf{99}$.



$\cup_{m=100}^{250} \mathrm{Spec}\,(\mathcal{P})$ for $c = \sqrt{5}/2$



Conjecture

Asymptotic and numerical evidence suggest the following:

Let $c > 0$. If $\lambda \in \mathrm{Spec}\left(\mathcal{P}\left(\lambda\right)\right) \setminus \mathbb{R}$, then $c < 2$ and

$$|\lambda \pm c| < 2.$$

This can also be translated in terms of Chebyshev polynomials via explicit expression for $\det\left(H_c^{(N)} - \lambda S\right)$:

Let $\sigma, \tau \in \mathbb{C}$, $\mathrm{Im}\left(\sigma\right) = \mathrm{Im}\left(\tau\right) > 0$. If, for some $n \in \mathbb{N}$,

$$U_{n+1}\left(\sigma\right) U_{n+1}\left(\tau\right) + U_n\left(\sigma\right) U_n\left(\tau\right) = 0,$$

then $|\sigma| < 1$ and $|\tau| < 1$.

Multi-Parametric Eigenvalue Problem

Pencil to Parametric problem

- Recall the pencil $\mathcal{P}$;

$$H_c^{(N)} - \lambda S = \begin{pmatrix} c-\lambda & 1 & & & & \\ 1 & \ddots & \ddots & & & \\ & \ddots & c-\lambda & 1 & & \\ & & 1 & c+\lambda & \ddots & \\ & & & \ddots & \ddots & 1 \\ & & & & 1 & c+\lambda \end{pmatrix}.$$

- Denote

$$\alpha = \lambda - c, \qquad \beta = -\lambda - c.$$

Pencil to Parametric problem

We will act by $\mathcal{P}$ on vectors which we will write as

$$(u_1, \ldots, u_n, v_n, \ldots, v_1)^T .$$

Then

$$\begin{pmatrix} H_0^{(n)} - \alpha I_n & B \\ B & H_0^{(n)} - \beta I_n \end{pmatrix} \begin{pmatrix} \overrightarrow{u} \\ \overrightarrow{v} \end{pmatrix} = \overrightarrow{0},$$

where $B = B^*$ with $B_{nn} = 1$ and all other entries of B are zeros.

Pencil to Parametric problem

We first generalize to the following: for any $\kappa > 0$, let $B = \kappa P$, $P = P^*$, $\|P\| = 1$,

$$\begin{pmatrix} H_0^{(n)} - \alpha I_n & \kappa P \\ \kappa P & H_0^{(n)} - \beta I_n \end{pmatrix} \begin{pmatrix} \overrightarrow{u} \\ \overrightarrow{v} \end{pmatrix} = \overrightarrow{0}. \tag{1}$$

which is a special case of

$$\begin{pmatrix} A - \alpha I_1 & C \\ C^* & D - \beta I_2 \end{pmatrix} \begin{pmatrix} \overrightarrow{u} \\ \overrightarrow{v} \end{pmatrix} = \overrightarrow{0}, \tag{2}$$

where, in general, $\overrightarrow{u} \in \mathcal{H}_1$ and $\overrightarrow{v} \in \mathcal{H}_2$,

- A, D are self-adjoint operators in $\mathcal{H}_1$, $\mathcal{H}_2$, respectively,
- C is a linear operator from $\mathcal{H}_2$ to $\mathcal{H}_1$,
- $\alpha, \beta \in \mathbb{C}$ are spectral parameters.

Two-parameter Matrix Eigenvalue problem

Denote

$$M = \begin{pmatrix} A & C \\ C^* & D \end{pmatrix},$$

so that the problem

$$\left(M - \begin{pmatrix} \alpha I_1 & \\ & \beta I_2 \end{pmatrix}\right)\begin{pmatrix} \vec{u} \\ \vec{v} \end{pmatrix} = \vec{0}, \tag{3}$$

where $\vec{u} \in \mathcal{H}_1$ and $\vec{v} \in \mathcal{H}_2$.

- $(\alpha, \beta) \in \mathbb{C}^2$ a *multi-eigenvalue* (or a *pair-eigenvalue*) of M if there exists a non-trivial solution $\begin{pmatrix} \vec{u} \\ \vec{v} \end{pmatrix} \in \mathcal{H}$ of (3).
- We denote by $\mathrm{Spec}_p(M)$ the *spectrum* of pair-eigenvalues of M.
- If α, $\beta \in \mathbb{R}$, then (α, β) is called as a *real pair-eigenvalue*, and otherwise it is a *non-real pair-eigenvalue* of (3).

$\beta(\alpha)$ problem

- The equation (3) can be re-written as

$$\begin{cases} (A - \alpha I_1)\overrightarrow{u} & = -C\overrightarrow{v}, \\ (D - \beta I_2)\overrightarrow{v} & = -C^*\overrightarrow{u}. \end{cases}$$

- If $\alpha \notin \mathrm{Spec}\,(A)$ and β is an eigenvalue of $\left(D - C^*\,(A - \alpha I_1)^{-1}\,C\right)$, then $(\alpha, \beta) \in \mathrm{Spec}_p\,(M)$.

  Note: α and β are interchangeable.

Restrictions

- Now, suppose that $\mathcal{H}_1, \mathcal{H}_2$ are finite dimensional, and therefore we are dealing with matrices.
- Additionally $\dim \mathcal{H}_1 = \dim \mathcal{H}_2$.
- C has rank 1, or $C = \kappa P$, where $\kappa > 0$ and P is a projection on a one-dimensional subspace $\mathrm{span}\,\{\overrightarrow{\varphi}\}$ of $\mathcal{H}$,

Notations

- The restriction of X on the space of vectors orthogonal to $\overrightarrow{\varphi}$ will be denoted by $X_{\perp,\perp}$.
- Eigenvalues of A and D will be denoted by

$$\begin{aligned} \widetilde{\alpha}_1 &\geq \widetilde{\alpha}_2 \geq \ldots \geq \widetilde{\alpha}_n, \\ \widetilde{\beta}_1 &\geq \widetilde{\beta}_2 \geq \ldots \geq \widetilde{\beta}_n, \end{aligned}$$

  respectively.
- Eigenvalues of $A_{\perp,\perp}$ and $D_{\perp,\perp}$ will be denoted by

$$\begin{aligned} \widehat{\alpha}_1 &\geq \widehat{\alpha}_2 \geq \ldots \geq \widehat{\alpha}_{n-1}, \\ \widehat{\beta}_1 &\geq \widehat{\beta}_2 \geq \ldots \geq \widehat{\beta}_{n-1}, \end{aligned}$$

  respectively.

Remark

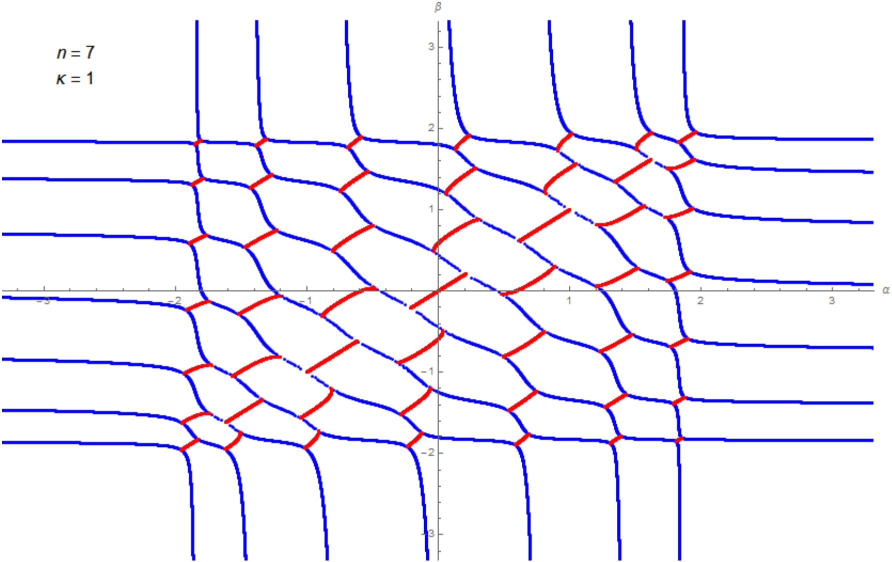
- All numerical examples will be related to

$$\begin{pmatrix} H_0^{(n)} - \alpha I & \kappa P \\ \kappa P & H_0^{(n)} - \beta I \end{pmatrix} \begin{pmatrix} \overrightarrow{u} \\ \overrightarrow{v} \end{pmatrix} = \overrightarrow{0}.$$

- All theoretical results will be related to

$$\begin{pmatrix} A - \alpha I & \kappa P \\ \kappa P & D - \beta I \end{pmatrix} \begin{pmatrix} \overrightarrow{u} \\ \overrightarrow{v} \end{pmatrix} = \overrightarrow{0}.$$

Spectral picture for $n = 7$



Spectral picture for $n = 7$

- Blue curves are all real eigencurves $\beta(\alpha)$, $\alpha \in \mathbb{R}$.
- Red curves are graphs of $\mathrm{Re}\beta(\mathrm{Re}\alpha)$ for eigenpairs such that

$$\mathrm{Im}(\alpha + \beta) = 0,$$

which keeps all $(\alpha, \beta) \in \mathbb{R}^2$ in the picture and some complex pair-eigenvalues.

Lemma

Blue and red lines intersect iff

$$\frac{d}{d\alpha}\beta(\alpha) = -1.$$

Characteristic equation

Theorem

If $\alpha \notin \mathrm{Spec}\,(A)$ *and* $\beta \notin \mathrm{Spec}\,(D)$, *then the characteristic equation for* $(\alpha, \beta) \in \mathrm{Spec}_p\,(M)$ *is*

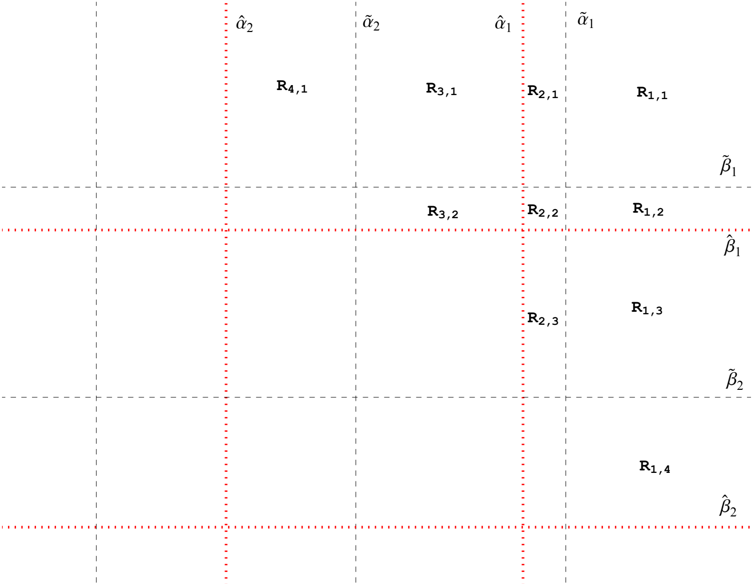
$$\kappa^2 \left\langle (A - \alpha I_n)^{-1} \overrightarrow{\varphi}, \overrightarrow{\varphi} \right\rangle \left\langle (D - \beta I_n)^{-1} \overrightarrow{\varphi}, \overrightarrow{\varphi} \right\rangle = 1, \tag{4}$$

which implies

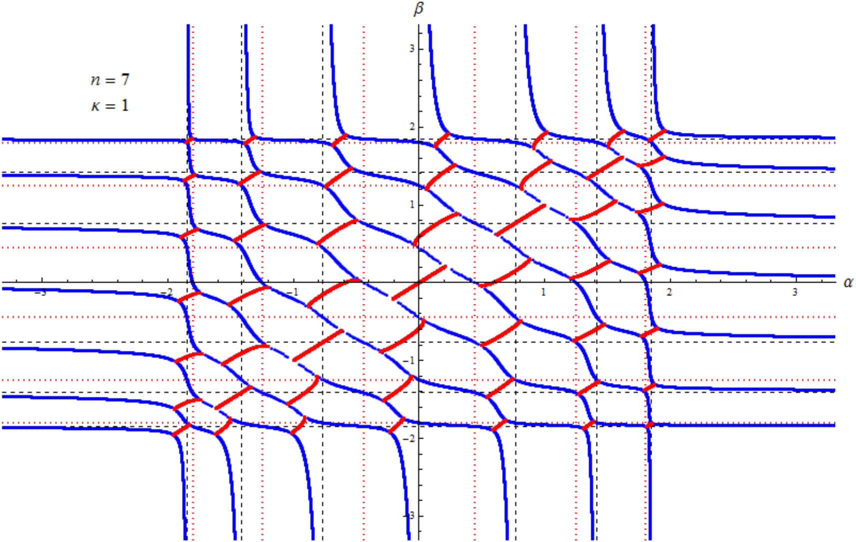
$$\beta' = -\kappa^2 \frac{\left\langle (A - \alpha I_n)^{-2} \overrightarrow{\varphi}, \overrightarrow{\varphi} \right\rangle \left(\left\langle (D - \beta I_n)^{-1} \overrightarrow{\varphi}, \overrightarrow{\varphi} \right\rangle \right)^2}{\left\langle (D - \beta I_n)^{-2} \overrightarrow{\varphi}, \overrightarrow{\varphi} \right\rangle},$$

and therefore $\frac{d\beta}{d\alpha} < 0$ *on each real branch of* $\beta\,(\alpha)$, $\alpha \in \mathbb{R}$.

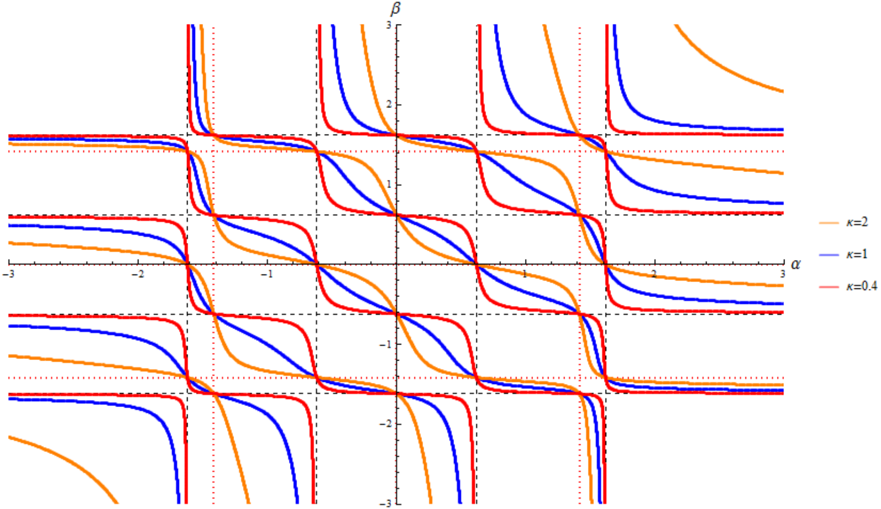
Mesh


$\hat{\alpha}_2$
$\tilde{\alpha}_2$
$\hat{\alpha}_1$
$\tilde{\alpha}_1$
$\mathbf{R_{4,1}}$
$\mathbf{R_{3,1}}$
$\mathbf{R_{2,1}}$
$\mathbf{R_{1,1}}$
$\tilde{\beta}_1$
$\mathbf{R_{3,2}}$
$\mathbf{R_{2,2}}$
$\mathbf{R_{1,2}}$
$\hat{\beta}_1$
$\mathbf{R_{2,3}}$
$\mathbf{R_{1,3}}$
$\tilde{\beta}_2$
$\mathbf{R_{1,4}}$
$\hat{\beta}_2$

Spectral picture for $n = 7$



$n = 4$ and particular values of κ



Chess Board Structure for $n = 6$

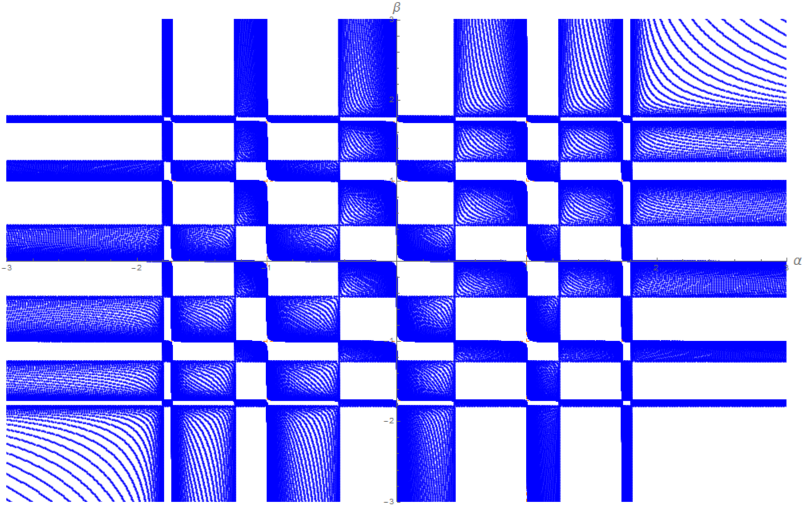


Figure : Superimposing the values of κ from 0.001 to 10 with the step-size of 0.1.

Chess Board Structure

Chess Board Theorem: Suppose that $\mathrm{Spec}\,(A) \bigcap \mathrm{Spec}\,(A_{\perp,\perp}) = \emptyset$ and $\mathrm{Spec}\,(D) \bigcap \mathrm{Spec}\,(D_{\perp,\perp}) = \emptyset$. Then all real pair-eigenvalues (α, β) of M lies in the region $R_{p,q}$ where $p + q$ is even, i.e.

$$(\alpha, \beta) \subset \mathbb{R}^2 \qquad \Rightarrow \qquad (\alpha, \beta) \in R_{p,q}.$$

When $\alpha \in \mathrm{Spec}\,(A)$

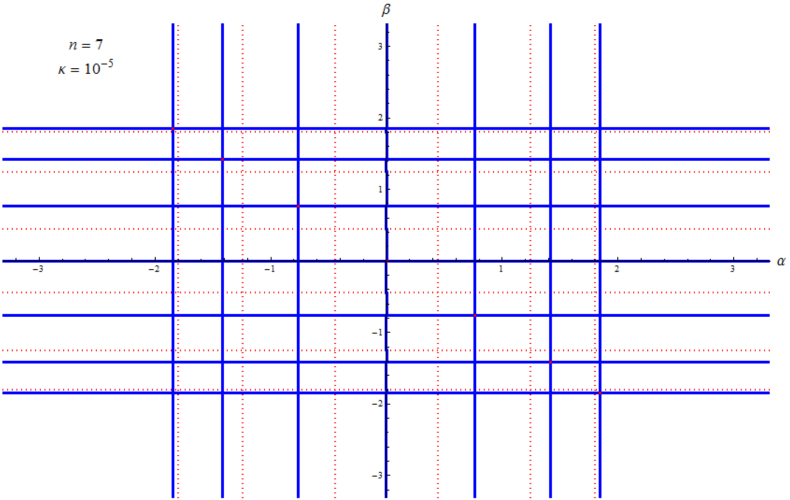
Lemma

Suppose $\alpha = \widetilde{\alpha}_i \in \mathrm{Spec}\,(A)$, $i = 1, \ldots, n$, *and let* $\overrightarrow{\psi_i}$ *be an eigenfunction corresponding to the eigenvalue* $\widetilde{\alpha}_i$ *of* A. *Assume that* $\left\langle \overrightarrow{\varphi}, \overrightarrow{\psi_i} \right\rangle \neq 0$. *Then, for any* $\kappa \in \mathbb{R} \setminus \{0\}$,

$$\begin{array}{ccc} (\alpha, \beta(\alpha)) & \in & \mathrm{Spec}_p\,(M) \\ & \Updownarrow & \\ \beta(\alpha) & \in & \mathrm{Spec}\,(D_{\perp\perp}), \end{array}$$

and additionally for $\alpha \approx \widetilde{\alpha}_i$, *there exists one* $(\alpha, \beta(\alpha)) \in \mathrm{Spec}_p\,(M)$ *such that* $\beta(\alpha) \to \pm\infty$ *as* $\alpha \to \widetilde{\alpha}_i^{\pm}$.

As $\kappa \to 0$



As $\kappa \to 0$

Theorem

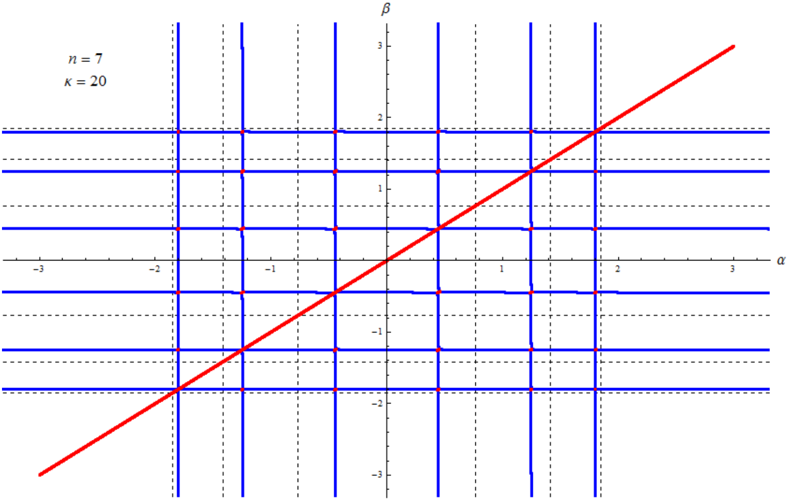
Let $\kappa \to 0$. *Then for every* $i \in \{1, \ldots, n\}$ *and every* $\beta \in \mathbb{R}$, *there exists a sequence* $\left\{(\alpha_k, \beta_k) \in \mathrm{Spec}_p(M)\right\}_k$ *such that*

$$\alpha_k \to \widetilde{\alpha}_i, \quad \beta_k \to \beta,$$

similarly for every $i \in \{1, \ldots, n\}$ *and every* $\alpha \in \mathbb{R}$, *there exists a sequence* $\left\{(\alpha_k, \beta_k) \in \mathrm{Spec}_p(M)\right\}_k$ *such that*

$$\alpha_k \to \alpha, \quad \beta_k \to \widetilde{\beta}_i.$$

As $\kappa \to +\infty$



As $\kappa \to +\infty$

Theorem

Let $\kappa \to \infty$*. Then for every* $j \in \{1, \dots, n-1\}$ *and every* $\beta \in \mathbb{R}$*, there exists a sequence* $\left\{(\alpha_k, \beta_k) \in \mathrm{Spec}_p\,(M)\right\}_k$ *such that*

$$\alpha_k \to \widehat{\alpha}_j, \quad \beta_k \to \beta,$$

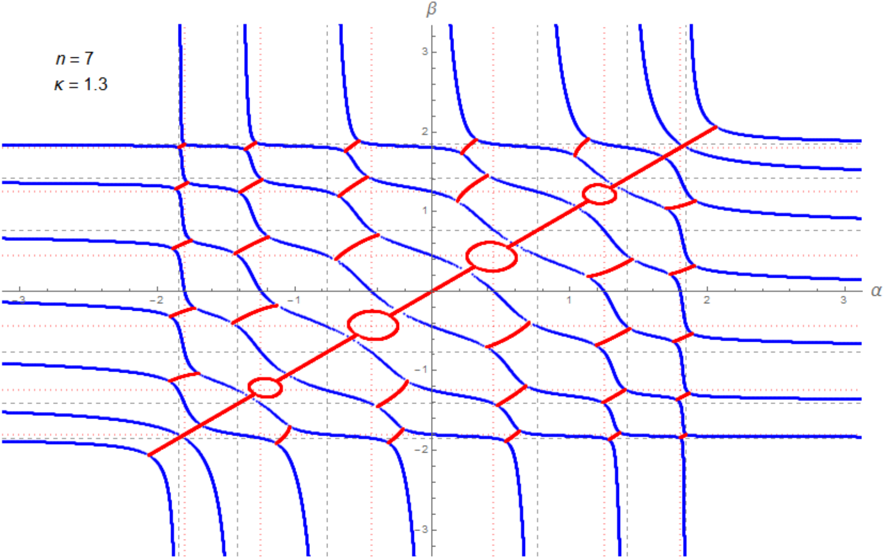
similarly for every $j \in \{1, \dots, n-1\}$ *and every* $\alpha \in \mathbb{R}$*, there exists a sequence* $\left\{(\alpha_k, \beta_k) \in \mathrm{Spec}_p\,(M)\right\}_k$ *such that*

$$\alpha_k \to \alpha, \quad \beta_k \to \widehat{\beta}_j.$$

In addition, there exists one non-real family of pair-eigenvalue (α, β) *of* M *such that*

$$\alpha = \overline{\beta}.$$

Modified problem: Non-real collisions



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