# Scalar and Vector Optimization with Composed Objective Functions and Constraints

Nicole Lorenz *  Gert Wanka †

**Abstract.** In this paper we consider scalar and vector optimization problems with objective functions being the composition of a convex function and a linear mapping and cone and geometric constraints. By means of duality theory we derive dual problems and formulate weak, strong and converse duality theorems for the scalar and vector optimization problems with the help of some generalized interior point regularity conditions and consider optimality conditions for a certain scalar problem.

**Keywords.** Duality, interior point regularity condition, optimality conditions.

**AMS subject classification.** 46N10, 49N15

## 1 Introduction

To a certain multiobjective optimization problem one can attach a scalar one whose optimal solution leads to solutions of the original problem. Different scalarization methods, especially *linear scalarization*, can be used to this purpose. Weak and strong duality results and required regularity conditions of the scalar and vector problem are associated with them. In the book of Boţ, Grad and Wanka (cf. [1]) a broad variety of scalar and vector optimization problems is considered. Related to the investigations within that book we consider here some different scalar and vector optimization problems associated with each other and show how the duals, weak and strong duality and some regularity conditions can be derived.

We assume $\mathcal{X}, \mathcal{Y}, \mathcal{V}$ and $\mathcal{Z}$ to be Hausdorff locally convex spaces, whereas in order to guarantee strong duality some of the regularity conditions contain the assumption that we have Fréchet spaces.

We consider the scalar optimization problem

$$(PS^{\Sigma}) \qquad \inf_{x \in \mathcal{A}} \left\{ \sum_{i=1}^{m} \lambda_i f_i(Ax) \right\}, \quad \mathcal{A} = \{x \in S : g_i(x) \leq 0, i = 1, \ldots, k\},$$

taking proper and convex functions $f_i : \mathcal{Y} \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}, i = 1, \ldots, m$, weighted by positive constants $\lambda_i, i = 1, \ldots, m$, further $g = (g_1, \ldots, g_k)^T : \mathcal{X} \to \mathbb{R}^k$, where $g_i, i = 1, \ldots, k$, is assumed to be convex, $S \subseteq \mathcal{X}$ is a non-empty convex set and $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, i.e. a linear continuous operator mapping from $\mathcal{X}$ to $\mathcal{Y}$. Another problem is the scalar one

$$(PS) \qquad \inf_{x \in \mathcal{A}} f(Ax), \quad \mathcal{A} = \{x \in S : g(x) \in -C\},$$

which is related to the first one. Here we use the proper and convex function $f : \mathcal{Y} \to \overline{\mathbb{R}}$ and the $C$-convex function $g : \mathcal{X} \to \mathcal{Z}$ and a nontrivial convex cone $C \subseteq \mathcal{Z}$.

*Faculty of Mathematics, Chemnitz University of Technology, D-09107 Chemnitz, Germany, e-mail: nicole.lorenz@mathematik.tu-chemnitz.de.

†Faculty of Mathematics, Chemnitz University of Technology, D-09107 Chemnitz, Germany, e-mail: gert.wanka@mathematik.tu-chemnitz.de

Further we consider two vector optimization problems to which scalar ones may be attached, whose dual problems are used to formulate duals to the vector optimization problems. This can be seen in the following sections.

For the space $\mathcal{X}$ partially ordered by the convex cone $K$ we denote by $\mathcal{X}^\bullet$ the space to which a greatest element $+\infty_K$ is attached (cf. [1]).

We consider the following vector optimization problem:

$$(PV^m) \quad \underset{x\in\mathcal{A}}{\mathrm{Min}} (f_1(Ax), \ldots, f_m(Ax))^T, \quad \mathcal{A} = \{x \in S : g_i(x) \leq 0, i = 1, \ldots, k\}.$$

Here we assume $f = (f_1, \ldots, f_m)^T : \mathcal{Y} \to \mathbb{R}^{m\bullet}$ to be a proper function with convex functions $f_i, i = 1, \ldots, m$, and $g_i : \mathcal{X} \to \mathbb{R}, i = 1, \ldots, k$, to be convex. Further we have $S \subseteq \mathcal{X}$. The problem $(PS^\Sigma)$ arises by linear scalarization of $(PV^m)$. Further we consider the following vector optimization problem related to the above one:

$$(PV) \qquad \underset{x\in\mathcal{A}}{\mathrm{Min}} f(Ax), \qquad \mathcal{A} = \{x \in S : g(x) \in -C\}.$$

Here $f : \mathcal{Y} \to \mathcal{V}^\bullet$ is a proper and $K$-convex function and $g : \mathcal{X} \to \mathcal{Z}$ is a $C$-convex function, using the nontrivial pointed convex cone $K \subseteq \mathcal{V}$ and the nontrivial convex cone $C \subseteq \mathcal{Z}$.

The conjugate dual problems to the scalar and vector optimization problem arise as a combination of the classical Fenchel and Lagrange duality. It is the so-called Fenchel-Lagrange duality introduced by Boţ and Wanka (cf. [2], [3], [10]).

For the primal-dual pair one has *weak duality*, where the values of the dual objective function at its feasible set do not surpass the values of the primal objective function at its feasible set. Further, for scalar optimization problems we have *strong duality* if there exists a solution of the dual problem such that the objective values coincide, whereas for vectorial ones in case of *strong duality* we assume the existence of solutions of the primal and dual problem such that the objective values coincide, and for *converse duality* we start with a solution of the dual and prove the existence of a primal solution such that the objective values coincide.

In order to have strong and converse duality we have to formulate regularity conditions. Since the classical Slater constraint qualifications (cf. [5] and [9]) are often not fulfilled, we will present generalized interior point regularity conditions, which are due to Rockafellar (cf. [9]). Conditions for some dual problems were given by Boţ, Grad and Wanka (cf. [1]). Thus we modify these conditions and resulting theorems to adopt them to the problems we study in this paper. Further, in [11] also some vector optimization problems and their duals having a composition in the objective function and the constraints were considered.

The central aim of this paper is to give an overview of special scalar and vector optimization problems. In addition, we point out the connections between them as well as the arising interior point regularity conditions.

The paper is organized as follows. In the following section we introduce some definitions and notations from the convex analysis we use within the paper. In Sect. 3 we consider two general scalar optimization problem, calculate the dual ones, give regularity conditions, further formulate weak and strong duality theorems and give optimality conditions for one of them. Moreover, we consider two vector optimization problems and also calculate the dual ones and formulate weak, strong and converse duality theorems, respectively.

## 2 Notations and Preliminaries

Let $\mathcal{X}$ be a Hausdorff locally convex space and $\mathcal{X}^*$ its topological dual space which we endow with the weak$^*$ topology $w(\mathcal{X}^*, \mathcal{X})$. We denote by $\langle x^*, x\rangle := x^*(x)$ the value of the linear continuous functional $x^* \in \mathcal{X}^*$ at $x \in \mathcal{X}$. For $\mathcal{X} = \mathbb{R}^n$ we have $\mathcal{X} = \mathcal{X}^*$ and for $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n, x^* = (x_1^*, \ldots, x_n^*)^T \in \mathbb{R}^n$ it holds $\langle x^*, x\rangle = (x^*)^T x = \sum_{i=1}^n x_i^* x_i$.

For $f:\mathcal{X}\to\mathcal{V}$ and $v^*\in\mathcal{V}^*$ we define the function $v^*f:\mathcal{X}\to\mathbb{R}$ by $v^*f(x):=\langle v^*,f(x)\rangle$ for $x\in\mathcal{X}$, where $\mathcal{V}$ is another Hausdorff locally convex space and $\mathcal{V}^*$ its topological dual space.

The zero vector will be denoted by $\mathbf{0}$, whereas the space we talk about will be clear from the context. By $e$ we denote the vector $(1,\dots,1)^T$.

For a set $D\subseteq\mathcal{X}$ the *indicator function* $\delta_D:\mathcal{X}\to\overline{\mathbb{R}}$ is defined by

$$\delta_D(x):=\begin{cases}0, & x\in D,\\ +\infty, & \text{otherwise.}\end{cases}$$

When $D\subseteq\mathcal{X}$ is non-empty and $f:\mathcal{X}\to\overline{\mathbb{R}}$ we denote by $f^*_D:\mathcal{X}^*\to\overline{\mathbb{R}}$ the function defined by

$$f^*_D(x^*)=(f+\delta_D)^*(x^*)=\sup_{x\in D}\{\langle x^*,x\rangle-f(x)\}.$$

One can see that for $D=\mathcal{X}$, $f^*_D$ becomes the *(Fenchel-Moreau) conjugate function of* $f$ which we denote by $f^*$. We have the so-called *Young* or *Young-Fenchel inequality*:

$$f(x)+f^*(x^*)\geq\langle x^*,x\rangle\,,\qquad\forall x\in\mathcal{X},\forall x^*\in\mathcal{X}^*. \tag{1}$$

The *support function* $\sigma_D:\mathcal{X}^*\to\overline{\mathbb{R}}$ is defined by $\sigma_D(x^*)=\sup_{x\in D}\langle x^*,x\rangle$ and it holds $\sigma_D=\delta^*_D$.

Let $K\subseteq\mathcal{X}$ be a nontrivial convex cone. The cone $K$ induces on $\mathcal{X}$ a partial ordering $\leqq_K$ defined for $x,y\in\mathcal{X}$ by $x\leqq_K y\Leftrightarrow y-x\in K$. Moreover, let us define $x\leq_K y$ if and only if $x\leqq_K y$ and $x\neq y$. The *dual cone* $K^*\subseteq\mathcal{X}^*$ and the *quasi interior of the dual cone* of $K^*$, respectively, are defined by

$$K^*:=\{x^*\in\mathcal{X}^*:\ \langle x^*,x\rangle\geq 0,\ \forall x\in K\},$$
$$K^{*0}:=\{x^*\in K^*:\ \langle x^*,x\rangle>0,\ \forall x\in K\setminus\{\mathbf{0}\}\}.$$

A convex cone $K$ is said to be *pointed* if its *linearity space* $l(K)=K\cap(-K)$ is the set $\{\mathbf{0}\}$. For a set $U\subseteq\mathcal{X}$ the *conic hull* is

$$\operatorname{cone}(U)=\bigcup_{\lambda\geq 0}\lambda U=\{\lambda u:u\in U,\lambda\geq 0\}.$$

If we assume that $\mathcal{X}$ is partially ordered by the convex cone $K$, we denote by $+\infty_K$ the *greatest element with respect to* $\leqq_K$ and by $\mathcal{X}^\bullet$ the set $\mathcal{X}\cup\{+\infty_K\}$. For any $x\in\mathcal{X}^\bullet$ it holds $x\leqq_K+\infty_K$ and $x\leq_K+\infty_K$ for any $x\in\mathcal{X}$. On $\mathcal{X}^\bullet$ we consider the following operations and conventions (cf. [1]): $x+(+\infty_K)=(+\infty_K)+x:=+\infty_K,\forall x\in\mathcal{X}\cup\{+\infty_K\},\lambda\cdot(+\infty_K):=+\infty_K,\forall\lambda\in(0,+\infty],0\cdot(+\infty_K):=+\infty_K$. Note that we define $+\infty_{\mathbb{R}_+}=:+\infty$ and further $\leqq_{\mathbb{R}_+}=:\leq$ and $\leq_{\mathbb{R}_+}=:<$.

By $B_{\mathcal{X}}(x,r)$ we denote the *open ball with radius* $r>0$ *and center* $x$ in $\mathcal{X}$ defined by $B_{\mathcal{X}}(x,r)=\{y\in\mathcal{X}:d(x,y)<r\}$, where $d:\mathcal{X}\times\mathcal{X}\to\mathbb{R}$ is the metric induced by the topology in $\mathcal{X}$ if $\mathcal{X}$ is metrizable.

The prefixes int, ri, icr, sqri and core are used for the *interior*, the *relative interior*, the *relative algebraic interior (or intrinsic core)*, the *strong quasi relative interior* and the *algebraic interior* or *core* of a set $U\subseteq\mathcal{X}$, respectively, where

$$\begin{aligned}
\operatorname{core}(U)&=\{x\in\mathcal{X}:\forall y\in\mathcal{X},\ \exists\delta>0\text{ such that }\forall\lambda\in[0,\delta]:x+\lambda y\in U\},\\
\operatorname{ri}(U)&=\{x\in\operatorname{aff}(U):\exists\varepsilon>0:B_{\mathcal{X}}(x,\varepsilon)\cap\operatorname{aff}(U)\subseteq U\},\\
\operatorname{icr}(U)&=\{x\in\mathcal{X}:\forall y\in\operatorname{aff}(U-U),\ \exists\delta>0\text{ s.t. }\ \forall\lambda\in[0,\delta]:x+\lambda y\in U\},\\
\operatorname{sqri}(U)&=\begin{cases}\operatorname{icr}(U), & \text{if }\operatorname{aff}(U)\text{is a closed set},\\ \emptyset, & \text{otherwise,}\end{cases}
\end{aligned}$$

and in case of having a convex set $U\subseteq\mathcal{X}$ we have

$$\operatorname{core}(U)=\{x\in U:\operatorname{cone}(U-x)=\mathcal{X}\},$$
$$\operatorname{sqri}(U)=\{x\in U:\operatorname{cone}(U-x)\text{ is a closed linear subspace}\}.$$

It holds $\operatorname{core}(U) \subseteq \operatorname{sqri}(U)$ and $\operatorname{aff}(U)$ is the *affine hull* of the set $U$,

$$\operatorname{aff}(U) = \left\{ \sum_{i=1}^{n} \lambda_i x_i : n \in \mathbb{N}, x_i \in U, \lambda_i \in \mathbb{R}, \sum_{i=1}^{n} \lambda_i = 1, i = 1, \ldots, n \right\}.$$

We assume $\mathcal{V}$ to be a Hausdorff locally convex space partially ordered by the nontrivial convex cone $C \subseteq \mathcal{V}$.

The *effective domain* of a function $f : \mathcal{X} \to \overline{\mathbb{R}}$ is $\operatorname{dom}(f) = \{x \in \mathcal{X} : f(x) < +\infty\}$ and we will say that $f$ is *proper* if $\operatorname{dom}(f) \neq \emptyset$ and $f(x) > -\infty$, $\forall x \in \mathcal{X}$. The *domain* of a vector function $f : \mathcal{X} \to \mathcal{V}^\bullet$ is $\operatorname{dom}(f) = \{x \in \mathcal{X} : f(x) \neq +\infty_C\}$. When $\operatorname{dom}(f) \neq \emptyset$ the vector function $f$ is called *proper*.

While a proper function $f : \mathcal{X} \to \overline{\mathbb{R}}$ is called *convex* if for all $x, y \in \mathcal{X}$ and all $\lambda \in [0,1]$ it holds $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$, a vector function $f : \mathcal{X} \to \mathcal{V}^\bullet$ is said to be *$C$-convex* if for all $x, y \in \mathcal{X}$ and all $\lambda \in [0,1]$ it holds $f(\lambda x + (1-\lambda)y) \leqq_C \lambda f(x) + (1-\lambda) f(y)$ (cf. [1]).

A function $f : \mathcal{X} \to \overline{\mathbb{R}}$ is called *lower semicontinuous at* $\overline{x} \in \mathcal{X}$ if $\liminf_{x \to \overline{x}} f(x) \geq f(\overline{x})$, while a function $f : \mathcal{X} \to \mathcal{V}^\bullet$ is *star $C$-lower semicontinuous at* $\overline{x} \in \mathcal{X}$ if $(v^* f)$ is lower semicontinuous at $\overline{x}$ for all $v^* \in C^*$. The latter notion was first given in [6].

For $f : \mathcal{X} \to \overline{\mathbb{R}}$ and $a \in \mathbb{R}$ we call $\operatorname{lev}_a(f) := \{x \in \mathcal{X} : f(x) \leq a\}$ the *level set* of $f$ at $a$.

By $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ we denote the set of linear continuous operators mapping from $\mathcal{X}$ into $\mathcal{Y}$. For $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ one can define the *adjoint operator*, $A^* : \mathcal{Y}^* \to \mathcal{X}^*$ by

$$\langle A^* y^*, x \rangle = \langle y^*, Ax \rangle, \quad \forall y^* \in \mathcal{Y}^*, x \in \mathcal{X}.$$

In the following we write min and max instead of inf and sup if we want to express that the infimum/supremum of a scalar optimization problem is attained.

**Definition 2.1** (infimal convolution). *For the proper functions $f_1, \ldots, f_k : \mathcal{X} \to \overline{\mathbb{R}}$, the function $f_1 \square \cdots \square f_k : \mathcal{X} \to \overline{\mathbb{R}}$ defined by*

$$(f_1 \square \cdots \square f_k)(p) = \inf \left\{ \sum_{i=1}^{k} f_i(p_i) : \sum_{i=1}^{k} p_i = p \right\}$$

*is called the infimal convolution of $f_i, i = 1, \ldots, k$.*

In order to state a theorem for the infimal convolution of conjugate functions we introduce additionally to a classical condition $(RC_1^\Sigma)$ the following generalized interior point regularity conditions $(RC_i^\Sigma), i \in \{2,3,4\}$:

$$(RC_1^\Sigma) \quad \left| \begin{array}{l} \exists x' \in \cap_{i=1}^k \operatorname{dom}(f_i) \text{ such that a number of } k-1 \text{ functions} \\ \text{of the functions } f_i, i = 1, \ldots, k, \text{ are continuous at } x', \end{array} \right. \tag{2}$$

$$(RC_2^\Sigma) \quad \left| \begin{array}{l} \mathcal{X} \text{ is Fréchet space}, f_i \text{ is lower semicontinuous}, i = 1, \ldots, k, \\ \text{and } \mathbf{0} \in \operatorname{sqri}\left( \Pi_{i=1}^k \operatorname{dom}(f_i) - \Delta_{\mathcal{X}^k} \right), \end{array} \right. \tag{3}$$

$$(RC_3^\Sigma) \quad \left| \begin{array}{l} \mathcal{X} \text{ is Fréchet space}, f_i \text{ is lower semicontinuous}, i = 1, \ldots, k, \text{ and} \\ \mathbf{0} \in \operatorname{core}\left( \Pi_{i=1}^k \operatorname{dom}(f_i) - \Delta_{\mathcal{X}^k} \right), \end{array} \right.$$

$$(RC_4^\Sigma) \quad \left| \begin{array}{l} \mathcal{X} \text{ is Fréchet space}, f_i \text{ is lower semicontinuous}, i = 1, \ldots, k, \text{ and} \\ \mathbf{0} \in \operatorname{int}\left( \Pi_{i=1}^k \operatorname{dom}(f_i) - \Delta_{\mathcal{X}^k} \right), \end{array} \right.$$

where for a set $M \subseteq \mathcal{X}$ we define $\Delta_{M^k} := \{(x, \ldots, x) \in \mathcal{X}^k : x \in M\}$. The following theorem holds (cf. [1, Theorem 3.5.8]):

**Theorem 2.2.** *Let $f_1, \dots, f_k : \mathcal{X} \to \overline{\mathbb{R}}$ be proper and convex functions. If one of the regularity conditions $(RC_i^\Sigma), i \in \{1,2,3,4\}$, is fulfilled, then it holds for all $p \in \mathcal{X}^*$*

$$\left(\sum_{i=1}^k f_i\right)^*(p) = (f_1^* \Box \cdots \Box f_k^*)(p) = \min\left\{\sum_{i=1}^k f_i^*(p_i) : \sum_{i=1}^k p_i = p\right\}. \tag{4}$$

**Remark 2.3.** *For $\mathcal{X} = \mathbb{R}^n$ formula (4) holds if $f_i, i = 1, \dots, k$, is proper and convex and $\cap_{i=1}^k \operatorname{ri}(\operatorname{dom}(f_i)) \neq \emptyset$, i.e. we do not need one of the conditions $(RC_i^\Sigma), i \in \{1,2,3,4\}$ (cf. [8, Theorem 20.1]).*

The function $f : \mathcal{X} \to \mathcal{V}^\bullet$ is called $C$-*epi closed* if its $C$-*epigraph*, namely $\operatorname{epi}_C f = \{(x,y) \in \mathcal{X} \times \mathcal{V} : f(x) \leqq_C y\}$, is a closed set (cf. [7]). For a real valued function $f : \mathcal{X} \to \overline{\mathbb{R}}$ and $C = \mathbb{R}_+$ we have $\operatorname{epi} f = \operatorname{epi}_C f$ and the following theorem holds (cf. [1, Theorem 2.2.9]):

**Theorem 2.4.** *Let the function $f : \mathcal{X} \to \overline{\mathbb{R}}$ be convex. Then the following statements are equivalent:*

$(i)$ *$f$ is lower semicontinuous,*
$(ii)$ *$\operatorname{epi} f$ is closed,*
$(iii)$ *the level set $\operatorname{lev}_a(f) = \{x \in \mathcal{X} : f(x) \leq a\}$ is closed for all $a \in \mathbb{R}$.*

# 3 Some Dual Optimization Problems

In this section we consider the optimization problems $(PS)$ and $(PV)$. These are related problems, the first one a scalar, the latter one a vectorial, having as objective function a composition of a convex (vector) function and a linear continuous operator and cone and geometric constraints. For these we formulate dual problems and state weak, strong and converse duality theorems under some classical and generalized interior point regularity conditions. Further, we consider two problems $(PS^\Sigma)$ and $(PV^m)$ related to the above ones and derive the same things.

For the whole section we assume that $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ and $\mathcal{V}$ are Hausdorff locally convex spaces, $\mathcal{Z}$ and $\mathcal{V}$ are assumed to be partially ordered by the nontrivial convex cone $C \subseteq \mathcal{Z}$ and the nontrivial pointed convex cone $K \subseteq \mathcal{V}$, respectively. Further, let $S \subseteq \mathcal{X}$ be a non-empty convex set and $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$.

## 3.1 The Scalar Optimization Problem $(PS)$

In this first subsection we consider a general scalar optimization problem. Therefore we assume the function $f : \mathcal{Y} \to \overline{\mathbb{R}}$ to be proper and convex and the vector function $g : \mathcal{X} \to \mathcal{Z}$ to be $C$-convex, fulfilling $A^{-1}(\operatorname{dom}(f)) \cap g^{-1}(-C) \cap S \neq \emptyset$. Consider the following primal scalar optimization problem:

$$(PS) \quad \inf_{x \in \mathcal{A}} f(Ax), \quad \mathcal{A} = \{x \in S : g(x) \in -C\}.$$

We derive here a dual problem which is called the Fenchel-Lagrange dual problem to $(PS)$. For this purpose we consider the perturbation function $\Phi_{FL} : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \overline{\mathbb{R}}$, given by

$$\Phi_{FL}(x,y,z) = \begin{cases} f(Ax+y), & x \in S, g(x) \in z - C, \\ +\infty, & \text{otherwise}, \end{cases} \tag{5}$$

where $\mathcal{X}$ is the space of feasible variables and $\mathcal{Y}$ and $\mathcal{Z}$ are the spaces of perturbation variables. First we calculate to $\Phi_{FL}$ the conjugate function $(\Phi_{FL})^* : \mathcal{X}^* \times \mathcal{Y}^* \times \mathcal{Z}^* \to \overline{\mathbb{R}}$:

$$
\begin{aligned}
&(\Phi_{FL})^*(x^*, y^*, z^*) \\
&= \sup_{\substack{(x,y,z)\in S\times\mathcal{Y}\times\mathcal{Z} \\ g(x)-z\in -C}} \{\langle x^*, x\rangle + \langle y^*, y\rangle + \langle z^*, z\rangle - f(Ax+y)\} \qquad (6)\\
&= \sup_{(x,r,s)\in S\times\mathcal{Y}\times -C} \{\langle x^*, x\rangle + \langle y^*, r - Ax\rangle + \langle z^*, g(x) - s\rangle - f(r)\} \\
&= \delta_{-C^*}(z^*) + \sup_{(x,r)\in S\times\mathcal{Y}} \{\langle x^* - A^*y^*, x\rangle + \langle y^*, r\rangle + (z^*g)(x) - f(r)\} \\
&= \delta_{-C^*}(z^*) + \sup_{x\in S}\{\langle x^* - A^*y^*, x\rangle + (z^*g)(x)\} + \sup_{r\in\mathcal{Y}}\{\langle y^*, r\rangle - f(r)\}. \qquad (7)
\end{aligned}
$$

It follows

$$-(\Phi_{FL})^*(0, y^*, z^*) = -\delta_{-C^*}(z^*) - (-z^*g)^*_S(-A^*y^*) - f^*(y^*).$$

The dual problem becomes (cf. [1] and take $z^* := -z^*$):

$$
\begin{aligned}
(DS_{FL}) \quad &\sup_{(y^*,z^*)\in\mathcal{Y}^*\times\mathcal{Z}^*} (-(\Phi_{FL})^*(0, y^*, z^*)) \\
&= \sup_{(y^*,z^*)\in\mathcal{Y}^*\times\mathcal{Z}^*} (-\delta_{-C^*}(z^*) - (-z^*g)^*_S(-A^*y^*) - f^*(y^*)) \\
&= \sup_{(y^*,z^*)\in\mathcal{Y}^*\times C^*} (-(z^*g)^*_S(-A^*y^*) - f^*(y^*)). \qquad (8)
\end{aligned}
$$

We denote by $v(PS)$ and $v(DS_{FL})$ the *optimal objective value* of $(PS)$ and $(DS_{FL})$, respectively. Then weak duality holds by construction (cf. [1]), i.e. $v(PS) \geq v(DS_{FL})$. In order to have strong duality we introduce some regularity conditions.

For a general optimization problem given by

$$(P) \quad \inf_{x\in\mathcal{X}} \Phi(x, 0),$$

depending on the perturbation function $\Phi : \mathcal{X}\times\mathcal{Y} \to \overline{\mathbb{R}}$ we introduce the following so-called *generalized interior point regularity conditions*, where we assume that $\Phi$ is a proper and convex function fulfilling $0 \in \Pr_{\mathcal{Y}}(\operatorname{dom}(\Phi))$ and $\Pr_{\mathcal{Y}} : \mathcal{X}\times\mathcal{Y} \to \mathcal{Y}$, defined for $(x, y) \in \mathcal{X}\times\mathcal{Y}$ by $\Pr_{\mathcal{Y}}(x, y) = y$, is the *projection operator* on $\mathcal{Y}$. Further, $\mathcal{X}$ is the space of feasible variables and $\mathcal{Y}$ is the space of perturbation variables (cf. [1]). The conditions have the following form:

$$
\begin{array}{l|l}
(RC_1^\Phi) & \exists x' \in \mathcal{X} \text{ such that } (x', 0) \in \operatorname{dom}(\Phi) \text{ and } \Phi(x', \cdot) \text{ is continuous at } 0, \\
(RC_2^\Phi) & \mathcal{X} \text{ and } \mathcal{Y} \text{ are Fréchet spaces, } \Phi \text{ is lower semicontinuous} \\
& \text{and } 0 \in \operatorname{sqri}(\Pr_{\mathcal{Y}}(\operatorname{dom}(\Phi))), \\
(RC_3^\Phi) & \mathcal{X} \text{ and } \mathcal{Y} \text{ are Fréchet spaces, } \Phi \text{ is lower semicontinuous and} \\
& 0 \in \operatorname{core}(\Pr_{\mathcal{Y}}(\operatorname{dom}(\Phi))), \\
(RC_4^\Phi) & \mathcal{X} \text{ and } \mathcal{Y} \text{ are Fréchet spaces, } \Phi \text{ is lower semicontinuous and} \\
& 0 \in \operatorname{int}(\Pr_{\mathcal{Y}}(\operatorname{dom}(\Phi))).
\end{array} \qquad (9)
$$

If $\mathcal{X}$ and $\mathcal{Y}$ are Fréchet spaces and $\Phi$ is lower semicontinuous, it holds

$$(RC_1^\Phi) \Rightarrow (RC_4^\Phi) \Leftrightarrow (RC_3^\Phi) \Rightarrow (RC_2^\Phi), \qquad (10)$$

i.e. the second is the weakest one (see also [1]).

If $(RC_1^\Phi)$ is fulfilled, the condition $0 \in \Pr_{\mathcal{Y}}(\mathrm{dom}(\Phi))$ holds since it is equivalent with $\exists x' \in \mathcal{X} : (x', 0) \in \mathrm{dom}(\Phi)$. If $(RC_i^\Phi), i \in \{2, 3, 4\}$, is fulfilled this obviously also holds since the sqri, core and int of $\Pr_{\mathcal{Y}}(\mathrm{dom}(\Phi))$ are subsets of the set $\Pr_{\mathcal{Y}}(\mathrm{dom}(\Phi))$.

We have to ensure that the perturbation function $\Phi_{FL}$ is proper and convex. The convexity follows by the convexity of $f, g$ and $S$. Further, $\Phi_{FL}$ is proper since $f$ is proper and $A^{-1}(\mathrm{dom}(f)) \cap S \cap g^{-1}(-C) \neq \emptyset$. These properties will be maintained in the following (sub)sections.

For the given perturbation function $\Phi_{FL}$ it holds

$$
\begin{aligned}
& (y, z) \in \Pr_{\mathcal{Y}\times\mathcal{Z}}(\mathrm{dom}(\Phi_{FL})) \\
\Leftrightarrow \quad & \exists x \in \mathcal{X} : \Phi_{FL}(x, y, z) < +\infty \\
\Leftrightarrow \quad & \exists x \in S : Ax + y \in \mathrm{dom}(f), g(x) \in z - C \\
\Leftrightarrow \quad & \exists x \in S : (y, z) \in (\mathrm{dom}(f) - Ax) \times (C + g(x)) \\
\Leftrightarrow \quad & (y, z) \in (\mathrm{dom}(f) \times C) - \bigcup_{x \in S} (Ax, -g(x)) \\
\Leftrightarrow \quad & (y, z) \in (\mathrm{dom}(f) \times C) - (A \times -g)(\Delta_{S^2}).
\end{aligned}
$$

The lower semicontinuity of $\Phi_{FL}$ is equivalent with the closeness of $\mathrm{epi}\,\Phi_{FL}$ (see Theorem 2.4) and it holds

$$
\begin{aligned}
& \mathrm{epi}\,\Phi_{FL} = \{(x, y, z, r) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \times \mathbb{R} : (Ax + y, r) \in \mathrm{epi}\, f\} \\
& \cap \{S \times \mathcal{Y} \times \mathcal{Z} \times \mathbb{R}\} \cap \{(x, y, z, r) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \times \mathbb{R} : (x, z) \in \mathrm{epi}_C\, g\}.
\end{aligned}
$$

The closeness of this set is guaranteed if $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ are Fréchet spaces, $f$ is lower semicontinuous, $S$ is closed and $g$ is $C$-epi closed. The regularity condition $(RC_2^\Phi)$ becomes:

$$
(RC_{2,FL}) \quad \left| \begin{array}{l} \mathcal{X}, \mathcal{Y} \text{ and } \mathcal{Z} \text{ are Fréchet spaces, } f \text{ is lower semi-} \\ \text{continuous, } S \text{ is closed, } g \text{ is } C\text{-epi closed and} \\ \mathbf{0} \in \mathrm{sqri}((\mathrm{dom}(f) \times C) - (A \times -g)(\Delta_{S^2}). \end{array} \right. \tag{11}
$$

Analogously one can rewrite the stronger conditions $(RC_3^\Phi)$ and $(RC_4^\Phi)$ using core and int, respectively, instead of sqri and get $(RC_{3,FL})$ and $(RC_{4,FL})$.

The regularity condition $(RC_1^\Phi)$ becomes under usage of the perturbation function $\Phi_{FL}$ in formula (5):

$$
(RC_{1,FL}) \quad \left| \begin{array}{l} \exists x' \in A^{-1}(\mathrm{dom}(f)) \cap S \text{ such that } f \text{ is continuous at} \\ Ax' \text{ and } g(x') \in -\mathrm{int}(C). \end{array} \right. \tag{12}
$$

We state now the following strong duality theorem:

**Theorem 3.1** (strong duality)**.** *Let the spaces $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$, the cone $C$, the functions $f$ and $g$, the set $S$ and the linear mapping $A$ be assumed as at the beginning of the (sub)section and further $A^{-1}(\mathrm{dom}(f)) \cap g^{-1}(-C) \cap S \neq \emptyset$.*

*If one of the regularity conditions $(RC_{i,FL}), i \in \{1, 2, 3, 4\}$, is fulfilled, then $v(PS) = v(DS_{FL})$ and the dual has an optimal solution.*

**Remark 3.2.** *If the function $f$ is continuous and the primal problem $(PS)$ has a compact feasible set $\mathcal{A}$, then there exists an optimal solution $\overline{x}$ to $(PS)$.*

## 3.2 The Scalar Optimization Problem $(PS^\Sigma)$

A multiobjective optimization problem with objective functions $f_i, i = 1, \ldots, m$, can be handled by weighting the functions and consider the sum of it, which is a linear scalarization. The arising problem is the subject of this subsection. Similar perturbations of the primal problem can be found in [4], where the authors consider an optimization problem having also cone constraints but still a weighted sum of convex functions without the composition with a linear continuous mapping.

Assume the functions $f_i : \mathcal{Y} \to \overline{\mathbb{R}}, i = 1, \ldots, m$, to be proper and convex and $g = (g_1, \ldots, g_k)^T : \mathcal{X} \to \mathbb{R}^k$ to be $C$-convex, $C = \mathbb{R}^k_+$. Further, let $\lambda$ be the fixed vector $\lambda = (\lambda_1, \ldots, \lambda_m)^T \in \operatorname{int}(\mathbb{R}^m_+)$. Let $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $A^{-1}(\bigcap_{i=1}^m \operatorname{dom}(f_i)) \cap g^{-1}(-\mathbb{R}^k_+) \cap S \neq \emptyset$. We consider the scalar optimization problem

$$(PS^\Sigma) \quad \inf_{x \in \mathcal{A}} \left\{ \sum_{i=1}^m \lambda_i f_i(Ax) \right\}, \quad \mathcal{A} = \{x \in S : g_i(x) \leq 0, i = 1, \ldots, k\},$$

and the following perturbation function $\Phi^\Sigma_{FL} : \mathcal{X} \times \mathcal{Y} \times \ldots \times \mathcal{Y} \times \mathcal{X} \times \ldots \times \mathcal{X} \to \overline{\mathbb{R}}$ in order to separate the conjugate functions of $f_i, i = 1, \ldots, m$, and the conjugate functions of $g_i, i = 1, \ldots, k$, in the dual:

$$\Phi^\Sigma_{FL}(x, y^1, \ldots, y^m, z^1, \ldots, z^k) = \begin{cases} \sum\limits_{i=1}^m \lambda_i f_i(Ax + y^i), & x \in S, g_i(x + z^i) \leq 0, i = 1, \ldots, k, \\ +\infty, & \text{otherwise.} \end{cases}$$

The conjugate function $(\Phi^\Sigma_{FL})^* : \mathcal{X}^* \times \mathcal{Y}^* \times \ldots \times \mathcal{Y}^* \times \mathcal{X}^* \times \ldots \times \mathcal{X}^* \to \overline{\mathbb{R}}$ is given by

$$\begin{aligned} &(\Phi^\Sigma_{FL})^*(x^*, y^{1*}, \ldots, y^{m*}, z^{1*}, \ldots, z^{k*}) \\ &= \sup_{\substack{x \in S, \\ y^i \in \mathcal{Y}, i=1,\ldots,m, \\ z^i \in \mathcal{X}, i=1,\ldots,k, \\ g_i(x+z^i) \leq 0, \\ i=1,\ldots,k}} \left\{ \langle x^*, x \rangle + \sum_{i=1}^m \left\langle y^{i*}, y^i \right\rangle + \sum_{i=1}^k \left\langle z^{i*}, z^i \right\rangle - \sum_{i=1}^m \lambda_i f_i(Ax + y^i) \right\}. \end{aligned}$$

By setting $Ax + y^i =: r^i \in \mathcal{Y}, i = 1, \ldots, m$, and $x + z^i =: s^i \in \mathcal{X}, i = 1, \ldots, k$, we get:

$$\begin{aligned} &- (\Phi^\Sigma_{FL})^*(0, y^{1*}, \ldots, y^{m*}, z^{1*}, \ldots, z^{k*}) \\ &= - \sup_{\substack{x \in S, \\ r^i \in \mathcal{Y}, i=1,\ldots,m, \\ s^i \in \mathcal{X}, i=1,\ldots,k, \\ g_i(s^i) \leq 0, i=1,\ldots,k}} \left\{ \sum_{i=1}^m \left\langle y^{i*}, r^i - Ax \right\rangle + \sum_{i=1}^k \left\langle z^{i*}, s^i - x \right\rangle - \sum_{i=1}^m \lambda_i f_i(r^i) \right\} \\ &= - \sup_{x \in S} \left\{ - \sum_{i=1}^m \left\langle y^{i*}, Ax \right\rangle - \sum_{i=1}^k \left\langle z^{i*}, x \right\rangle \right\} - \sum_{i=1}^m \sup_{r^i \in \mathcal{Y}} \left\{ \left\langle y^{i*}, r^i \right\rangle - \lambda_i f_i(r^i) \right\} \\ &\quad - \sum_{i=1}^k \sup_{\substack{s^i \in \mathcal{X}, \\ g_i(s^i) \leq 0}} \left\langle z^{i*}, s^i \right\rangle \\ &= -\delta^*_S \left( -A^* \sum_{i=1}^m y^{i*} - \sum_{i=1}^k z^{i*} \right) - \sum_{i=1}^m (\lambda_i f_i)^* (y^{i*}) - \sum_{i=1}^k \sup_{\substack{s^i \in \mathcal{X}, \\ g_i(s^i) \leq 0}} \left\langle z^{i*}, s^i \right\rangle . \end{aligned}$$

We have $(\lambda_i f_i)^*(y^{i*}) = \lambda_i f_i^* \left( \frac{y^{i*}}{\lambda_i} \right)$ since $\lambda_i > 0$ for all $i = 1, \ldots, k$, and by setting $y^{i*} := \frac{y^{i*}}{\lambda_i}, i =$

$1,\dots,k$, we get the following dual problem to $(PS^{\Sigma})$:

$$(DS^{\Sigma}_{FL}) \qquad \sup_{\substack{(y^{1*},\dots,y^{m*},z^{1*},\dots,z^{k*})\\ \in\mathcal{Y}^*\times\dots\times\mathcal{Y}^*\times\\ \mathcal{X}^*\times\dots\times\mathcal{X}^*}} \left\{-(\Phi^{\Sigma}_{FL})^*(0,y^{1*},\dots,y^{m*},z^{1*},\dots,z^{k*})\right\}$$

$$= \sup_{\substack{(y^{1*},\dots,y^{m*},z^{1*},\dots,z^{k*})\\ \in\mathcal{Y}^*\times\dots\times\mathcal{Y}^*\times\\ \mathcal{X}^*\times\dots\times\mathcal{X}^*}} \left\{ -\delta^*_S\left(-A^*\sum_{i=1}^m y^{i*} - \sum_{i=1}^k z^{i*}\right) - \sum_{i=1}^m (\lambda_i f_i)^*(y^{i*}) - \sum_{i=1}^k \sup_{\substack{s^i\in\mathcal{X},\\ g_i(s^i)\leq 0}} \langle z^{i*}, s^i\rangle \right\}$$

$$= \sup_{\substack{(y^{1*},\dots,y^{m*},\\ z^{1*},\dots,z^{k*})\\ \in\mathcal{Y}^*\times\dots\times\mathcal{Y}^*\times\\ \mathcal{X}^*\times\dots\times\mathcal{X}^*}} \left\{ -\delta^*_S\left(-A^*\sum_{i=1}^m \lambda_i y^{i*} - \sum_{i=1}^k z^{i*}\right) - \sum_{i=1}^m \lambda_i f_i^*(y^{i*}) - \sum_{i=1}^k \sup_{\substack{s^i\in\mathcal{X},\\ g_i(s^i)\leq 0}} \langle z^{i*}, s^i\rangle \right\}. \tag{13}$$

The following theorem holds according to the general approach described in Subsect. 3.1 and because of the previous calculations.

**Theorem 3.3** (weak duality)**.** *Between $(PS^{\Sigma})$ and $(DS^{\Sigma}_{FL})$ weak duality holds, i.e. $v(PS^{\Sigma}) \geq v(DS^{\Sigma}_{FL})$.*

In order to formulate a strong duality theorem we consider the regularity conditions given in Sect. 3.1. The continuity of $\Phi^{\Sigma}_{FL}(x',\cdot,\dots,\cdot)$ at $\mathbf{0}$ is equivalent with the continuity of $f_i$ at $Ax', i=1,\dots,k$, further $g(x') \in -\operatorname{int}(\mathbb{R}^k_+)$ and the continuity of $g$ at $x'$ (which is equivalent with the continuity of $g_i, i=1,\dots,k$, at $x'$). So the first regularity condition becomes:

$$(RC^{\Sigma}_{1,FL}) \quad \left| \begin{array}{l} \exists x' \in A^{-1}\left(\bigcap\limits_{i=1}^m \operatorname{dom}(f_i)\right) \cap S \text{ such that } f_i \text{ is} \\ \text{continuous at } Ax', i=1,\dots,m,\ g_i \text{ is continuous at } x', \\ i=1,\dots,k, \text{ and } g(x') \in -\operatorname{int}(\mathbb{R}^k_+). \end{array}\right. \tag{14}$$

We further have, using the definition of the level set:

$$\begin{aligned}
&(y^1,\dots,y^m,z^1,\dots,z^k) \in \operatorname{Pr}_{\mathcal{Y}\times\dots\times\mathcal{Y}\times\mathcal{X}\times\dots\times\mathcal{X}}(\operatorname{dom}(\Phi^{\Sigma}_{FL}))\\
\Leftrightarrow\ & \exists x \in \mathcal{X}:\ \Phi^{\Sigma}_{FL}(x,y^1,\dots,y^m,z^1,\dots,z^k) < +\infty,\\
\Leftrightarrow\ & \exists x \in S:\ Ax + y^i \in \operatorname{dom}(f_i), i=1,\dots,m,\ g_i(x+z^i) \leq 0, i=1,\dots,k,\\
\Leftrightarrow\ & \exists x \in S:\ y^i \in \operatorname{dom}(f_i) - Ax, i=1,\dots,m,\ x+z^i \in \operatorname{lev}_0(g_i), i=1,\dots,k,\\
\Leftrightarrow\ & \exists x \in S:\ (y^1,\dots,y^m,z^1,\dots,z^k) \in \prod_{i=1}^m (\operatorname{dom}(f_i) - Ax) \times \prod_{i=1}^k (\operatorname{lev}_0(g_i) - x),\\
\Leftrightarrow\ & \exists x \in S:\ (y^1,\dots,y^m,z^1,\dots,z^k) \in \prod_{i=1}^m \operatorname{dom}(f_i) \times \prod_{i=1}^k \operatorname{lev}_0(g_i) - (Ax,\dots,Ax,x,\dots,x),\\
\Leftrightarrow\ & (y^1,\dots,y^m,z^1,\dots,z^k) \in \prod_{i=1}^m \operatorname{dom}(f_i) \times \prod_{i=1}^k \operatorname{lev}_0(g_i) - \left(\prod_{i=1}^m A \times \prod_{i=1}^k \operatorname{id}_{\mathcal{X}}\right)(\Delta_{S^{m+k}}).
\end{aligned} \tag{15}$$

The lower semicontinuity of $\Phi^{\Sigma}_{FL}$, we need for the further regularity conditions, is equivalent with the closeness of $\operatorname{epi}\Phi^{\Sigma}_{FL}$ (see Theorem 2.4) and it holds:

**Lemma 3.4.** *The set*

$$\operatorname{epi}\Phi^{\Sigma}_{FL} = \Bigg\{(x, y^1, \ldots, y^m, z^1, \ldots, z^k, r) \in \mathcal{X} \times \mathcal{Y} \times \ldots \times \mathcal{Y} \times \mathcal{X} \times \ldots \times \mathcal{X} \times \mathbb{R} : \sum_{i=1}^{m} \lambda_i f_i(Ax + y^i) \le r\Bigg\} \cap \{S \times \mathcal{Y} \times \ldots \times \mathcal{Y} \times \mathcal{X} \times \ldots \times \mathcal{X} \times \mathbb{R}\}$$
$$\bigcap_{i=1}^{k} \{(x, y^1, \ldots, y^m, z^1, \ldots, z^k, r) \in \mathcal{X} \times \mathcal{Y} \times \ldots \times \mathcal{Y} \times \mathcal{X} \times \ldots \times \mathcal{X} \times \mathbb{R} : x + z^i \in \operatorname{lev}_0(g_i)\}$$

*is closed if $\mathcal{X}$ and $\mathcal{Y}$ are Fréchet spaces, $f_i$ is lower semicontinuous, $i = 1, \ldots, m$, $S$ is closed and* $\operatorname{lev}_0(g_i)$ *is closed,* $i = 1, \ldots, k$.

*Proof.* Let the sequence $(x_n, y^1_n, \ldots, y^m_n, z^1_n, \ldots, z^k_n, r_n) \in \operatorname{epi}(\Phi^{\Sigma}_{FL})$ converge to $(x, y^1, \ldots, y^m, z^1, \ldots, z^k, r) \in \mathcal{X} \times \mathcal{Y} \times \ldots \times \mathcal{Y} \times \mathcal{X} \times \ldots \times \mathcal{X} \times \mathbb{R}$. We show that it holds $(x, y^1, \ldots, y^m, z^1, \ldots, z^k, r) \in \operatorname{epi}(\Phi^{\Sigma}_{FL})$ in order to get the closeness of $\operatorname{epi}(\Phi^{\Sigma}_{FL})$.

We have $\sum_{i=1}^{m} \lambda_i f_i(Ax_n + y^i_n) \le r_n$. Further it holds $x_n \in S$ and $x_n + z^i_n \in \operatorname{lev}_0(g_i)$ and we get by the lower semicontinuity of $f_i, i = 1, \ldots, m$,

$$\sum_{i=1}^{m} \lambda_i f_i(Ax + y^i) \le \liminf_{n\to\infty} \sum_{i=1}^{m} \lambda_i f_i(Ax_n + y^i_n) \le \liminf_{n\to\infty} r_n = r.$$

Since $x \in S$, which follows by the closeness of $S$, and $\lim\limits_{n\to\infty}(x_n + z^i_n) = x + z^i \in \operatorname{lev}_0(g_i)$, which follows by the closeness of $\operatorname{lev}_0(g_i)$, the assertion follows. □

**Remark 3.5.** *The fact that* $\operatorname{lev}_0(g_i), i = 1, \ldots, k$, *is closed is implied by the lower semicontinuity of* $g_i, i = 1, \ldots, k$.

With this lemma we get (cf. formula (9)):

$$(RC^{\Sigma}_{2,FL}) \left|\begin{array}{l} \mathcal{X} \text{ and } \mathcal{Y} \text{ are Fréchet spaces, } f_i \text{ is lower semicontinuous,} \\ i = 1, \ldots, m, S \text{ is closed, } \operatorname{lev}_0(g_i) \text{ is closed, } i = 1, \ldots, k, \text{ and} \\ \mathbf{0} \in \operatorname{sqri}\left(\prod\limits_{i=1}^{m} \operatorname{dom}(f_i) \times \prod\limits_{i=1}^{k} \operatorname{lev}_0(g_i) - \left(\prod\limits_{i=1}^{m} A \times \prod\limits_{i=1}^{k} \operatorname{id}_{\mathcal{X}}\right)(\Delta_{S^{m+k}})\right). \end{array}\right. \tag{16}$$

The conditions $(RC^{\Sigma}_{3,FL})$ and $(RC^{\Sigma}_{4,FL})$ can be formulated analogously using core and int instead of sqri. Then the following theorem holds:

**Theorem 3.6** (strong duality)**.** *Let the spaces $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z} = \mathbb{R}^k$, the cone $C = \mathbb{R}^k_+$, the functions $f_i, i = 1, \ldots, m$, and $g_i, i = 1, \ldots, k$, and the linear mapping $A$ be assumed as at the beginning of the (sub)section and further $A^{-1}(\bigcap_{i=1}^{m} \operatorname{dom}(f_i)) \cap g^{-1}(-C) \cap S \ne \emptyset$.*

*If one of the regularity conditions $(RC^{\Sigma}_{i,FL}), i \in \{1, 2, 3, 4\}$, is fulfilled, then $v(PS^{\Sigma}) = v(DS^{\Sigma}_{FL})$ and the dual has an optimal solution.*

Here Remark 3.2 also holds.

**Remark 3.7.** *The dual problem $(DS^{\Sigma}_{FL})$ given in formula (13) contains terms of the form*

$$-\sup_{s^i \in \mathcal{X}, g_i(s^i) \le 0} \left\langle z^{i*}, s^i \right\rangle = \inf_{s^i \in \mathcal{X}, g_i(s^i) \le 0} \left\langle -z^{i*}, s^i \right\rangle.$$

*We use now Lagrange duality. In case of having strong duality it holds*

$$\inf_{s^i \in \mathcal{X}, g_i(s^i) \le 0} \left\langle -z^{i*}, s^i \right\rangle = \sup_{\mu^{i*} \ge 0} \inf_{s^i \in \mathcal{X}} \{-\left\langle z^{i*}, s^i \right\rangle + \mu^{i*} g_i(s^i)\} = \sup_{\mu^{i*} \ge 0} \left(-(\mu^{i*} g_i)^*(z^{i*})\right). \tag{17}$$

*In order to have strong duality the following regularity condition has to be fulfilled for $i = 1, \dots, k$ (cf. [1, Subsection 3.2.3]):*

$$(RC_L^i) \quad \Big| \; \exists x' \in \mathcal{X} : \quad g_i(x') < 0.$$

*Assuming that $(RC_{i,FL}^{\Sigma}), i \in \{1, 2, 3, 4\}$, is fulfilled we additionally only have to ask $(RC_L^i), i = 1, \dots, k$, to be fulfilled in order to get the following dual problem (cf. formula (13)) and strong duality between $(PS^{\Sigma})$ and $(DS_{FL}^{\Sigma'})$:*

$$\begin{aligned}
&(DS_{FL}^{\Sigma'}) \\
&\sup_{\substack{(y^{1*},\dots,y^{m*},\\ z^{1*},\dots,z^{k*})\\ \in \mathcal{Y}^* \times \dots \times \mathcal{Y}^* \times \\ \mathcal{X}^* \times \dots \times \mathcal{X}^*}} \left\{ -\delta_S^*\left( -A^* \sum_{i=1}^m \lambda_i y^{i*} - \sum_{i=1}^k z^{i*} \right) - \sum_{i=1}^m \lambda_i f_i^*(y^{i*}) + \sum_{i=1}^k \sup_{\mu^{i*} \geq 0} \left( -(\mu^{i*} g_i)^*(z^{i*}) \right) \right\} \\
= &\sup_{\substack{(y^{1*},\dots,y^{m*},\\ z^{1*},\dots,z^{k*},\mu^{1*},\dots,\mu^{k*})\\ \in \mathcal{Y}^* \times \dots \times \mathcal{Y}^* \times \\ \mathcal{X}^* \times \dots \times \mathcal{X}^* \times \mathbb{R}_+ \times \dots \times \mathbb{R}_+}} \left\{ -\delta_S^*\left( -A^* \sum_{i=1}^m \lambda_i y^{i*} - \sum_{i=1}^k \mu^{i*} z^{i*} \right) - \sum_{i=1}^m \lambda_i f_i^*(y^{i*}) - \sum_{i=1}^k \mu^{i*} g_i^*(z^{i*}) \right\}. \quad (18)
\end{aligned}$$

*The last equality holds by the following consideration. In case of $\mu^{i*} > 0$ we have $(\mu^{i*} g_i)^*(z^{i*}) = \mu^{i*} g_i^*\left(\frac{z^{i*}}{\mu^{i*}}\right)$ and take $z^{i*} := \frac{z^{i*}}{\mu^{i*}}$ such that the term becomes $\mu^{i*} g_i^*(z^{i*})$ for $i = 1, \dots, k$. For $\mu^{i*} = 0$ it holds*

$$(0 \cdot g_i)^*(z^{i*}) = \begin{cases} 0, & z^{i*} = 0, \\ +\infty, & \textit{otherwise}. \end{cases}$$

*Consequently we can always use $\mu^{i*} g_i^*(z^{i*})$ (notice the conventions $0 \cdot (+\infty) := +\infty$ and $0 \cdot (-\infty) := -\infty$ (cf. [1])).*

*In analogy with Theorem 3.3 between $(PS^{\Sigma})$ and $(DS_{FL}^{\Sigma'})$ weak duality holds, i.e. $v(PS^{\Sigma}) \geq v(DS_{FL}^{\Sigma'})$. Further we have:*

**Theorem 3.8** (strong duality)**.** *Let the spaces $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z} = \mathbb{R}^k$, the cone $C = \mathbb{R}_+^k$, the functions $f_i, i = 1, \dots, m$, and $g_i, i = 1, \dots, k$, and the linear mapping $A$ be assumed as at the beginning of the (sub)section and further $A^{-1}(\bigcap_{i=1}^m \operatorname{dom}(f_i)) \cap g^{-1}(-C) \cap S \neq \emptyset$.*

*If one of the regularity conditions $(RC_{i,FL}^{\Sigma}), i \in \{1, 2, 3, 4\}$, is fulfilled and $(RC_L^i)$ is fulfilled for $i = 1, \dots, k$, then $v(PS^{\Sigma}) = v(DS_{FL}^{\Sigma'})$ and the dual has an optimal solution.*

With respect to the fact mentioned in the above remark, the following theorem providing optimality conditions holds.

**Theorem 3.9.** *(a) If one of the regularity conditions $(RC_{i,FL}^{\Sigma}), i \in \{1, 2, 3, 4\}$, is fulfilled, $(RC_L^i)$ is fulfilled for $i = 1, \dots, k$, and $(PS^{\Sigma})$ has an optimal solution $\overline{x}$, then $(DS_{FL}^{\Sigma'})$ has an optimal solution $(\overline{y}^{1*}, \dots, \overline{y}^{m*}, \overline{z}^{1*}, \dots, \overline{z}^{k*}, \overline{\mu}^{1*}, \dots, \overline{\mu}^{k*}) \in \mathcal{Y}^* \times \dots \times \mathcal{Y}^* \times \mathcal{X}^* \times \dots \times \mathcal{X}^* \times \mathbb{R}_+ \times \dots \times \mathbb{R}_+$ such that the following optimality conditions are fulfilled:*

$$\begin{aligned}
&(i) && f_i(A\overline{x}) + f_i^*(\overline{y}^{i*}) - \langle \overline{y}^{i*}, A\overline{x} \rangle = 0, \quad i = 1, \dots, m, \\
&(ii) && \overline{\mu}^{i*} g_i(\overline{x}) = 0, \quad i = 1, \dots, k, \\
&(iii) && \overline{\mu}^{i*}(g_i^*(\overline{z}^{i*}) - \langle \overline{z}^{i*}, \overline{x} \rangle) = 0, \quad i = 1, \dots, k, \\
&(iv) && \sum_{i=1}^m \lambda_i \langle \overline{y}^{i*}, A\overline{x} \rangle + \sum_{i=1}^k \overline{\mu}^{i*} \langle \overline{z}^{i*}, \overline{x} \rangle = \inf_{x \in S} \left\{ \sum_{i=1}^m \lambda_i \langle \overline{y}^{i*}, Ax \rangle + \sum_{i=1}^k \overline{\mu}^{i*} \langle \overline{z}^{i*}, x \rangle \right\}.
\end{aligned}$$

*(b) Let $\overline{x}$ be feasible to $(PS^{\Sigma})$ and $(\overline{y}^{1*},\ldots,\overline{y}^{m*},\overline{z}^{1*},\ldots,\overline{z}^{k*},\overline{\mu}^{1*},\ldots,\overline{\mu}^{k*}) \in \mathcal{Y}^* \times \ldots \times \mathcal{Y}^* \times \mathcal{X}^* \times \ldots \times \mathcal{X}^* \times \mathbb{R}_+ \times \ldots \times \mathbb{R}_+$ be feasible to $(DS^{\Sigma'}_{FL})$ fulfilling the optimality conditions $(i)-(iv)$. Then $\overline{x}$ is an optimal solution for $(PS^{\Sigma})$, $(\overline{y}^{1*},\ldots,\overline{y}^{m*},\overline{z}^{1*},\ldots,\overline{z}^{k*},\overline{\mu}^{1*},\ldots,\overline{\mu}^{k*})$ is an optimal solution for $(DS^{\Sigma'}_{FL})$ and $v(PS^{\Sigma}) = v(DS^{\Sigma'}_{FL})$.*

*Proof.* (a) Since $(PS^{\Sigma})$ has an optimal solution $\overline{x} \in S$, one of the conditions $(RC^{\Sigma}_{i,FL}), i \in \{1,2,3,4\}$, is fulfilled and $(RC^{i}_{L})$ is fulfilled for $i = 1,\ldots,k$, Theorem 3.8 guarantees the existence of an optimal solution for $(DS^{\Sigma'}_{FL})$, namely $(\overline{y}^{1*},\ldots,\overline{y}^{m*},\overline{z}^{1*},\ldots,\overline{z}^{k*},\overline{\mu}^{1*},\ldots,\overline{\mu}^{k*})$, such that

$$\begin{aligned}
&v(PS^{\Sigma}) = v(DS^{\Sigma'}_{FL})\\
\Leftrightarrow &\sum_{i=1}^m \lambda_i f_i(A\overline{x}) = -\sum_{i=1}^m \lambda_i f_i^*(\overline{y}^{i*}) - \sum_{i=1}^k \overline{\mu}^{i*} g_i^*(\overline{z}^{i*}) - \delta_S^*\left(-A^*\sum_{i=1}^m \lambda_i \overline{y}^{i*} - \sum_{i=1}^k \overline{\mu}^{i*}\overline{z}^{i*}\right)\\
\Leftrightarrow &\sum_{i=1}^m \lambda_i \left[f_i(A\overline{x}) + f_i^*(\overline{y}^{i*}) - \left\langle \overline{y}^{i*}, A\overline{x}\right\rangle\right] + \sum_{i=1}^m \lambda_i \left\langle \overline{y}^{i*}, A\overline{x}\right\rangle\\
&\quad + \sum_{i=1}^k \overline{\mu}^{i*}\left[g_i^*(\overline{z}^{i*}) + g_i(\overline{x}) - \left\langle \overline{z}^{i*}, \overline{x}\right\rangle\right] - \sum_{i=1}^k \overline{\mu}^{i*} g_i(\overline{x})\\
&\quad + \sum_{i=1}^k \overline{\mu}^{i*}\left\langle \overline{z}^{i*}, \overline{x}\right\rangle + \delta_S^*\left(-A^*\sum_{i=1}^m \lambda_i \overline{y}^{i*} - \sum_{i=1}^k \overline{\mu}^{i*}\overline{z}^{i*}\right) = 0.
\end{aligned}$$

By applying Young's inequality (cf. formula (1)) and having $\overline{\mu}^{i*} \geq 0$ and $g_i(\overline{x}) \leq 0$, this sum which is equal to zero consists of $m + 2k + 1$ nonnegative terms. Thus the inequalities have to be fulfilled with equality and we get the following equivalent formulation:

$$\left\{\begin{array}{ll}
(i) & f_i(A\overline{x}) + f_i^*(\overline{y}^{i*}) - \left\langle \overline{y}^{i*}, A\overline{x}\right\rangle = 0, \quad i = 1,\ldots,m,\\
(ii) & \overline{\mu}^{i*} g_i(\overline{x}) = 0, \quad i = 1,\ldots,k,\\
(iii) & \overline{\mu}^{i*}(g_i^*(\overline{z}^{i*}) + g_i(\overline{x}) - \left\langle \overline{z}^{i*}, \overline{x}\right\rangle) = 0, \quad i = 1,\ldots,k,\\
(iv) & \sum\limits_{i=1}^m \lambda_i \left\langle \overline{y}^{i*}, A\overline{x}\right\rangle + \sum\limits_{i=1}^k \overline{\mu}^{i*}\left\langle \overline{z}^{i*}, \overline{x}\right\rangle + \delta_S^*\left(-A^*\sum\limits_{i=1}^m \lambda_i \overline{y}^{i*} - \sum\limits_{i=1}^k \overline{\mu}^{i*}\overline{z}^{i*}\right) = 0,
\end{array}\right.$$

$$\Leftrightarrow \left\{\begin{array}{ll}
(i) & f_i(A\overline{x}) + f_i^*(\overline{y}^{i*}) - \left\langle \overline{y}^{i*}, A\overline{x}\right\rangle = 0, \quad i = 1,\ldots,m,\\
(ii) & \overline{\mu}^{i*} g_i(\overline{x}) = 0, \quad i = 1,\ldots,k,\\
(iii) & \overline{\mu}^{i*}(g_i^*(\overline{z}^{i*}) - \left\langle \overline{z}^{i*}, \overline{x}\right\rangle) = 0, \quad i = 1,\ldots,k,\\
(iv) & \sum\limits_{i=1}^m \lambda_i \left\langle \overline{y}^{i*}, A\overline{x}\right\rangle + \sum\limits_{i=1}^k \overline{\mu}^{i*}\left\langle \overline{z}^{i*}, \overline{x}\right\rangle = \inf\limits_{x\in S}\left\{\sum\limits_{i=1}^m \lambda_i \left\langle \overline{y}^{i*}, Ax\right\rangle + \sum\limits_{i=1}^k \overline{\mu}^{i*}\left\langle \overline{z}^{i*}, x\right\rangle\right\}.
\end{array}\right.$$

(b) All calculations in part (a) can be carried out in reverse direction. □

## 3.3 The Vector Optimization Problem $(PV)$

In this subsection we consider a vector optimization problem with an objective function being the composition of a convex function $f$ and a linear continuous operator $A$ and cone and geometric constraints in analogy with the scalar problem in Sect. 3.1.

The properties of the spaces and sets were defined at the beginning of the section. Assume the function $f : \mathcal{Y} \to \mathcal{V}^{\bullet}$ to be proper and $K$-convex and $g : \mathcal{X} \to \mathcal{Z}$ to be $C$-convex, fulfilling $A^{-1}(\text{dom}(f)) \cap g^{-1}(-C) \cap S \neq \emptyset$.

By $\text{Min}(V,K)$ we denote the *set of minimal points of* $V$, where $y \in V \subseteq \mathcal{V}$ is said to be a *minimal point* of the set $V$ if $y \in V$ and there exists no $y' \in V$ such that $y' \leq_K y$. The set $\text{Max}(V,K)$ of *maximal points* of $V$ is defined analogously.

We consider the following vector optimization problem:

$$(PV) \quad \underset{x\in\mathcal{A}}{\operatorname{Min}} f(Ax), \quad \mathcal{A} = \{x \in S : g(x) \in -C\}.$$

We investigate a duality approach with respect to *properly efficient solutions* in the sense of linear scalarization (cf. [1]), that are defined as follows:

**Definition 3.10** (properly efficient solution)**.** *An element $\overline{x} \in \mathcal{A}$ is said to be a properly efficient solution to $(PV)$ if $\overline{x} \in A^{-1}(\operatorname{dom}(f))$ and $\exists v^* \in K^{*0}$ such that $\langle v^*, f(A\overline{x})\rangle \leq \langle v^*, f(Ax)\rangle$, $\forall x \in \mathcal{A}$.*

Further, we define *efficient solutions*:

**Definition 3.11** (efficient solution)**.** *An element $\overline{x} \in \mathcal{A}$ is said to be an efficient solution to $(PV)$ if $\overline{x} \in A^{-1}(\operatorname{dom}(f))$ and $f(A\overline{x}) \in \operatorname{Min}((f \circ A)(A^{-1}(\operatorname{dom}(f)) \cap \mathcal{A}), K)$. This means that if $\overline{x} \in A^{-1}(\operatorname{dom}(f)) \cap \mathcal{A}$ then for all $x \in A^{-1}(\operatorname{dom}(f)) \cap \mathcal{A}$ from $f(Ax) \leqq_K f(A\overline{x})$ follows $f(A\overline{x}) = f(Ax)$.*

Depending on the perturbation function $\Phi_{FL}$, the dual problem to $(PV)$ can be given by (cf. [1, Section 4.3.1]):

$$(DV_{FL}) \qquad \underset{(v^*,y^*,z^*,v)\in\mathcal{B}_{FL}}{\operatorname{Max}} v,$$

where

$$\mathcal{B}_{FL} = \{(v^*, y^*, z^*, v) \in K^{*0} \times \mathcal{Y}^* \times Z^* \times \mathcal{V} : \langle v^*, v\rangle \leq -(v^*\Phi_{FL})^*(0, -y^*, -z^*)\}.$$

Here we consider the perturbation function $\Phi_{FL} : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{V}^\bullet$, analogously as given in the scalar case in Sect. 3.1:

$$\Phi_{FL}(x, y, z) = \begin{cases} f(Ax + y), & x \in S, g(x) \in z - C, \\ +\infty_K, & \text{otherwise.} \end{cases} \tag{19}$$

The formula for the conjugate function of $v^*\Phi_{FL} : \mathcal{X}^* \times \mathcal{Y}^* \times \mathcal{Z}^* \to \overline{\mathbb{R}}$ follows from the calculations above (cf. formulas (6) and (7)):

$$-(v^*\Phi_{FL})^*(x^*, y^*, z^*) = -(-z^*g)^*_S(x^* - A^*y^*) - (v^*f)^*(y^*) - \delta_{-C^*}(z^*).$$

From this formula, the dual problem of $(PV)$ can be deduced. It is given by

$$(DV_{FL}) \underset{(v^*,y^*,z^*,v)\in\mathcal{B}_{FL}}{\operatorname{Max}} v, \tag{20}$$

where

$$\begin{aligned} \mathcal{B}_{FL} &= \{(v^*, y^*, z^*, v) \in K^{*0} \times \mathcal{Y}^* \times Z^* \times \mathcal{V} : \langle v^*, v\rangle \leq -(v^*\Phi_{FL})^*(0, -y^*, -z^*)\} \\ &= \{(v^*, y^*, z^*, v) \in K^{*0} \times \mathcal{Y}^* \times C^* \times \mathcal{V} : \langle v^*, v\rangle \leq -(v^*f)^*(-y^*) - (z^*g)^*_S(A^*y^*)\}. \end{aligned}$$

Weak duality follows from [1, Theorem 4.3.1]:

**Theorem 3.12** (weak duality)**.** *There is no $x \in \mathcal{A}$ and no $(v^*, y^*, z^*, v) \in \mathcal{B}_{FL}$ such that $f(Ax) \leq_K v$.*

To formulate a strong and converse duality theorem we have to state a regularity condition. The conditions $(RC_{1,FL})$ and $(RC_{2,FL})$ from above (cf. formula (11) and (12)) (as well as $(RC_{3,FL})$ and $(RC_{4,FL})$) can, under some small modifications, be applied for the vectorial case. It holds (see [1, Remark 4.3.1]):

**Remark 3.13.** *For having strong duality we only have to assume that for all $v^* \in K^{*0}$ the scalar optimization problem $\inf_{x\in\mathcal{X}}(v^*\Phi_{FL})(x, 0, 0)$ is stable.*

*This can be guaranteed by assuming that $\mathcal{X}$ and the spaces of perturbation variables, $\mathcal{Y}$ and $\mathcal{Z}$, are Fréchet spaces, $f$ is star $K$-lower semicontinuous, $S$ is closed, $g$ is $C$-epi closed and $\mathbf{0} \in \operatorname{sqri}((\operatorname{dom}(f) \times C) - (A \times -g)(\Delta_{S^2}))$ since $\operatorname{dom}(f) = \operatorname{dom}(v^*f)$. This follows by Theorem 3.1.*

Further, this fact can be seen in the proof of the strong and converse duality Theorem 3.15. We have:

$$(RCV_{1,FL}) \quad \Big| \ \exists x' \in A^{-1}(\mathrm{dom}(f)) \cap S \text{ such that } f \text{ is continuous at } Ax' \text{ and } g(x') \in -\operatorname{int}(C),$$

which is identical with $(RC_{1,FL})$ (cf. formula (12)) and

$$(RCV_{2,FL}) \quad \left| \begin{array}{l} \mathcal{X}, \mathcal{Y} \text{ and } \mathcal{Z} \text{ are Fréchet spaces, } f \text{ is star } K\text{-lowersemicontinuous, } S \text{ is closed,} \\ g \text{ is } C\text{-epi closed and } \mathbf{0} \in \mathrm{sqri}((\mathrm{dom}(f) \times C) - (A \times -g)(\Delta_{S^2})). \end{array} \right.$$

Analogously we formulate $(RCV_{3,FL})$ and $(RCV_{4,FL})$ by using core and int instead of sqri.

Before we proof a strong and converse duality theorem we want to formulate the following preliminary result (in analogy with [1, Theorem 4.3.3], to which we also refer for the proof):

**Lemma 3.14.** *Assume that $\mathcal{B}_{FL}$ is non-empty and that one of the regularity conditions $(RCV_{i,FL}), i \in \{1, 2, 3, 4\}$, is fulfilled. Then*

$$\mathcal{V} \setminus \mathrm{cl}\big((f \circ A)(A^{-1}(\mathrm{dom}(f)) \cap \mathcal{A}) + K\big) \subseteq \mathrm{core}(h(\mathcal{B}_{FL})),$$

*where $h : K^{*0} \times \mathcal{Y}^* \times C^* \times \mathcal{V} \to \mathcal{V}$ is defined by $h(v^*, y^*, z^*, v) = v$.*

Now we get the following theorem (in analogy with [1, Theorem 4.3.7]):

**Theorem 3.15** (strong and converse duality)**.** *(a) If one of the conditions $(RCV_{i,FL}), i \in \{1, 2, 3, 4\}$, is fulfilled and $\overline{x} \in \mathcal{A}$ is a properly efficient solution to $(PV)$, then there exists $(\overline{v}^*, \overline{y}^*, \overline{z}^*, \overline{v}) \in \mathcal{B}_{FL}$, an efficient solution to $(DV_{FL})$, such that $f(A\overline{x}) = \overline{v}$.*

*(b) If one of the conditions $(RCV_{i,FL}), i \in \{1, 2, 3, 4\}$, is fulfilled, $(f \circ A)(A^{-1}(\mathrm{dom}(f)) \cap \mathcal{A}) + K$ is closed and $(\overline{v}^*, \overline{y}^*, \overline{z}^*, \overline{v})$ is an efficient solution to $(DV_{FL})$, then there exists $\overline{x} \in \mathcal{A}$, a properly efficient solution to $(PV)$, such that $f(A\overline{x}) = \overline{v}$.*

The following proof of the theorem will be done in analogy with the one of [1, Theorem 4.3.2 and 4.3.4]:

*Proof.* (a) Since $\overline{x} \in \mathcal{A}$ is a properly efficient solution, there exists $\overline{v}^* \in K^{*0}$ such that $\overline{x}$ is an optimal solution to the scalarized problem

$$\inf_{x \in \mathcal{A}} \langle \overline{v}^*, f(Ax) \rangle .$$

Using that one of the regularity conditions $(RCV_{i,FL}), i \in \{1, 2, 3, 4\}$, is fulfilled we can apply Theorem 3.1. Therefore we have to show that the problem $\inf_{x \in \mathcal{A}} \langle \overline{v}^*, f(Ax) \rangle$ with the assumptions given by $(RCV_{i,FL})$ fulfills the regularity condition $(RC_{i,FL})$ for fixed $i \in \{1, 2, 3, 4\}$.

Let us consider $(RCV_{i,FL})$, $i \in \{2, 3, 4\}$. Since $f$ is assumed to be star $K$-lower semicontinuous, $v^*f$ is lower semicontinuous by definition. The assumptions regarding $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, S$ and $g$ hold analogously. Further, we have $\mathrm{dom}(v^*f) = \mathrm{dom}(f)$ and therefore

$$\mathrm{sqri}((\mathrm{dom}(v^*f) \times C) - (A \times -g)(\Delta_{S^2})) = \mathrm{sqri}((\mathrm{dom}(f) \times C) - (A \times -g)(\Delta_{S^2}))$$

and analogously for core and int. Thus the conditions $(RC_{i,FL})$, $i \in \{2, 3, 4\}$, hold.

The continuity of $v^*f$ follows by the continuity of $f$ and since $\mathrm{dom}(v^*f) = \mathrm{dom}(f)$ the fulfillment of $(RC_{1,FL})$ follows by assuming $(RCV_{1,FL})$.

From the mentioned theorem it follows that there exist $\overline{z}^* \in C^*$ and $\overline{y}^* \in \mathcal{Y}^*$ such that $\langle \overline{v}^*, f(A\overline{x}) \rangle = -(\overline{v}^*f)^*(-\overline{y}^*) - (\overline{z}^*g)^*_S(A^*\overline{y}^*)$. It follows that for $\overline{v} = f(A\overline{x})$ the element $(\overline{v}^*, \overline{y}^*, \overline{z}^*, \overline{v})$ is feasible to the dual problem $(DV_{FL})$. By weak duality, that was given in Theorem 3.12, it follows that $(\overline{v}^*, \overline{y}^*, \overline{z}^*, \overline{v})$ is an efficient solution.

(b) Assume that $\overline{v} \notin (f \circ A)\left(A^{-1}(\mathrm{dom}(f)) \cap \mathcal{A}\right) + K$. From Lemma 3.14 it follows that $\overline{v} \in \mathrm{core}(h(\mathcal{B}_{FL}))$. By definition of the core for $k \in K \setminus \{0\}$ there exists $\lambda > 0$ such that $v_\lambda := \overline{v} + \lambda k \geq_K \overline{v}$

and $v_\lambda \in h(\mathcal{B}_{FL})$. This contradicts the fact that $(\overline{v}^*, \overline{y}^*, \overline{z}^*, \overline{v})$ is an efficient solution for $(DV_{FL})$ since $v_\lambda$ is in the image set of $(DV_{FL})$ and $v_\lambda \geq_K \overline{v}$.

Thus we have $\overline{v} \in (f \circ A)\left(A^{-1}(\mathrm{dom}(f)) \cap \mathcal{A}\right) + K$, which means that there exists $\overline{x} \in A^{-1}(\mathrm{dom}(f)) \cap \mathcal{A}$ and $\overline{k} \in K$ such that $\overline{v} = f(A\overline{x}) + \overline{k}$. By Theorem 3.12 there is no $x \in \mathcal{A}$ and no $(v^*, y^*, z^*, v) \in \mathcal{B}_{FL}$ such that $f(Ax) \leq_K v$ and hence it holds $\overline{k} = 0$. Consequently we have $f(A\overline{x}) = \overline{v}$ and $\overline{x}$ is a properly efficient solution to $(PV)$ which follows by the following calculation. It holds

$$\langle \overline{v}^*, f(A\overline{x})\rangle = \langle \overline{v}^*, \overline{v}\rangle \leq -(\overline{v}^* f)^*(-\overline{y}^*) - (\overline{z}^* g)^*_S(A^*\overline{y}^*) = -(\overline{v}^*\Phi_{FL})^*(0, -\overline{y}^*, -\overline{z}^*) \leq \inf_{x\in\mathcal{A}} \langle \overline{v}^*, f(Ax)\rangle .$$

Here the last inequality follows by weak duality for the scalarized problem (cf. Sect. 3.1) and thus $\overline{x}$ turns out to be a properly efficient solution to $(PV)$ by Definition 3.10 fulfilling $\overline{v} = f(A\overline{x})$. □

## 3.4 The Vector Optimization Problem $(PV^m)$

We assume that the spaces $\mathcal{V}$ and $\mathcal{Z}$ are finite dimensional, especially $\mathcal{V} = \mathbb{R}^m, K = \mathbb{R}^m_+, \mathcal{Z} = \mathbb{R}^k$ and $C = \mathbb{R}^k_+$. Further, let the functions $f_i : \mathcal{Y} \to \overline{\mathbb{R}}, i = 1, \ldots, m$, be proper and convex and $g = (g_1, \ldots, g_k)^T : \mathcal{X} \to \mathbb{R}^k$ be $\mathbb{R}^k_+$-convex, fulfilling $A^{-1}(\bigcap_{i=1}^m \mathrm{dom}(f_i)) \cap g^{-1}(-\mathbb{R}^k_+) \cap S \neq \emptyset$. We consider the following vector optimization problem:

$$(PV^m) \quad \operatorname*{Min}_{x\in\mathcal{A}} \begin{pmatrix} f_1(Ax) \\ \vdots \\ f_m(Ax) \end{pmatrix}, \qquad \mathcal{A} = \{x \in S : g_i(x) \leq 0, i = 1, \ldots, k\}.$$

The perturbation function $\Phi^m_{FL} : \mathcal{X} \times \mathcal{Y} \times \ldots \times \mathcal{Y} \times \mathcal{X} \times \ldots \times \mathcal{X} \to \mathbb{R}^{m\bullet}$ is similar to the one in Sect. 3.2 in order to separate the conjugate functions of $f_i, i = 1, \ldots, m$, and the conjugate functions of $g_i, i = 1, \ldots, k$, in the dual problem:

$$\begin{aligned} &\Phi^m_{FL}(x, y^1, \ldots, y^m, z^1, \ldots, z^k) \\ &= \begin{cases} (f_1(Ax + y^1), \ldots, f_m(Ax + y^m))^T, & x \in S, g_i(x + z^i) \leq 0, i = 1, \ldots, k, \\ +\infty, & \text{otherwise.} \end{cases} \end{aligned}$$

Thus the dual problem becomes by taking $v := (v_1, \ldots, v_m)^T \in \mathbb{R}^m$ and $v^* = (v_1^*, \ldots, v_m^*)^T \in \mathrm{int}(\mathbb{R}^m_+)$ (cf. formula (20)):

$$(DV^m_{FL}) \quad \operatorname*{Max}_{(v^*, y^*, z^*, v)\in\mathcal{B}^m_{FL}} v,$$

where

$$\begin{aligned} \mathcal{B}^m_{FL} = \Big\{ &(v^*, y^{1*}, \ldots, y^{m*}, z^{1*}, \ldots, z^{k*}, v) \in \mathrm{int}(\mathbb{R}^m_+) \times \mathcal{Y}^* \times \ldots \times \mathcal{Y}^* \times \mathcal{X}^* \times \ldots \times \mathcal{X}^* \times \mathbb{R}^m : \\ &v^T v^* \leq -(v^*\Phi^m_{FL})^*(0, -y^{1*}, \ldots, -y^{m*}, -z^{1*}, \ldots, -z^{k*}) \Big\}. \end{aligned}$$

Especially it holds

$$\begin{aligned} &-(v^*\Phi^m_{FL})^*(0, -y^{1*}, \ldots, -y^{m*}, -z^{1*}, \ldots, -z^{k*}) \\ &= -\sup_{\substack{x\in S, \\ y^i\in\mathcal{Y}, i=1,\ldots,m, \\ z^i\in\mathcal{X}, i=1,\ldots,k, \\ g_i(x+z^i)\leq 0, i=1,\ldots,k}} \left\{ -\sum_{i=1}^m v_i^* f_i(Ax + y^i) - \sum_{i=1}^m \left\langle y^{i*}, y^i\right\rangle - \sum_{i=1}^k \left\langle z^{i*}, z^i\right\rangle \right\} \\ &= -\delta^*_S\left(A^* \sum_{i=1}^m v_i^* y^{i*} + \sum_{i=1}^k z^{i*}\right) - \sum_{i=1}^m v_i^* f_i^*(-y^{i*}) - \sum_{i=1}^k \sup_{\substack{s^i\in\mathcal{X}, \\ g_i(s^i)\leq 0}} \left\langle -z^{i*}, s^i\right\rangle , \end{aligned}$$

which arises from formula (13). The dual becomes:

$$(DV_{FL}^m) \quad \underset{(v^*,y^{1*},\ldots,y^{k*},z^{1*},\ldots,z^{k*},v)\in\mathcal{B}_{FL}^m}{\mathrm{Max}} v, \tag{21}$$

where

$$\mathcal{B}_{FL}^m = \Bigg\{(v^*,y^{1*},\ldots,y^{m*},z^{1*},\ldots,z^{k*},v) \in \mathrm{int}(\mathbb{R}_+^m)\times\mathcal{Y}^*\times\ldots\times\mathcal{Y}^*\times\mathcal{X}^*\times\ldots\times\mathcal{X}^*\times\mathbb{R}^m : \\ v^T v^* \leq -\delta_S^*\left(A^*\sum_{i=1}^m v_i^* y^{i*} + \sum_{i=1}^k z^{i*}\right) - \sum_{i=1}^m v_i^* f_i^*(-y^{i*}) - \sum_{i=1}^k \sup_{\substack{s^i\in\mathcal{X},\\ g_i(s^i)\leq 0}} \left\langle -z^{i*}, s^i\right\rangle\Bigg\}.$$

The following weak duality theorem holds:

**Theorem 3.16** (weak duality)**.** *Between $(PV^m)$ and $(DV_{FL}^m)$ weak duality holds, i.e. there is no $x\in\mathcal{A}$ and no $(v^*,y^{1*},\ldots,y^{m*},z^{1*},\ldots,z^{k*},v)\in\mathcal{B}_{FL}^m$ such that $f(Ax)\leq_K v$.*

In order to formulate a strong and converse duality theorem, we have to state some regularity conditions. Therefore let us first consider the following lemma:

**Lemma 3.17.** *Let be $f=(f_1,\ldots,f_m)^T:\mathcal{Y}\to\mathbb{R}^{m\bullet}$. If $f_i, i=1,\ldots,m$, is lower semicontinuous, then $f$ is star $K$-lower semicontinuous, where $K=\mathbb{R}_+^m$.*

*Proof.* Let be $v^*=(v_1^*,\ldots,v_m^*)^T\in K=\mathbb{R}_+^m$. If we assume that $f_i, i=1,\ldots,m$, is lower semicontinuous then $\langle v^*,f\rangle=\sum_{i=1}^m v_i^* f_i$ is lower semicontinuous since it is a sum of lower semicontinuous functions and $v_i^*\geq 0, i=1,\ldots,m$ (cf. [1, Prop. 2.2.11]). This means by definition that $f$ is star $K$-lower semicontinuous. □

As mentioned in the last subsection it is possible to apply the regularity conditions given in the scalar case under some modifications. So formulas (14) and (16) become:

$$(RCV_{1,FL}^m)\left|\begin{array}{l} \exists x'\in A^{-1}\left(\bigcap\limits_{i=1}^m \mathrm{dom}(f_i)\right)\cap S \text{ such that } f_i \text{ is continuous at } Ax', i=1,\ldots,m,\\ g_i \text{ is continuous at } x', i=1,\ldots,k, \text{ and } g(x')\in-\mathrm{int}(\mathbb{R}_+^k),\end{array}\right. \tag{22}$$

$$(RCV_{2,FL}^m)\left|\begin{array}{l} \mathcal{X} \text{ and } \mathcal{Y} \text{ are Fréchet spaces, } f_i \text{ is lower semicontinuous,}\\ i=1,\ldots,m, S \text{ is closed, } \mathrm{lev}_0(g_i) \text{ is closed, } i=1,\ldots,k, \text{and}\\ \mathbf{0}\in\mathrm{sqri}\left(\prod\limits_{i=1}^m \mathrm{dom}(f_i)\times\prod\limits_{i=1}^k \mathrm{lev}_0(g_i) - \left(\prod\limits_{i=1}^m A\times\prod\limits_{i=1}^k \mathrm{id}_{\mathcal{X}}\right)(\Delta_{S^{m+k}})\right).\end{array}\right. \tag{23}$$

The conditions $(RCV_{3,FL}^m)$ and $(RCV_{4,FL}^m)$ can be formulated analogously using core and int instead of sqri. The following theorem holds:

**Theorem 3.18.** *(a) If one of the conditions $(RCV_{i,FL}^m), i\in\{1,2,3,4\}$, is fulfilled and $\overline{x}\in\mathcal{A}$ is a properly efficient solution to $(PV^m)$, then there exists $(\overline{v}^*,\overline{y}^{1*},\ldots,\overline{y}^{m*},\overline{z}^{1*},\ldots,\overline{z}^{k*},\overline{v})\in\mathcal{B}_{FL}^m$, an efficient solution to $(DV_{FL}^m)$, such that $f(A\overline{x})=\overline{v}$.*
*(b) If one of the conditions $(RCV_{i,FL}^m), i\in\{1,2,3,4\}$, is fulfilled, $(f\circ A)(A^{-1}(\bigcap_{i=1}^m\mathrm{dom}(f_i))\cap\mathcal{A})+K$ is closed and $(\overline{v}^*,\overline{y}^{1*},\ldots,\overline{y}^{m*},\overline{z}^{1*},\ldots,\overline{z}^{k*},\overline{v})$ is an efficient solution to $(DV_{FL}^m)$, then there exists $\overline{x}\in\mathcal{A}$, a properly efficient solution to $(PV^m)$, such that $f(A\overline{x})=\overline{v}$.*

**Remark 3.19.** *Remark 3.7 can be applied here which leads to the dual problem (cf. formula (18))*

$$(DV_{FL}^{m'}) \quad \underset{(v^*,y^{1*},\ldots,y^{k*},z^{1*},\ldots,z^{k*},\mu^{1*},\ldots,\mu^{k*},v)\in\mathcal{B}_{FL}^m}{\mathrm{Max}} v, \tag{24}$$

*where*

$$\mathcal{B}_{FL}^m = \Bigg\{ (v^*, y^{1*}, \ldots, y^{m*}, z^{1*}, \ldots, z^{k*}, \mu^{1*}, \ldots, \mu^{k*}, v) \in \operatorname{int}(\mathbb{R}_+^m) \times \mathcal{Y}^* \times \ldots \times \mathcal{Y}^* \times \mathcal{X}^* \times \ldots \times \mathcal{X}^* \times \mathbb{R}_+ \times \ldots \times \mathbb{R}_+ \times \mathbb{R}^m : v^T v^* \leq -\delta_S^* \left( A^* \sum_{i=1}^m v_i y^{i*} - \sum_{i=1}^k \mu^{i*} z^{i*} \right) - \sum_{i=1}^m v_i f_i^*(-y^{i*}) + \sum_{i=1}^k \mu^{i*} g_i^*(-z^{i*}) \Bigg\}.$$

*Further, weak duality holds by construction and Theorem 3.18 holds analogously under the assumption that one of the regularity conditions* $(RCV_{i,FL}^m), i \in \{1,2,3,4\}$, *is fulfilled and* $(RC_L^i)$ *is fulfilled for* $i = 1, \ldots, k$.

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