On random walks and Fourier decay

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Definition of the Fourier transform

Question 0 What does Fourier decay have to do with random walks?

Fix $\nu \in \mathcal{P}(\mathbb{R})$, and $q \in \mathbb{R}$.
Def Fourier transform of ν at q is

$$\mathcal{F}_q(\nu) := \int \exp(2\pi i q x) d\nu(x)$$

Fourier decay problem $\mathcal{F}_q(\nu) = o(1)$? (is ν Rajchman) Rate of decay?

Shmerkin (2014) $\exists \beta > 0$, $|\mathcal{F}_q(\nu)| = O\left(\frac{1}{|q|^\beta}\right) \Rightarrow \forall \mu \in \mathcal{P}(\mathbb{R})$ with $\dim \mu = 1$, $\mu * \nu \ll \lambda$
$\lambda :=$ Leb. mea. on $\mathbb{R}$.

Some examples

1. Riemann-Lebesgue Lemma: if $\nu \ll \lambda \Rightarrow \nu$ Rajchman.
2. Heuristics for dynamically defined measures: $\mathcal{F}_q(\nu) \neq o(1)$ $\Rightarrow \nu$ has some (approximate) arithmetic structure.
3. Heuristics for smoothly defined measures:
   Poly. Fourier decay $\iff$
   spectral gap for derivative cocycle $\iff$
   exp. fast renewal Theorem for derivative cocycle.

Bernoulli Convolutions

For $r \in (0,1)$, $\nu_r \sim \sum \pm r^n$, $\pm$ IID unbiased.

Question For which $\frac{1}{2} < r < 1$ is $\nu_r \ll \lambda$?

Theorem (Erdős, 1939) If r^{-1} Pisot $\Rightarrow \nu_r$ not Rajchman $\Rightarrow \nu_r$ not abs. continuous.

Idea of proof $\mathcal{F}_q(\nu_r) = \prod_{j=0}^{\infty} \cos(2\pi r^j q)$.
Use: r^{-1} Pisot $\Rightarrow \exists a = a(r) \in (0,1)$,
dist$(r^j, \mathbb{Z}) < a^{|j|}$, $\forall j \in \mathbb{Z}$.

Self-conformal measures

$\Phi = \{f_1, ..., f_n\} \subseteq C^{1+\gamma}([0,1])$ non-sing. contractions (IFS),

$\mathbf{p} = (p_1, ..., p_n)$ prob. vector, $p_i > 0$.

$\exists! K \subseteq [0,1], K \neq \emptyset$ compact s.t. $K = \bigcup_{i=1}^n f_i(K)$

$\exists! \nu_\mathbf{p} \in$ Prob. mea.(K) s.t. $\nu_\mathbf{p} = \sum_{i=1}^n p_i \cdot f_i \nu_\mathbf{p}$

$\nu_\mathbf{p} =$ self-conformal measure.

Self-similar measures

$\Phi \subseteq \text{Aff}([0,1]) \Rightarrow \nu =$ self similar measure.

Example Ber. conv. ν_r, $\Phi = \{r \cdot x - 1, r \cdot x + 1\}$, $\mathbf{p} = (\frac{1}{2}, \frac{1}{2})$

Solomyak 2019 Poly. F-decay is typical for self-similar measures.

What is a non linear IFS?

Def of linear We call $\Phi \subset C^2$ linear if:
$\forall f \in \Phi, x \in K, \qquad f''(x) = 0$

Note $\Phi \subseteq C^\omega$ and Φ linear $\Rightarrow$ Φ is self-similar.

Question (Hochman) $\exists$ linear $\Phi \subseteq C^\infty$, Φ not conjugate to self-similar?

A. Ben-Ovadia, Shannon Yes, even s.t. $f'|_K \equiv c = c_\Phi, \forall f \in \Phi$.

Main results

Thm (A. - Rodriguez Hertz - Wang, 2023)

1. $\Phi \subset C^2$ not cong. to linear $\Rightarrow \forall$ self-conformal mea. poly. F-decay.
2. $\Phi \subset C^\omega$ and $\exists f \in \Phi \setminus \text{Aff}(\mathbb{R}) \Rightarrow \forall$ self-conformal mea. poly. F-decay.

Related Jordan-Sahlsten (2016), Bourgain-Dyatlov (2017), Li (2022), Sahlsten-Stevens (2022), Baker-Sahlsten (2023), Algom-Chang-Wu-Wu (2024), Baker-Banaji (2024), Baker-Khalil-Sahlsten (2024).

A random walk and stopping time

$\Phi = \{f_1, f_2, ..., f_n\} \subset C^2$, $K = K_\Phi$, $\mathbf{p}$ s.t. $\nu = \nu_{\mathbf{p}}$.

A random walk Fix $x \in K$, $I \in \{1, ..., n\}^m$,

$S_m(I) := -\log |f_I'(x)|$, Note $S_m \to \infty$.

Stopping time For $k > 0$, $\omega \in \{1, ..., n\}^{\mathbb{N}}$,

$\tau_k(\omega) := \min\{m : S_m(\omega|_m) > k\}$.

Def $S_{\tau_k}(\omega) = S_{\tau_k(\omega)}\left(\omega|_{\tau_k(\omega)}\right) =$
RV on $[k, k+1]$ w.r.t $\mathbf{p}^{\mathbb{N}}$ on $\{1, ..., n\}^{\mathbb{N}}$.

F-decay for self conformal measures - outline of method

Want: $|\mathcal{F}_q(\nu)| = O\left(\frac{1}{|q|^\alpha}\right)$. Assume: $\Phi \subset C^2$ not conj. to linear.

Pick $k \approx \log|q|$.

Linearization $|\mathcal{F}_q(\nu)| \lessapprox \int \left|\mathcal{F}_{qe^{-S_{\tau_k}(\omega)}}(\nu)\right| d\mathbf{p}^{\mathbb{N}}(\omega)$

Use: self conformality.

Equidistribution $\exists \epsilon = \epsilon(\Phi, \mathbf{p})$, $\kappa = \kappa(\Phi, \mathbf{p}) \ll_C \lambda_{[0,1]}$ s.t.: dist $\left(S_{\tau_k} - k,\ \kappa\right) = O\left(e^{-k \cdot \epsilon}\right)$

Use: Renewal Theorem with exp. error term.

Related: Renewal Theorem of Li (2019).

$|\mathcal{F}_q(\nu)| \lessapprox \int_k^{k+1} \left|\mathcal{F}_{q \cdot e^{-z}}(\nu)\right| dz$ up to errors

High oscillations $\leq O\left(e^{-k \cdot \dim_\infty \nu \cdot \epsilon'}\right)$. Use: Lemma of Hochman.