Fourier decay for nonlinear pushforwards of self-similar measures

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[1]Based on joint work in progress with Han Yu

Fourier transform of measures

- **Fourier transform** of a measure μ on $\mathbb{R}^k$ is the function $\widehat{\mu}\colon \mathbb{R}^k \to \mathbb{C}$,
$$\widehat{\mu}(\xi) = \int_{\mathbb{R}^k} e^{-2\pi i \langle \xi, x \rangle} \, d\mu(x).$$

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- Decay rates of $|\widehat{\mu}(\xi)| \to 0$ as $|\xi| \to \infty$ have applications to normality of μ-typical points, Fourier uniqueness, fractal uncertainty principles, Fourier restriction...
- If μ is Cantor–Lebesgue measure,
$$\widehat{\mu}(1) = \widehat{\mu}(3) = \widehat{\mu}(9) = \cdots$$
but $|\widehat{\mu}(\xi)|$ decays outside a "zero-dimensional" set of ξ.

Nonlinear pushforwards

Principle: **nonlinear** dynamically defined measures often have polynomial Fourier decay.

Theorem (Kaufman, 1984)

If μ is Cantor–Lebesgue and $f(x) = x^2$ then the pushforward satisfies

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Conjecture (perhaps should be a question!)

Arbitrary $\eta < \frac{1}{2}\frac{\log 2}{\log 3} \approx 0.32$ works and $\operatorname{supp}(f_*\mu)$ is a Salem set.

Self-similar measures

Fix $r_1, \ldots, r_m \in (0,1)$, **commuting** $k \times k$ orthogonal maps $O_1, \ldots, O_m$, vectors $t_1, \ldots, t_m \in \mathbb{R}^k$, and weights $p_1, \ldots, p_m \in (0,1)$ with $p_1 + \cdots + p_m = 1$. The **self-similar measure** μ satisfies

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Notation:

- A: Assouad dim of $\mathrm{supp}(\mu)$
- B: Box/Hausdorff dim of $\mathrm{supp}(\mu)$
- F: Frostman exponent of μ
- $\kappa_p := \sup\left\{ s \geq 0 : \int_{B(0,R)} |\widehat{\mu}(\xi)|^p d\xi \ll R^{k-s} \right\}$: Fourier ℓ^p dimension of μ.

Always $\kappa_1 \leq F \leq \kappa_2 \leq B \leq A$.

If OSC & measure of max dim: $\kappa_1 \leq F = \kappa_2 = B = A$.

Quantitative decay

A Assouad; B Box; F Frostman; κ_p Fourier ℓ^p

For μ as above, $f\colon \mathbb{R}^k \to \mathbb{R}$ is C^2 with graph $\{x, f(x)\}_{x \in \mathbb{R}^k} \subset \mathbb{R}^{k+1}$ having **positive Gauss curvature** over supp(μ), and $2^n \ll |\xi| \ll 2^{2n}$,

$$|\widehat{f_* \mu}(\xi)| \lesssim \frac{1}{2^{Fn}} (2^{2n}/|\xi|)^A \sum_{D_0 \in \mathcal{C}_n} \int_{D_0} |\widehat{\mu}(\xi')| d\xi' + |\xi|/2^{2n}.$$

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theorem (B.–Yu)

If $F > k/2$ then $|\widehat{f_*\mu}(\xi)| \lesssim |\xi|^{-\eta}$ for $\eta > 0$ indep. of f. Can take any

$$\eta < \max\{\sigma(k-\kappa_1), \sigma((k+B-\kappa_2)/2)\}$$

where

$$\sigma(x) = \frac{\frac{F-x}{1+A-x}}{2-\frac{F-x}{1+A-x}}.$$

Missing digit measures

- Let μ_b be the natural self-similar measure on the set of numbers whose base-b expansion misses digit 0.
- Chow–Varjú–Yu (2024+): $\kappa_1 \to 1$ as $b \to \infty$.

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Corollary

If $f\colon \mathbb{R} \to \mathbb{R}$ has $f'' > 0$, then $|\widehat{f_*\mu_b}(\xi)| \lesssim |\xi|^{-\eta}$, where we can take $\eta = 1/3 - o_b(1)$.

If μ_b is Salem then one could take $\eta = 1/2 - o_b(1)$.

Nonlinear arithmetic

- Arithmetic product of $X, Y \subseteq \mathbb{R}$ is $X \cdot Y \coloneqq \{xy : x \in X, y \in Y\} \subseteq \mathbb{R}$.
- Multiplicative convolution $\mu \cdot \nu$ of measures on $\mathbb{R}$ is pushforward of $\mu \times \nu$ under $f(x, y) = xy$.

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- If X, Y are "large" (dimension), when does this imply $X \cdot Y$ is "large" (positive Lebesgue measure)?
- Generalised Marstrand projection theorem (Peres–Schlag, 2000): $K \subset \mathbb{R}^2$ compact, $\dim_{\mathrm{H}} K > 1$, P_a a family of smooth maps (satisfying conditions), then $P_a(K)$ has positive measure for "almost every" a.

Arithmetic of self-similar sets

When can exceptional directions be removed?

Theorem (Hochman–Shmerkin, 2012)

If self-similar set E has some contraction ratio s and F has a contraction ratio t with $\log s / \log t \notin \mathbb{Q}$ then $\dim_{\mathrm{H}}(A \cdot B) = \min\{\dim_{\mathrm{H}} A + \dim_{\mathrm{H}} B, 1\}$.

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Conjectures

- Let E, F be self-similar sets in $\mathbb{R}$. If $\dim_{\mathrm{H}} E + \dim_{\mathrm{H}} F > 1$, then $\mathrm{Leb}(E \cdot F) > 0$.
- Let μ, ν be self-similar measures on $\mathbb{R}$. If $\dim_{\mathrm{H}} \mu + \dim_{\mathrm{H}} \nu > 1$ then $\mu \cdot \nu$ is absolutely continuous.

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Note $f(x, y)$ is a **nonlinear** projection! Won't work for linear projections or $E + F$ (recall Ian Morris's talk).

Progress towards conjecture for measures

theorem (B.–Yu)

- If μ is self-similar with SSC and natural weights and
$$\dim_{\mathrm{H}} \mu > (\sqrt{65} - 5)/4 \approx 0.766...$$
then $\mu \cdot \mu$ has an L^2 density.
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Idea: apply quantitative theorem. If ν on $\mathbb{R}$ has $|\widehat{\nu}(\xi)| \lesssim |\xi|^{-1/2-\varepsilon}$ then $\int_{\mathbb{R}} |\widehat{\nu}(\xi)|^2 d\xi < \infty$. By Plancherel ν has an L^2 density.

Progress for sets

No separation conditions!

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- If $F \subset \mathbb{R}$ is self-similar with $\min\{\dim_{\mathrm{H}} E, \dim_{\mathrm{H}} F\} > (\sqrt{65} - 5)/4$ then $\mathrm{Leb}(F \cdot F) > 0$. Also if $R_{(a,b)}$ is the radial projection from (a, b) then $\mathrm{Leb}(R_{a,b}(F \times F)) > 0$ for **all** $(a, b) \notin F \times F$.

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- If $F_1, \ldots, F_n \subset \mathbb{R}$ are self-similar and $\sum_{i=1}^{n} \dim_{\mathrm{H}} F_i > 1 + (n-1)^{-1}$ then $\mathrm{Leb}(F_1 \cdot \ldots \cdot F_n) > 0$

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Proof: take a SSC subsystem with ε dimension loss, put natural measure on it, apply corollary for measures.

Qualitative decay

theorem (B.–Yu)

Let μ be a self-similar measure on $\mathbb{R}^k$ not supported in an affine subspace, with commuting linear parts.
Let U be an open neighbourhood of $\text{supp}(\mu)$.
Let $f\colon U \to \mathbb{R}^d$ be analytic; assume the graph of f is not contained in a proper affine subspace of $\mathbb{R}^{k+d}$.
Then $|f_*\mu(\xi)| \lesssim |\xi|^{-\eta}$ for some $\eta > 0$.

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Application: combining with Algom – Rodriguez Hertz – Wang (2023+,2024+) this proves that in many cases **nonlinear analytically-self-conformal measures** in dimension 1 and 2 have polynomial Fourier decay.

Thank you for listening!