Inverse and scattering problems on extension domains

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- G. Claret, M. Hinz, A. Rozanova-Pierrat, A. Teplyaev, Layer potential operators for transmission problems on extension domains, submitted, https://arxiv.org/pdf/2403.11601.pdf

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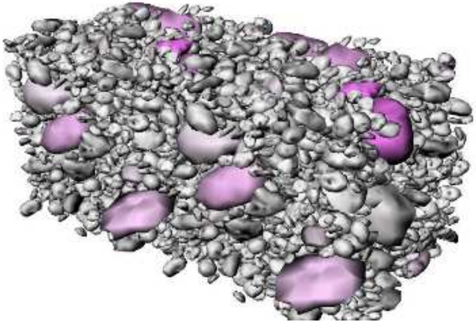
Nature complexity and their models

Porous materials



Traffic noise absorbing wall

“Fractal wall” TM, porous material is the cement-wood (acoustic absorbent),

Patent Ecole Polytechnique-Colas, Canadian and US patent

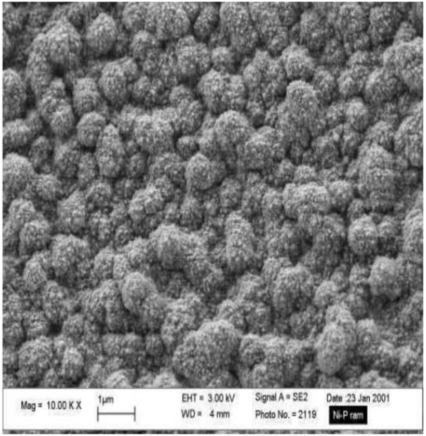
Acoustic anechoic chambers

Test anechoic chamber

Microsoft anechoic chamber -20db noise level,
the quietest place on earth

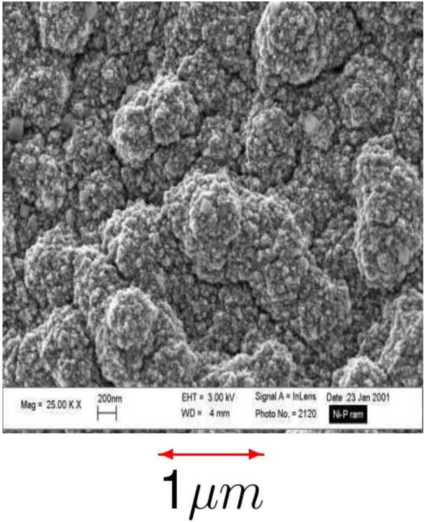
Test semi-anechoic chamber

Irregularity of boundaries

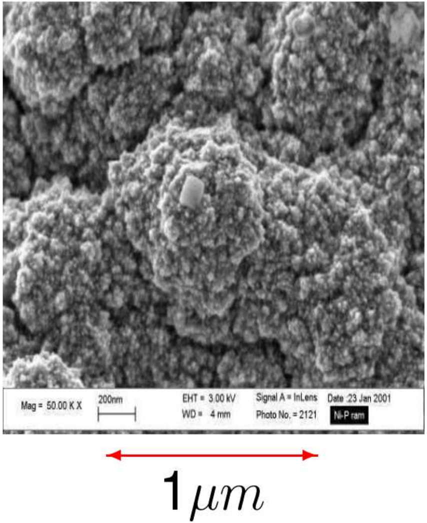


$1\,\mu m$

Irregularity of boundaries

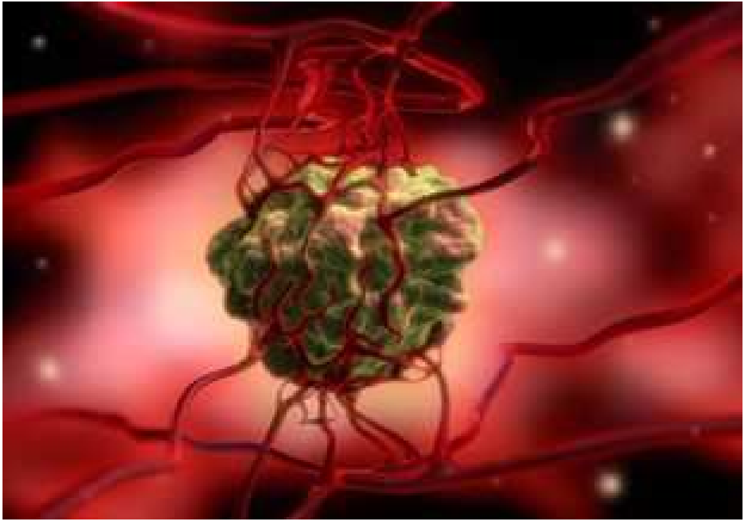


Irregularity of boundaries

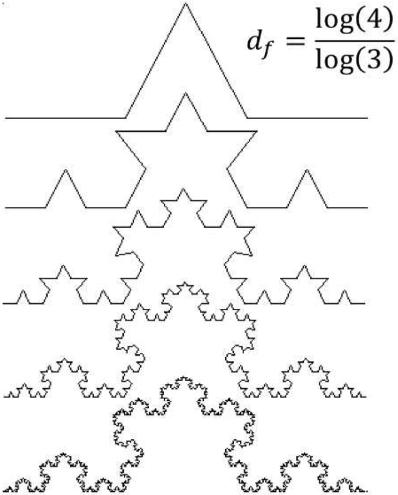


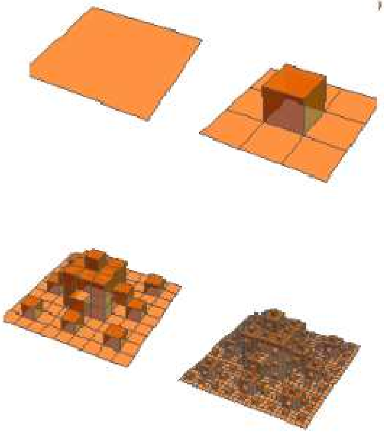
Irregularity of boundaries

Antigiogenesis of cancerous tumours



Examples of self-similar fractal boundaries





$2 < d = \frac{\log(13)}{\log(3)} \approx 2.33 < 3$ (Wikipedia)

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(One-sided-)extension and admissible domains

Definition

*A domain $\Omega \subset \mathbb{R}^n$ is called a H^1-**extension domain** if there exists a bounded linear extension operator $E_\Omega : H^1(\Omega) \to H^1(\mathbb{R}^n)$:*

$$\forall u \in H^1(\Omega) \quad \exists v = E_\Omega u \in H^1(\mathbb{R}^n) \text{ with } v|_\Omega = u \text{ and } C(\Omega) > 0 :$$

$$\|v\|_{H^1(\mathbb{R}^n)} \leq C\|u\|_{H^1(\Omega)}.$$

*H^1-extension domain Ω is called H^1-**admissible** if its boundary $\partial\Omega$ has positive capacity.*

Jones [1981]: If Ω is an uniform (or (ε, ∞)-) domain, then it is Sobolev extension domain.

Hajłasz, Koskela and Tuominen [2008]: $\Omega \subset \mathbb{R}^n$ is a H^1-extension domain $\Longleftrightarrow$ Ω is an n-set and $H^1(\Omega) = C^{1,2}(\Omega)$ (space of the fractional sharp maximal functions) with norms' equivalence.

Examples, remarks

Domains with boundaries $\partial\Omega$ as

- d-sets: $\dim_H \partial\Omega = d > 0$
  $\exists c_1, c_2 > 0,$

$$c_1 r^d \leq \mu(\partial\Omega \cap \overline{B_r(x)}) \leq c_2 r^d, \quad \text{for } \forall\, x \in \partial\Omega,\ 0 < r \leq 1,$$

- Lipschitz and more regular boundaries
- bounded dimension boundaries

$$n - 2 < \dim_H \partial\Omega \leq n$$

Trace operator

Proposition
For a H^1-admissible domain Ω of $\mathbb{R}^n$, given $u \in H^1(\Omega)$, let

$$\operatorname{Tr}_i u := (E_\Omega u)^\sim|_{\partial\Omega}$$

be the restriction of any quasi continuous representative $(E_\Omega u)^\sim$ of $E_\Omega u$. Then the ***(interior) trace operator***

$$\operatorname{Tr}_i : H^1(\Omega) \to \mathcal{B}(\partial\Omega)$$

is a well-defined linear surjection.

Consequently, q. e.

$$x \in \partial\Omega \qquad \operatorname{Tr}_i u(x) = \lim_{r\to 0} \frac{1}{\lambda^n(\Omega \cap B_r(x))} \int_{\Omega\cap B_r(x)} u(y)dy.$$

M. Biegert, 2009, Theorem 6.1, Remark 6.2 and Corollary 6.3

Trace theorem

Let $\Omega \subset \mathbb{R}^n$ be an H^1-admissible bounded domain.

$$H^1(\Omega) = H^1_0(\Omega) \oplus V_1(\Omega), \quad V_1(\Omega) = \{u \in H^1(\Omega) | -\Delta u + u = 0 \text{ weakly}\}$$

(i) The space $H^1_0(\Omega) := \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{H^1(\Omega)}}$ is the kernel of Tr_i, that is, $H^1_0(\Omega) = \ker \mathrm{Tr}_i$.

(ii) Endowed with the norm

$$\|f\|_{\mathcal{B}(\partial\Omega)} := \min\{\|v\|_{H^1(\Omega)} \mid v \in H^1(\Omega) \text{ and } \mathrm{Tr}_i\, v = f\}, \quad (1)$$

the space $\mathcal{B}(\partial\Omega)$ is a Hilbert space.

(iii) $\|\mathrm{Tr}_i\|_{\mathcal{L}(H^1(\Omega),\mathcal{B}(\partial\Omega))} = 1$.
Its restriction $\mathrm{tr}_i : V_1(\Omega) \to \mathcal{B}(\partial\Omega)$ to $V_1(\Omega)$ is an isometry and onto.

Green formula

Let $\Omega \subset \mathbb{R}^n$ be H^1-admissible.

$$H^1_\Delta(\Omega) := \left\{u \in H^1(\Omega) \;\middle|\; \Delta u \in L^2(\Omega)\right\}$$

Given $u \in H^1_\Delta(\Omega)$, there is a unique element g of $\mathcal{B}'(\partial\Omega)$ such that

$$\langle g, \mathrm{Tr}_i\, v\rangle_{\mathcal{B}'(\partial\Omega),\, \mathcal{B}(\partial\Omega)} = \int_\Omega (\Delta u) v \,\mathrm{d}x + \int_\Omega \nabla u \nabla v \,\mathrm{d}x, \quad v \in H^1(\Omega).$$

We call this element g the *weak interior normal derivative* of u (with respect to Ω) and denote it by $\frac{\partial_i u}{\partial \nu} := g$.

$\frac{\partial_i}{\partial \nu} : H^1_\Delta(\Omega) \to \mathcal{B}'(\partial\Omega)$ is linear and bounded:

$$\left\| \frac{\partial_i u}{\partial \nu} \right\|_{\mathcal{B}'(\partial\Omega)} \leq \|u\|_{H^1(\Omega)} + \|\Delta u\|_{L^2(\Omega)}.$$

(thanks to multiple works of M. R. Lancia (d-sets, Jonsson measures))

Some important corollaries

1. $\forall f \in \mathcal{B}(\partial\Omega)$ the assigment

$$\iota(f)(h) := \langle f, h \rangle_{\mathcal{B}(\partial\Omega)}, \quad h \in \mathcal{B}(\partial\Omega),$$

   defines an isometric isomorphism ι from $\mathcal{B}(\partial\Omega)$ onto $\mathcal{B}'(\partial\Omega)$.
   The dual pairing is defined by

$$\langle g, f \rangle_{\mathcal{B}'(\partial\Omega),\mathcal{B}(\partial\Omega)} := \langle \iota^{-1}(g), f \rangle_{\mathcal{B}(\partial\Omega)} = \langle g, \iota(f) \rangle_{\mathcal{B}'(\partial\Omega)}, \; f \in \mathcal{B}(\partial\Omega), \; g \in \mathcal{B}'(\partial\Omega).$$

   We may identify $\mathcal{B}(\partial\Omega)$ with its image $\iota(\mathcal{B}(\partial\Omega)) \subset \mathcal{B}'(\partial\Omega)$ under ι.
2. **Gelfand triple:**

$$\boxed{B(\partial\Omega) \hookrightarrow L^2(\partial\Omega, \mu) = (L^2(\partial\Omega, \mu))' \hookrightarrow B'(\partial\Omega), \quad B''(\partial\Omega) = B(\partial\Omega)}$$

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The adjoint trace operator $\mathrm{Tr}_i^* : \mathcal{B}'(\partial\Omega) \to (H^1(\Omega))'$, is defined:

$$\forall v \in H^1(\Omega), \quad \forall g \in \mathcal{B}'(\partial\Omega), \quad \langle g, \mathrm{Tr}_i \, v \rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)} = \langle \mathrm{Tr}_i^* \, g, v \rangle_{(H^1(\Omega))', H^1(\Omega)}.$$

Dirichlet type or harmonic extensions for $-\Delta + 1$ on admissible domains

$V_1(\Omega)$ is also the space of weak solutions of the **Dirichlet boundary-value problem**

$$\begin{cases} -\Delta u + u & = 0 \quad \text{in } \Omega \\ u|_{\partial\Omega} & = f \in \mathcal{B}(\partial\Omega) \end{cases}$$

$$E^D : \mathcal{B}(\partial\Omega) \to E^D(\mathcal{B}(\partial\Omega)) = V_1(\Omega) \subset H^1(\Omega)$$
$$f \mapsto u^f = E^D(f),$$

where u^f is the unique weak solution to the Dirichlet boundary problem

Proposition
$E^D : \mathcal{B}(\partial\Omega) \to V_1(\Omega)$ *is an isometry:* $\quad \forall f \in \mathcal{B}(\partial\Omega) \quad \|f\|_{\mathcal{B}(\partial\Omega)} = \|E^D f\|_{H^1(\Omega)}$.
In this sense $E^D = \mathrm{tr}_i^{-1}$.

Neumann problem for $-\Delta + 1$ on admissible domains

Let $\Omega \subset \mathbb{R}^n$ be an H^1-admissible bounded domain.

$$\begin{cases} -\Delta u + u & = 0 \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu}|_{\partial\Omega} & = g \in \mathcal{B}'(\partial\Omega) \end{cases}$$

$$\forall v \in H^1(\Omega) \quad \exists! u \in H^1(\Omega) \quad \langle u, v \rangle_{H^1(\Omega)} = \langle g, \mathrm{Tr}_i\, v \rangle_{\mathcal{B}'(\partial\Omega),\mathcal{B}(\partial\Omega)}.$$

$V_1(\Omega)$ is the space of the weak solutions of the **Neumann boundary value problem**.

$$E^N : \frac{\partial_i u}{\partial \nu} \in \mathcal{B}'(\partial\Omega) \mapsto u \in V_1(\Omega) \subset H^1(\Omega)$$

where u is the unique weak solution of the Neumann boundary value problem for $-\Delta + 1$, is an isometry, $(E^N)^{-1} = \frac{\partial}{\partial \nu_i}$ on $V_1(\Omega)$

$$\forall g \in \mathcal{B}'(\partial\Omega) \quad \|E^N g\|_{H^1(\Omega)} = \|\mathrm{tr}_i^* g\|_{(H^1(\Omega))'} = \|g\|_{\mathcal{B}'(\partial\Omega)}.$$

Corollary

Corollary

Let Ω be H^1-admissible.

(i) *Both $E^N : \mathcal{B}'(\partial\Omega) \to V_1(\Omega)$ and $\frac{\partial_i}{\partial\nu} : V_1(\Omega) \to \mathcal{B}'(\partial\Omega)$ are isometries and onto, and we have $\frac{\partial_i}{\partial\nu} = (E^N)^{-1}$ on $V_1(\Omega)$.*

(ii) *For $u, v \in V_1(\Omega)$ we have*

$$\left\langle \frac{\partial_i u}{\partial\nu}, \operatorname{tr}_i v \right\rangle_{\mathcal{B}'(\partial\Omega),\mathcal{B}(\partial\Omega)} = \langle u, v \rangle_{H^1(\Omega)} = \left\langle \frac{\partial_i v}{\partial\nu}, \operatorname{tr}_i u \right\rangle_{\mathcal{B}'(\partial\Omega),\mathcal{B}(\partial\Omega)}.$$

(iii) *The dual $\left(\frac{\partial_i}{\partial\nu}\right)^* : \mathcal{B}(\partial\Omega) \to (V_1(\Omega))'$ of $\frac{\partial_i}{\partial\nu}$ on $V_1(\Omega)$ is an isometry and onto.*

Poincaré-Steklov operator for admissible domains

Theorem
Let Ω be a H^1-admissible bounded domain and $k \in \mathbb{R} \setminus \sigma(-\Delta_D)$. Then the ***Poincaré-Steklov operator***

$$A : \mathcal{B}(\partial\Omega) \to \mathcal{B}'(\partial\Omega)$$
$$\operatorname{Tr} u \mapsto \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega}$$

associated with the weak solutions from

$$u \in H^1_\Delta(\Omega) := \left\{ v \in H^1(\Omega) \;\middle|\; \Delta v \in L^2(\Omega) \right\}$$

$$(-\Delta + k)u = 0 \text{ on } \Omega \quad \text{with} \quad \operatorname{Tr} u|_{\partial\Omega} = f \in \mathcal{B}(\partial\Omega), \tag{2}$$

is a linear bounded operator with $\operatorname{Ker} A \neq \{0\}$ *and it coincides with its adjoint.*

---

d-sets K. Arfi, A.R.-P. 2019, A.R.-P. 2020, Lipschitz case by W. Arendt, A. F. M. ter Elst 2011, 2015

Poincaré-Steklov operator as an isometry for admissible domains

Lemma

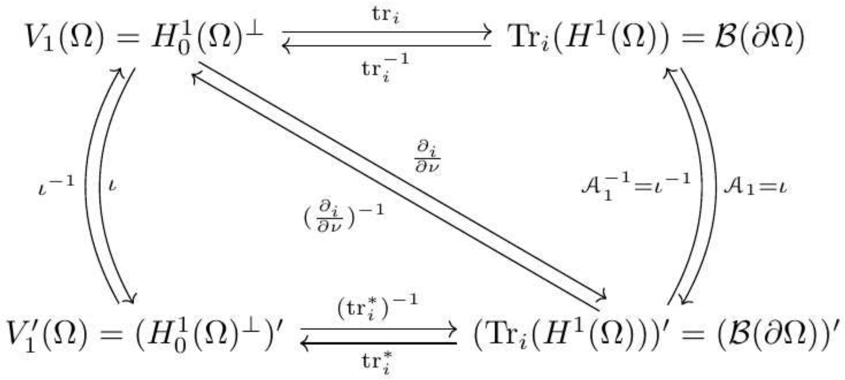
Let Ω be a bounded H^1-admissible domain.

(i) *For any $k \in \mathbb{R} \setminus \sigma(\Delta_D)$ the Poincaré-Steklov operator $A_k : \mathcal{B}(\partial\Omega) \to \mathcal{B}'(\partial\Omega)$ is injective if and only if k is not an eigenvalue of the self-adjoint Neumann Laplacian for Ω.*

(ii) *The Poincaré-Steklov operator $A_1 : \mathcal{B}(\partial\Omega) \to \mathcal{B}'(\partial\Omega)$ is an isometry and*

$$A_1 = \frac{\partial_i}{\partial \nu} \circ (\mathrm{tr}_i)^{-1}.$$

Different isometries



$(E^N)^{-1} = \frac{\partial}{\partial \nu_i}$, $E^D = \mathrm{tr}_i^{-1}$

$$\forall g \in \mathcal{B}'(\partial\Omega) \quad \|g\|_{\mathcal{B}'(\partial\Omega)} = \|\,\mathrm{tr}_i^*\, g\|_{(H^1(\Omega))'},$$
$$\forall u \in V_1(\Omega) \quad \|u\|_{H^1(\Omega)} = \|\,\mathrm{tr}_i\, u\|_{\mathcal{B}(\partial\Omega)}.$$

in complement to S. N. Chandler Wilde, D. P. Hewett, A. Moiola, 2017-...

Framework of harmonic problems for $-\Delta$

Definition

$$\dot{H}^1(\Omega) = \{u \in L^2_{loc}(\Omega) |\ \nabla u \in L^2(\Omega, \mathbb{R}^n) \textit{ modulo locally constant functions}\}$$

is the Hilbert space, endowed with the scalar product $\langle u, v\rangle_{\dot{H}^1(\Omega)} = \int_\Omega \nabla u \nabla v \, \mathrm{d}x$.

- A domain $\Omega \subset \mathbb{R}^n$ is an $\dot{H}^1$***-extension domain*** if there is a bounded linear extension operator $\dot{E}_\Omega : \dot{H}^1(\Omega) \to \dot{H}^1(\mathbb{R}^n)$.

Chen, Fukushima 2012; Deny-Lions 1954; Mazja 1985

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- Ω is $\dot{H}^1$***-admissible*** if 1) Ω is an $\dot{H}^1$-extension domain and 2) $\partial\Omega$ is compact and of positive capacity.

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Framework of harmonic problems for $-\Delta$

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- Ω is $\dot{H}^1$***-admissible*** if 1) Ω is an $\dot{H}^1$-extension domain and
  2) $\partial\Omega$ is compact and of positive capacity.
- If so, $\dot{\mathcal{B}}(\partial\Omega)$ is the vector space modulo constants of all q.e. equivalence classes of pointwise restrictions $\tilde{w}|_{\partial\Omega}$ of quasi continuous representatives $\tilde{w}$ of classes $u \in \dot{H}^1(\mathbb{R}^n)$.

Chen, Fukushima 2012; Deny-Lions 1954; Mazja 1985

Precisions on $\dot{H}^1(\Omega)$ and $\dot{\mathcal{B}}(\partial\Omega)$

(i) Let Ω be a bounded H^1-extension domain. Then

$$\dot{H}^1(\Omega) \approx \{u \in H^1(\Omega) | \int_\Omega u(x)\,\mathrm{d}x = 0\}$$

$$H^1(\Omega) \approx \dot{H}^1(\Omega) \oplus \mathbb{R}$$

(ii) Let Ω be a bounded H^1- and $\dot{H}^1$-admissible domain. Then

$$\mathcal{B}(\partial\Omega) \approx \dot{\mathcal{B}}(\partial\Omega) \oplus \mathbb{R}$$

Several Examples

For $n \geq 2$:

(i) **by JONES-1981:** Any (ε, δ)-domain $\Omega \subset \mathbb{R}^n$ is an H^1-domain; any (ε, ∞)-domain is $\dot{H}^1$-extension domain.

(ii) Any (ε, ∞)-domain $\Omega \subset \mathbb{R}^n$ with $\mathbb{R}^n \backslash \overline{\Omega} \neq \varnothing$ is H^1-admissible, and if one of the two open sets is bounded, it is also $\dot{H}^1$-admissible.

(iii) $\Omega = \mathbb{R}^n \backslash \{0\}$ is not H^1-admissible.

For $n = 1$:

(a) (a, b) with a or b finite is H^1-admissible;

(b) if a and b finite, (a, b) also $\dot{H}^1$-admissible.

Conductivity problem

Let Ω be a bounded $\dot{H}^1$-admissible domain. $\gamma \in L^\infty_{\gg 0}(\Omega)$ continuous near $\partial\Omega$.

$$\begin{cases} \nabla \cdot (\gamma \nabla u) = 0 & \text{on } \Omega, \\ \gamma \frac{\partial_i}{\partial \nu} u = g \in \dot{\mathcal{B}}'(\partial\Omega). \end{cases}$$

Variational formulation: $\forall v \in \dot{H}^1(\Omega), \quad (\gamma \nabla u, \nabla v)_{L^2(\Omega)} = \langle g, \operatorname{tr}_i v \rangle_{\dot{\mathcal{B}}'(\partial\Omega), \dot{\mathcal{B}}(\partial\Omega)}.$

$$A_\gamma : \operatorname{Tr} u|_{\partial\Omega} \mapsto \gamma \frac{\partial_i}{\partial \nu} u \bigg|_{\partial\Omega}$$

Calderón's problem

Knowing A_γ, can we recover γ?

Theorem
Let Ω be a bounded $\dot{H}^1$-admissible domain of $\mathbb{R}^n$ such that

$$\exists \delta, \rho > 0,\ \forall x_0 \in \partial\Omega,\ \forall r < \rho,\ \exists z \in \Omega^c, \quad \delta r < d(z, \partial\Omega) \leq |z - x_0| < r.$$

Let $\ell, L > 0$. Let $\gamma_1, \gamma_2 \in W^{1,\infty}(\Omega)$ be such that $\ell \leq \gamma_{1,2}$ and $\|\gamma_{1,2}\|_{W^{1,\infty}(\Omega)} \leq L$. Then, it holds

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\partial\Omega)} \leq c\, \|A_{\gamma_1} - A_{\gamma_2}\|_{\mathcal{L}(\dot{\mathcal{B}}(\partial\Omega), \dot{\mathcal{B}}'(\partial\Omega))},$$

where $c > 0$ depends on ℓ, L, n, $diam(\Omega)$ and c_n^Ω, where

$$\forall x \in \Omega,\ \forall r \in]0, 1], \quad \lambda^n(B_r(x) \cap \Omega) \geq c_n^\Omega\, r^n.$$

Generalization of: Alessandrini, *Singular solutions of elliptic equations and the determination of conductivity by boundary measurements*, 1991.

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Classical case: Lipschitz boundaries

Transmission problem:

$$\begin{cases} -\Delta u = 0 & \text{on } \mathbb{R}^n \backslash \partial\Omega, \\ [\![\operatorname{tr} u]\!] = f \in L^2_0(\partial\Omega, \lambda^{(n-1)}), \\ \left[\!\!\left[\dfrac{\partial u}{\partial \nu}\right]\!\!\right] = g \in L^2_0(\partial\Omega, \lambda^{(n-1)}). \end{cases}$$

Solution given by $u = \mathcal{S}_{\partial\Omega} g - \mathcal{D}_{\partial\Omega} f$ where:

$$\mathcal{S}_{\partial\Omega} g(x) := \int_{\partial\Omega} G(x-y) g(y)\, \lambda^{(n-1)}(\mathrm{d}y), \quad x \in \mathbb{R}^n$$

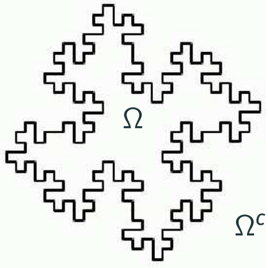
is the **single layer potential operator**, and

$$\mathcal{D}_{\partial\Omega} f(x) := \int_{\partial\Omega} \frac{\partial G}{\partial \nu_y}(x-y) f(y)\, \lambda^{(n-1)}(\mathrm{d}y), \quad x \in \mathbb{R}^n \backslash \partial\Omega$$

is the **double layer potential operator**, with G the fundamental solution to $-\Delta$ on $\mathbb{R}^n$.

Verchota, 1984.

Two-sided $\dot{H}^1$-admissible domains



$$\begin{cases} -\Delta u & = 0 \quad \text{on } \mathbb{R}^n \setminus \partial\Omega \\ u_i|_{\partial\Omega} - u_e|_{\partial\Omega} & = -f \in \dot{\mathcal{B}}(\partial\Omega) \\ \frac{\partial_i u_i}{\partial\nu}|_{\partial\Omega} - \frac{\partial_e u_e}{\partial\nu}|_{\partial\Omega} & = g \in \dot{\mathcal{B}}'(\partial\Omega) \end{cases}$$

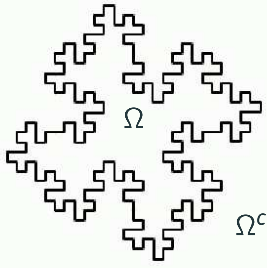
Definition

*$\Omega \subset \mathbb{R}^n$ is a **two-sided $\dot{H}^1$-admissible domain** if*

1. *$\Omega \neq \varnothing$ and $\Omega^c \neq \varnothing$ are $\dot{H}^1$-extension domains*
2. *the Lebesgue measure of $\partial\Omega$ is zero.*

$\Rightarrow \dim_{\mathcal{H}}(\partial\Omega) \geq n-1$, hence its capacity is positive.

Two-sided $\dot{H}^1$-admissible domains



$$\begin{cases} -\Delta u & = 0 \quad \text{on } \mathbb{R}^n \setminus \partial\Omega \\ u_i|_{\partial\Omega} - u_e|_{\partial\Omega} & = -f \in \dot{\mathcal{B}}(\partial\Omega) \\ \frac{\partial_i u_i}{\partial\nu}|_{\partial\Omega} - \frac{\partial_e u_e}{\partial\nu}|_{\partial\Omega} & = g \in \dot{\mathcal{B}}'(\partial\Omega) \end{cases}$$

$$\dot{H}^1(\mathbb{R}^n \backslash \partial\Omega) = \dot{H}^1(\Omega) \oplus \dot{H}^1(\Omega^c) = \dot{H}^1_0(\mathbb{R}^n \backslash \partial\Omega) \oplus \dot{V}_0(\mathbb{R}^n \backslash \partial\Omega)$$

$$u \in \dot{H}^1(\mathbb{R}^n \backslash \partial\Omega), \quad [\![\dot{\mathrm{T}}\mathrm{r}\, u]\!] := \dot{\mathrm{T}}\mathrm{r}_i\, u - \dot{\mathrm{T}}\mathrm{r}_e\, u$$

$$\dot{\mathrm{T}}\mathrm{r}_i : \dot{H}^1(\Omega) \to \dot{\mathcal{B}}(\partial\Omega), \quad \dot{\mathrm{T}}\mathrm{r}_e : \dot{H}^1(\Omega^c) \to \dot{\mathcal{B}}(\partial\Omega)$$

$\dot{\mathcal{B}}(\partial\Omega)$ is the Hilbert space with norm

$$\|f\|_{\dot{\mathcal{B}}(\partial\Omega),t} := \left(\|f\|^2_{\dot{\mathcal{B}}(\partial\Omega),i} + \|f\|^2_{\dot{\mathcal{B}}(\partial\Omega),e}\right)^{1/2}$$

Weak exterior normal derivative

Ω is a two-sided $\dot{H}^1$-admissible domain.

$$\forall v \in \dot{H}^1(\Omega), \quad \left\langle \frac{\dot{\partial}_i u}{\partial \nu}, \dot{\mathrm{Tr}}_i\, v \right\rangle_{\dot{\mathcal{B}}', \dot{\mathcal{B}}} = \int_\Omega (\Delta u) v \,\mathrm{d}x + \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x,$$

where $u \in \dot{H}^1(\Omega)$ with $\Delta u \in L^2(\Omega)$. Then, $\frac{\dot{\partial}_i u}{\partial \nu} \in \dot{\mathcal{B}}'(\partial\Omega)$.

$$\forall v \in \dot{H}^1(\Omega^c), \quad \left\langle \frac{\dot{\partial}_e u}{\partial \nu}, \dot{\mathrm{Tr}}_e\, v \right\rangle_{\dot{\mathcal{B}}', \dot{\mathcal{B}}} = -\int_{\Omega^c} (\Delta u) v \,\mathrm{d}x - \int_{\Omega^c} \nabla u \cdot \nabla v \,\mathrm{d}x,$$

where $u \in \dot{H}^1(\Omega^c)$ with $\Delta u \in L^2(\Omega^c)$. Then, $\frac{\dot{\partial}_e u}{\partial \nu} \in \dot{\mathcal{B}}'(\partial\Omega)$.
We denote $\left[\!\left[\frac{\dot{\partial} u}{\partial \nu}\right]\!\right] := \frac{\dot{\partial}_i u}{\partial \nu} - \frac{\dot{\partial}_e u}{\partial \nu}$.

Proposition

$\frac{\dot{\partial}_i}{\partial \nu} : \dot{H}^1_\Delta(\Omega) \to \dot{\mathcal{B}}'(\partial\Omega)$ and $\frac{\dot{\partial}_e}{\partial \nu} : \dot{H}^1_\Delta(\Omega^c) \to \dot{\mathcal{B}}'(\partial\Omega)$ are continuous.

M.R. Lancia, *A Transmission Problem with a Fractal Interface*, 2002.

Jump trace properties

Suppose that $\Omega \subset \mathbb{R}^n$ is a two-sided $\dot{H}^1$-admissible domain.

$$\dot{V}_{0,\dot{\mathcal{S}}}(\mathbb{R}^n \backslash \partial\Omega) := \{u \in \dot{V}_0(\mathbb{R}^n \backslash \partial\Omega) \mid [\![\dot{\mathrm{tr}}\, u]\!] = 0\}$$

and

$$\dot{V}_{0,\dot{\mathcal{D}}}(\mathbb{R}^n \backslash \partial\Omega) := \{u \in \dot{V}_0(\mathbb{R}^n \backslash \partial\Omega) \mid \left[\!\!\left[\frac{\dot{\partial} u}{\partial \nu}\right]\!\!\right] = 0\}.$$

Lemma

(i) $\dot{\mathrm{tr}} : \dot{V}_{0,\dot{\mathcal{S}}}(\mathbb{R}^n \backslash \partial\Omega) \to \dot{\mathcal{B}}(\partial\Omega)$, *is a linear isometry and onto.*

(ii) $\frac{\dot{\partial}}{\partial \nu} : \dot{V}_{0,\dot{\mathcal{D}}}(\mathbb{R}^n \backslash \partial\Omega) \to \mathcal{B}'(\partial\Omega)$, *defined as* $\frac{\dot{\partial}}{\partial \nu} := \frac{\dot{\partial}_i}{\partial \nu} = \frac{\dot{\partial}_e}{\partial \nu}$, *is a linear isometry and onto.*

(iii) $\dot{V}_0(\mathbb{R}^n \backslash \partial\Omega) = \dot{V}_{0,\dot{\mathcal{S}}}(\mathbb{R}^n \backslash \partial\Omega) \oplus \dot{V}_{0,\dot{\mathcal{D}}}(\mathbb{R}^n \backslash \partial\Omega)$.

Case of two-sided $\dot{H}^1$ admissible domains

Transmission problem:

$$\begin{cases} -\Delta u & = 0 \quad \text{on } \mathbb{R}^n \setminus \partial\Omega \\ u_i|_{\partial\Omega} - u_e|_{\partial\Omega} & = -f \in \dot{\mathcal{B}}(\partial\Omega) \\ \frac{\partial_i u_i}{\partial\nu}|_{\partial\Omega} - \frac{\partial_e u_e}{\partial\nu}|_{\partial\Omega} & = g \in \dot{\mathcal{B}}'(\partial\Omega) \end{cases} \tag{3}$$

Weak formulation:

$\forall v \in \dot{H}^1(\mathbb{R}^n \backslash \partial\Omega)$ with $[\![\operatorname{tr} v]\!] = 0$,
$u \in \dot{H}^1(\mathbb{R}^n \backslash \partial\Omega)$ a *weak solution* in the $\dot{H}^1$-sense if

$$\langle u, v\rangle_{\dot{H}^1(\mathbb{R}^n\backslash\partial\Omega)} = 0 \quad \forall v \in C_c^\infty(\mathbb{R}^n\backslash\partial\Omega),$$

$$\langle u, v\rangle_{\dot{H}^1(\mathbb{R}^n\backslash\partial\Omega)} = \langle g, \dot{\operatorname{tr}}\, v\rangle_{\dot{\mathcal{B}}'(\partial\Omega),\dot{\mathcal{B}}(\partial\Omega)} \quad \forall v \in \dot{V}_{0,\dot{\mathcal{S}}}(\mathbb{R}^n\backslash\partial\Omega) \text{ and}$$

$$\langle u, v\rangle_{\dot{H}^1(\mathbb{R}^n\backslash\partial\Omega)} = \langle [\![\frac{\dot{\partial} v}{\partial\nu}]\!], f\rangle_{\dot{\mathcal{B}}'(\partial\Omega),\dot{\mathcal{B}}(\partial\Omega)} \quad \forall v \in \dot{V}_{0,\dot{\mathcal{D}}}(\mathbb{R}^n\backslash\partial\Omega).$$

Case of two-sided $\dot{H}^1$ admissible domains

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$$\begin{cases} -\Delta u & = 0 \quad \text{on } \mathbb{R}^n \setminus \partial\Omega \\ u_i|_{\partial\Omega} - u_e|_{\partial\Omega} & = -f \in \dot{\mathcal{B}}(\partial\Omega) \\ \frac{\partial_i u_i}{\partial\nu}|_{\partial\Omega} - \frac{\partial_e u_e}{\partial\nu}|_{\partial\Omega} & = g \in \dot{\mathcal{B}}'(\partial\Omega) \end{cases} \tag{3}$$

For $g = 0$:

Lemma

Let $\Omega \subset \mathbb{R}^n$ be two-sided $\dot{H}^1$-admissible. For all $f \in \dot{\mathcal{B}}(\partial\Omega)$, $\exists!$ weak solution $u^f \in \dot{V}_{0,\dot{\mathcal{D}}}(\mathbb{R}^n \backslash \partial\Omega)$ of (3) in the $\dot{H}^1$-sense s.t. $\|u^f\|_{\dot{H}^1(\mathbb{R}^n \backslash \partial\Omega)} \leq \|f\|_{\dot{\mathcal{B}}(\partial\Omega)}$.

Double layer potential operator:

$$\dot{\mathcal{D}} : \dot{\mathcal{B}}(\partial\Omega) \to \dot{V}_{0,\dot{\mathcal{D}}}(\mathbb{R}^n \backslash \partial\Omega), \quad \dot{\mathcal{D}}f := u^f$$

is linear bounded bijective, and its inverse is $\dot{\mathcal{D}}^{-1} = -[\![\dot{\mathrm{tr}}]\!]$.

Case of two-sided $\dot{H}^1$ admissible domains

Transmission problem:

$$\begin{cases} -\Delta u & = 0 \quad \text{on } \mathbb{R}^n \setminus \partial\Omega \\ u_i|_{\partial\Omega} - u_e|_{\partial\Omega} & = -f \in \dot{\mathcal{B}}(\partial\Omega) \\ \frac{\partial_i u_i}{\partial\nu}|_{\partial\Omega} - \frac{\partial_e u_e}{\partial\nu}|_{\partial\Omega} & = g \in \dot{\mathcal{B}}'(\partial\Omega) \end{cases} \tag{3}$$

For $f = 0$:

Lemma

Let $\Omega \subset \mathbb{R}^n$ be two-sided $\dot{H}^1$-admissible. For all $g \in \dot{\mathcal{B}}'(\partial\Omega)$, $\exists!$ weak solution $u_g \in \dot{V}_{0,\dot{\mathcal{S}}}(\mathbb{R}^n \backslash \partial\Omega)$ of (3) in the $\dot{H}^1$-sense s.t. $\|u_g\|_{\dot{H}^1(\mathbb{R}^n\backslash\partial\Omega)} \leq \|g\|_{\dot{\mathcal{B}}'(\partial\Omega)}$.

Single layer potential operator:

$$\dot{\mathcal{S}} : \dot{\mathcal{B}}'(\partial\Omega) \to \dot{V}_{0,\dot{\mathcal{S}}}(\mathbb{R}^n \backslash \partial\Omega), \quad \dot{\mathcal{S}}g := u_g$$

is linear bounded bijective, and its inverse is $\dot{\mathcal{S}}^{-1} = [\![\frac{\partial}{\partial\nu}]\!]$.

Case of two-sided $\dot{H}^1$ admissible domains

Transmission problem:

$$\begin{cases} -\Delta u & = 0 \quad \text{on } \mathbb{R}^n \setminus \partial\Omega \\ u_i|_{\partial\Omega} - u_e|_{\partial\Omega} & = -f \in \dot{\mathcal{B}}(\partial\Omega) \\ \frac{\partial_i u_i}{\partial\nu}|_{\partial\Omega} - \frac{\partial_e u_e}{\partial\nu}|_{\partial\Omega} & = g \in \dot{\mathcal{B}}'(\partial\Omega) \end{cases} \tag{3}$$

Corollary

Let Ω be two-sided $\dot{H}^1$-admissible.

$$\forall f \in \dot{\mathcal{B}}(\partial\Omega) \text{ and } \forall g \in \dot{\mathcal{B}}'(\partial\Omega), \quad \exists! u \in \dot{V}_0(\mathbb{R}^n \setminus \partial\Omega)$$

in the $\dot{H}^1$-sense of (3) is given by ***Green's third identity:***

$$u = \dot{\mathcal{S}}g - \dot{\mathcal{D}}f$$

and satisfies

$$\|u\|_{\dot{H}^1(\mathbb{R}^n \setminus \partial\Omega)} \leq \|f\|_{\dot{\mathcal{B}}(\partial\Omega)} + \|g\|_{\dot{\mathcal{B}}'(\partial\Omega)}.$$

Representations of layer potentials

Let K denote the fundamental solution (or Green's function) to Δ on $\mathbb{R}^n$.
Given a finite signed Borel measure ν, $K * \nu := K * \nu^+ - K * \nu^-$ on $\mathbb{R}^n \backslash \partial\Omega$.

$$K * \nu(x) := \int_{\mathbb{R}^n} K(x-y)\nu(\mathrm{d}y), \quad x \in \mathbb{R}^n$$

We call ν *centered* if $\nu(\mathbb{R}^n) = 0$. The measure ν is of *finite energy* if $\exists c > 0$ such that

$$\int_{\partial\Omega} |v| \, \mathrm{d}|\nu| \leq c \, \|v\|_{H^1(\mathbb{R}^n)}, \quad v \in H^1(\mathbb{R}^n) \cap C_c(\mathbb{R}^n). \tag{4}$$

Proposition
Let $n \geq 2$ and let Ω be a two-sided $\dot{H}^1$-admissible domain in $\mathbb{R}^n$. Then

1. $\overline{\mathcal{S}} = \mathcal{I} \circ \overline{\mathrm{tr}}^*$,
   where the Newton potential operator $u \mapsto \mathcal{I}u = (|\xi|^{-2}\hat{u})^\vee$ extended to an isometric isomorphism $\dot{H}^{-1}(\mathbb{R}^n) \to \dot{H}^1(\mathbb{R}^n)$ ($\mathcal{I}\nu = K * \nu$).

Representations of layer potentials

Let K denote the fundamental solution (or Green's function) to Δ on $\mathbb{R}^n$.
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Proposition
Let $n \geq 2$ and let Ω be a two-sided $\dot{H}^1$-admissible domain in $\mathbb{R}^n$. Then

2. *Let ν be a centered finite signed Borel measure on $\partial\Omega$ of finite energy. Then sets of zero capacity have zero ν-measure, and ν defines an element of $\dot{\mathcal{B}}'(\partial\Omega)$ by*

$$\langle \nu, f \rangle_{\dot{\mathcal{B}}'(\partial\Omega), \dot{\mathcal{B}}(\partial\Omega)} := \int_{\partial\Omega} f \,\mathrm{d}\nu, \quad f \in \dot{\mathcal{B}}(\partial\Omega). \tag{5}$$

Some properties of the generalized layer potentials

The operators $\mathcal{S}_{\partial\Omega} : \dot{\mathcal{B}}'(\partial\Omega) \to \dot{H}^1(\mathbb{R}^n \backslash \partial\Omega)$ and $\mathcal{D}_{\partial\Omega} : \dot{\mathcal{B}}(\partial\Omega) \to \dot{H}^1(\mathbb{R}^n \backslash \partial\Omega)$ are **linear and continuous**.

Green's third identity: the unique weak solution $u \in \dot{H}^1(\mathbb{R}^n \backslash \partial\Omega)$ to (3) is

$$u := u_{\mathcal{S}} - u_{\mathcal{D}} = \mathcal{S}_{\partial\Omega} g - \mathcal{D}_{\partial\Omega} f.$$

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Green's third identity: the unique weak solution $u \in \dot{H}^1(\mathbb{R}^n \backslash \partial\Omega)$ to (3) is

$$u := u_{\mathcal{S}} - u_{\mathcal{D}} = \mathcal{S}_{\partial\Omega} g - \mathcal{D}_{\partial\Omega} f.$$

It holds:

$$\forall g \in \dot{\mathcal{B}}'(\partial\Omega), \quad \mathcal{S}_{\partial\Omega} g = G *_{\mathbb{R}^n} \mathrm{tr}^* g.$$

In particular, if μ is a ***d*-upper regular measure** on $\partial\Omega$ and if $g \in L^2(\partial\Omega, \mu)$:

$$\mathcal{S}_{\partial\Omega} g(x) := \int_{\partial\Omega} G(x-y) g(y) \, \mu(\mathrm{d}y), \quad x \in \mathbb{R}^n.$$

Neumann-Poincaré operator for $-\Delta$ with $[[\operatorname{tr} u]] = -f$ and $\left[\left[\frac{\partial u}{\partial \nu}\right]\right] = 0$

Definition
If Ω is a bounded two-sided-admissible domain of $\mathbb{R}^n$, let us define:

$\dot{\mathcal{K}} : \dot{\mathcal{B}}(\partial\Omega) \to \dot{\mathcal{B}}(\partial\Omega)$, *defined by*

$$\dot{\mathcal{K}} := \frac{1}{2}(\dot{\operatorname{tr}}_i + \dot{\operatorname{tr}}_e) \circ \dot{\mathcal{D}},$$

is the Neumann-Poincaré operator for the problem associated to $-\Delta$:

$$\begin{aligned} \dot{\mathcal{K}} : \dot{\mathcal{B}}(\partial\Omega) &\to \dot{\mathcal{B}}(\partial\Omega) \\ -f = [[\dot{\operatorname{tr}}\, u]] &\mapsto \frac{1}{2}(\dot{\operatorname{tr}}_i + \dot{\operatorname{tr}}_e)u = \frac{1}{2}(\dot{\operatorname{tr}}_i + \dot{\operatorname{tr}}_e) \circ \dot{\mathcal{D}} f. \end{aligned}$$

Adjoint Neumann-Poincaré operator for $-\Delta$ with $g = \left[\!\left[\frac{\partial u}{\partial \nu}\right]\!\right]$ and $[\![\mathrm{tr}(u)]\!] = 0$

$\dot{\mathcal{K}}^* : \dot{\mathcal{B}}'(\partial\Omega) \to \dot{\mathcal{B}}'(\partial\Omega)$ denotes the adjoint operator to $\dot{\mathcal{K}} : \dot{\mathcal{B}}(\partial\Omega) \to \dot{\mathcal{B}}(\partial\Omega)$.

Theorem

Let Ω be two-sided $\dot{H}^1$-admissible. Then

(i) $\dot{\mathrm{tr}}_i \circ \dot{\mathcal{D}} = -\frac{1}{2}I + \dot{\mathcal{K}}$ *and* $\dot{\mathrm{tr}}_e \circ \dot{\mathcal{D}} = \frac{1}{2}I + \dot{\mathcal{K}}$.

(ii) $\frac{\dot{\partial}_i}{\partial \nu} \circ \dot{\mathcal{S}} = \frac{1}{2}I + \dot{\mathcal{K}}^*$ *and* $\frac{\dot{\partial}_e}{\partial \nu} \circ \dot{\mathcal{S}} = -\frac{1}{2}I + \dot{\mathcal{K}}^*$. *In particular,*

$$\dot{\mathcal{K}}^* = \frac{1}{2}\left(\frac{\dot{\partial}_i}{\partial \nu} + \frac{\dot{\partial}_e}{\partial \nu}\right) \circ \dot{\mathcal{S}}$$

Moreover, $\dot{\mathcal{K}} : \dot{\mathcal{B}}(\partial\Omega) \to \dot{\mathcal{B}}(\partial\Omega)$ *and* $\dot{\mathcal{K}}^* : \dot{\mathcal{B}}'(\partial\Omega) \to \dot{\mathcal{B}}'(\partial\Omega)$ *are linear and continuous.*

Spectral properties of the Neumann-Poincaré operator on $\dot{\mathcal{B}}(\partial\Omega)$

Lipschitz case on L^2 of G. Verchota 1984

Theorem

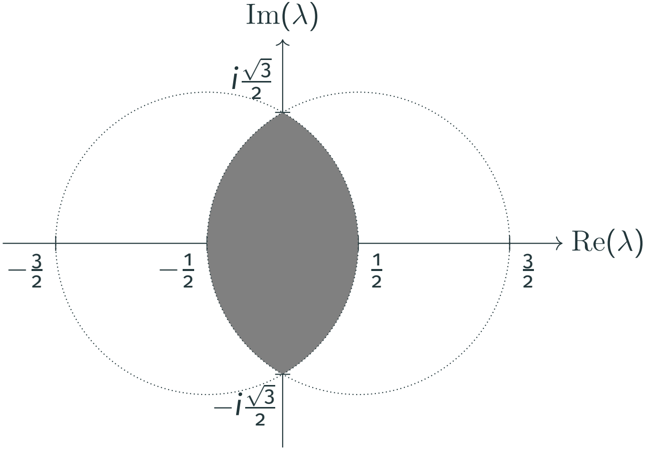
Let Ω be two-sided $\dot{H}^1$-admissible.

For $\lambda \in \mathbb{C}$, if $|\lambda - \frac{1}{2}| \geq 1$ or $|\lambda + \frac{1}{2}| \geq 1$, then the operators $\lambda \mathbf{I} + \dot{\mathcal{K}}$ and $\lambda \mathbf{I} + \dot{\mathcal{K}}^*$ are invertible on $\dot{\mathcal{B}}(\partial\Omega)$ and $\dot{\mathcal{B}}'(\partial\Omega)$ respectively.

In particular, their real spectra are included in $(-\frac{1}{2}, \frac{1}{2})$.

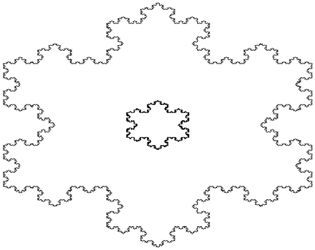
Spectral properties of the Neumann-Poincaré operator on $\dot{\mathcal{B}}(\partial\Omega)$

Lipschitz case on L^2 of G. Verchota 1984



Two-phased transmission problem Lipschitz case of H. Ammari, H. Kang 2004

Let (Ω, μ) and (D, η) be two-sided-admissible domains of $\mathbb{R}^n$, $D \subset\subset \Omega$, $k \in]0, 1[\cup]1, +\infty[$.



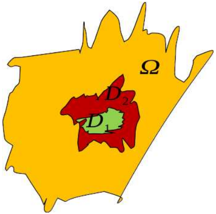
$$\begin{cases} \nabla \cdot \Big(\big(1 + (k-1)\mathbb{1}_D\big) \nabla u \Big) = 0 \quad \text{on } \Omega, \\ \dfrac{\partial_i u}{\partial \nu}\Big|_{\partial\Omega} = g \in \dot{\mathcal{B}}'(\partial\Omega) \end{cases} \tag{6}$$

Subdomain identification: uniqueness with one measurement in the monotone case

Theorem

Let $D_1 \subset D_2 \subset\subset \Omega$ *be tree bounded two-sided-admissible domains of* $\mathbb{R}^n$. *Let* $k \in]0, 1[\cup]1, +\infty[$ *and* u_1 *and* u_2 *be the solutions to the two-phased transmission problem, respectively associated to* D_1 *and* D_2.

If, for some Neumann condition $g \in \dot{\mathcal{B}}'(\partial\Omega)\backslash\{0\}$, $\mathrm{tr}_i^{\partial\Omega} u_1 = \mathrm{tr}_i^{\partial\Omega} u_2$, *then* $D_1 = D_2$.



Conclusion

Results independent on the boundary measure

Poincaré-Steklov and layer potentials on such boundaries

Transmission problem and imagery application by the Neumann-Poincaré operator

Thank you very much for your attention!