Assouad-like Φ-dimensions and Cantor sets

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Those I have had the great pleasure to work with on this:

Cabrelli, Molter, Shonkwiler, Hare, Zubermann, García

Overview

I will talk about these "generalized" Assouad-like dimensions, but I decided to mostly talk about the context in which we stumbled upon them.

This will hopefully show one natural context in which they provide useful information.

The talk is deliberately not highly technical.

Linear compact sets

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Let $a_i = |\mathcal{O}_i|$; these are the *gap lengths*. We usually assume that $a_1 \geq a_2 \geq a_3 \geq a_4 \geq \cdots$.

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By our assumptions, $\sum_i a_i = 1$.

Rearrangements

Clearly E is determined by both the gap lengths $\{a_i\}$ and the locations of the $\mathcal{O}_i$.

Let $\mathscr{C}_a$ be the set of all such sets E with gap lengths $\{a_i\}$.

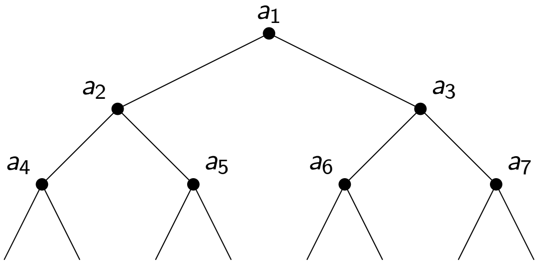
Any two elements of $\mathscr{C}_a$ are *rearrangements* of each other (that is, we have rearranged the gaps to construct one from the other).

How to specify $E \in \mathscr{C}_a$

There are (at least) two useful ways of specifying $E \in \mathscr{C}_a$.

The first is to label the nodes of a binary tree by the lengths a_n (or use 0 to remove the node) and arrange the gaps accordingly.

The "Cantor" arrangement – $\mathcal{C}_a$



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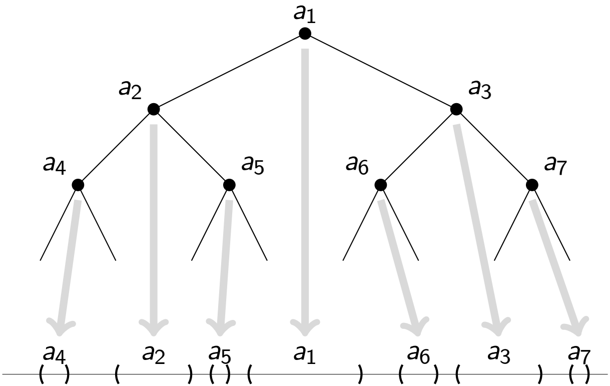

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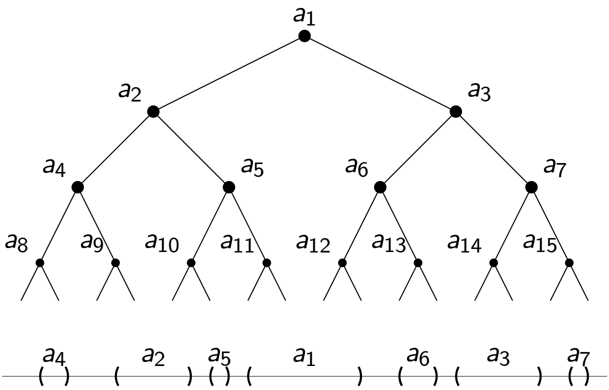

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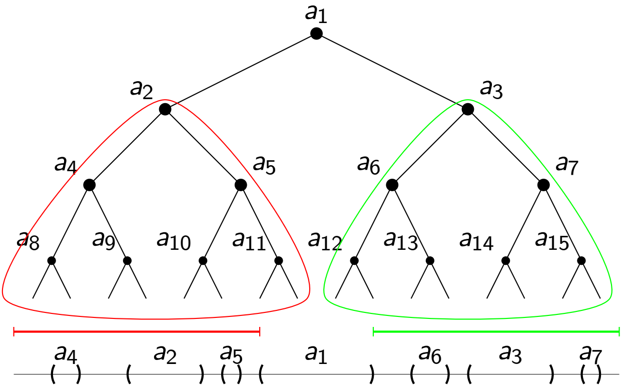

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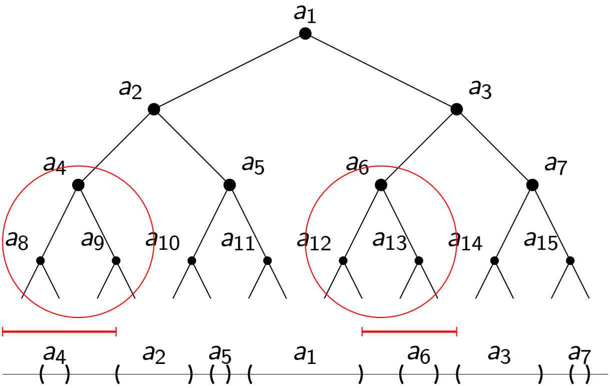

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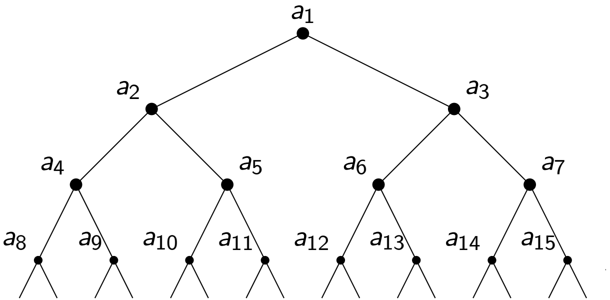
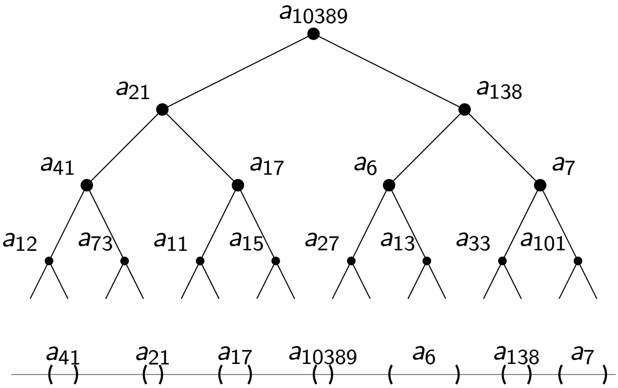
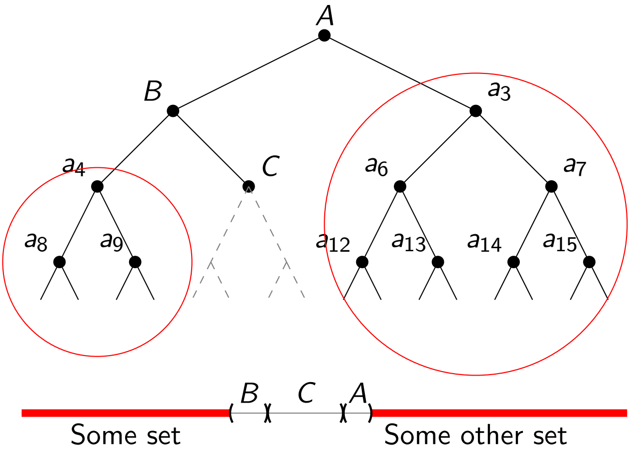


This makes a perfect set which is as "balanced" and homogeneous as possible (given a_n).

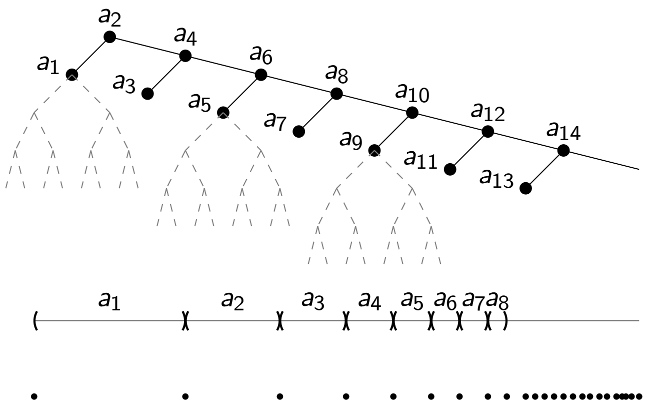
Another arrangement which yields a perfect set



An arrangement with isolated points



The "decreasing" arrangement – $\mathcal{D}_a$



It is perhaps worth mentioning that the mapping from labeled tree to compact set E is not injective, though this won't matter to us.

How to specify $E \in \mathscr{C}_a$

There are (at least) two useful ways of specifying $E \in \mathscr{C}_a$.

The first is to label the nodes of a binary tree by the lengths a_n and arrange the gaps accordingly.

The second is to see that any arrangement of the $\mathcal{O}_i$ induces a linear order on $\mathbb{N}$.

Arrangement via total order on $\mathbb{N}$

Every $E \in \mathscr{C}_a$ defines a total order on $\mathbb{N}$ by $i \prec j$ iff $x < y$ for all $x \in \mathcal{O}_i$ and $y \in \mathcal{O}_j$.

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Conversely given a total order $\prec$ on $\mathbb{N}$ and the lengths $\{a_i\}$, we can construct a rearrangement set $E \in \mathscr{C}_a$ by

$$E = \{\sum_{i \in L} a_i : L, R \text{ a } \mathit{cut} \text{ of } \mathbb{N}\}.$$

The location of this point is defined by the gaps which are to the left of it.

Arrangement via total order on $\mathbb{N}$

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$\mathcal{C}_a$ corresponds to the order (given in binary) $1\omega 0\alpha \prec 1\omega \prec 1\omega 1\beta$ where ω, α, β are finite binary words.

No element of $\mathbb{N}$ has an immediate successor or predecessor under this order.

Something to think about if you need a distraction

What is a nice/useful/convenient way of specifying a compact $E \subset [0, 1]$ of positive Lebesgue measure?

That is, what is a nice way to encode the additional information needed to specify the mass distribution.

Box dimensions and $\mathscr{C}_a$

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This is a consequence of the relation between $\dim_B E$ and E_ϵ.

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Hausdorff dimension of rearrangements

In 1954 Besicovitch and Taylor proved:

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If

1. $0 < s < \dim_H(\mathcal{C}_a)$ and $0 \leq \gamma \leq \infty$, or
2. $s = \dim_H(\mathcal{C}_a)$ and $0 \leq \gamma \leq \mathcal{H}^s(\mathcal{C}_a)$

then there is $E \in \mathscr{C}_a$ with $\mathcal{H}^s(E) = \gamma$.

Thus $\{\dim_H(E) : E \in \mathscr{C}_a\} = [0, \dim_H(\mathcal{C}_a)]$

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Their proof of this quite nice.

From general results, there exists $F \subset \mathcal{C}_a$ with $\mathcal{H}^s(F) = \gamma$.

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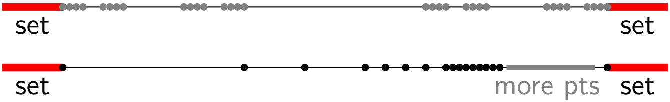
From general results, there exists $F \subset \mathcal{C}_a$ with $\mathcal{H}^s(F) = \gamma$.

The "gaps" of F are "unions of $\mathcal{O}_i$'s". We add points to F in a decreasing order to get the missing gaps so that $E \in \mathcal{C}_a$.



Hausdorff dimension of $\mathcal{C}_a$

When $C \subset \mathbb{R}$ is a *central* Cantor set, $\underline{\dim}_B C = \dim_H C$.

This can be used to show $\underline{\dim}_B \mathcal{C}_a = \dim_H \mathcal{C}_a$, which is how we got the previous formula for $\underline{\dim}_B E$.

Packing dimension of rearrangements

$\mathcal{P}_0^s(E) \leq 2\mathcal{P}_0^s(\mathcal{D}_a)$ and $\dim_P(E) \leq \dim_P(\mathcal{C}_a)$ for all $E \in \mathscr{C}_a$.

If $0 < s < \dim_P(\mathcal{C}_a)$ and $0 \leq \gamma \leq \infty$ then there is some $E \in \mathscr{C}_a$ with $\mathcal{P}^s(E) = \gamma$.

Thus, again, $\{\dim_P(E) : E \in \mathscr{C}_a\} = [0, \dim_P(\mathcal{C}_a)]$ (and with a similar proof).

In fact, $\dim_P(\mathcal{C}_a) = \overline{\dim}_B(\mathcal{C}_a) = \overline{\lim} \frac{\log m}{-\log a_m}$.

Assoud dimensions

The Assouad dimensions are related to the extreme local behaviour of the box-counting dimensions.

The upper dimension was defined by Assouad to study the problem of embedding metric spaces in $\mathbb{R}^n$.

They have been the subject of intensive study recently in the fractals literature (in particular by Fraser and his collaborators/students; see his recent book *Assouad dimension and fractal geometry*).

Assouad dimensions: "localize" in space and scale

Choose x and $0 < r < R$

Cover $B(x, R) \cap F$ with $B(x_i, r)$

Compare $N(x, r, R)$ to $(R/r)^d$

Extremize d over r, x, R as $R \to 0$

Assouad dimensions

Let $N_r(S)$ be the minimal number of balls of radius $r > 0$ which cover S.

(Upper) Assouad dimension

$$\dim_A(F) = \inf\{\alpha > 0 : \exists C, \rho > 0, \forall 0 < r < R \leq \rho, \sup_{x \in F} N_r(B(x,R) \cap F) \leq C \left(\frac{R}{r}\right)^{\alpha}\}.$$

$\dim_A(F)$ gives the largest local growth rate of N_r between any two scales.

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Lower (Assouad) dimension

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Assoud cousin – the quasi-Assouad dimension

The quasi-Assouad dimension is a refinement of the Assouad dimension and is preserved under quasi-Lipschitz mappings (unlike $\dim_A$).

$\dim_{qA} F = \lim_{\delta \to 0} h_\delta(F)$ where h_δ is defined like $\dim_A$ but with $0 < r < R^{1+\delta}$.

$\dim_{qA} F$ measures the largest local growth rate of N_r but only between two scales which are "far enough" apart.

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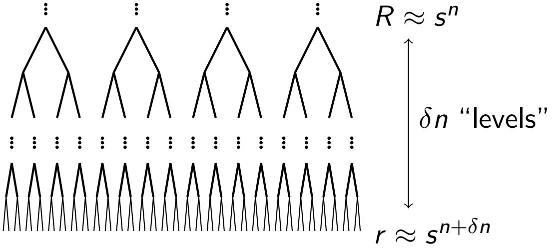
$\dim_{qA} F = \lim_{\delta \to 0} h_\delta(F)$ where h_δ is defined like $\dim_A$ but with $0 < r < R^{1+\delta}$.

There is also a dual lower version, $\dim_{qL} F$.

$$\dim_L(F) \le \dim_{qL}(F) \le \dim_H(F) \le \underline{\dim}_B(F) \le \overline{\dim}_B(F) \le \dim_{qA}(F) \le \dim_A(F)$$

Assoud cousin – the quasi-Assouad dimension

If $R \approx s^n$ then $R^{1+\delta} \approx s^{n+\delta n}$.



So $\dim_{qA} F$ “reaches deep into the tree” for its comparisons.

Assouad dimensions of rearrangements

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$\{\dim_A(E) : E \in \mathscr{C}_a\} = [\dim_A \mathcal{C}_a, \dim_A \mathcal{D}_a]$ $(= \{0\}$ if $\dim_A \mathcal{D}_a = 0)$.

The proof is constructive and works by building "approximate" discrete central Cantor sets of the appropriate dimension.

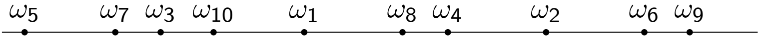
This requires a_n to be *doubling*: $a_n \leq \kappa a_{2n}$.

Random rearrangements

Random rearrangements: probability model (Hawkes, 84)

Take $\boldsymbol{\omega} = (\omega_n) \in [0,1]^{\mathbb{N}}$ with Lebesgue measure on each factor.

$\boldsymbol{\omega}$ defines a random total order $\prec_{\boldsymbol{\omega}}$ on $\mathbb{N}$ by $i \preceq_{\boldsymbol{\omega}} j$ iff $\omega_i \leq \omega_j$.



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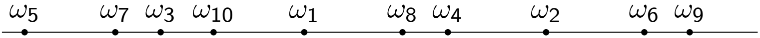

This gives a "uniformly" random choice of sets from $\mathscr{C}_a$.

The random rearrangement will almost surely be a perfect set.

Since dimensional calculations are permutable events (only depend on very fine scales), each dimension will have a constant value almost surely.

Random rearrangements

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This requires some regularity of a_n (doubling is fine).

In fact, (Hu 1995) proved that for $a_n = 1/3, 1/9, 1/9, 1/27, \ldots$, and $d = \log 2/\log 3$, the function $\varphi(x) = x^d(\log\log(1/x))^{1-d}$ is the a.s. exact Hausdorff dimension function.

Random rearrangements for doubling $\{a_n\}$

$$\left.\begin{aligned}\dim_A(E) &= \dim_A(\mathcal{D}_a)\\ \dim_{qA}(E) &= \dim_{qA}(\mathcal{C}_a)\end{aligned}\right.\quad \text{a.s. (Garcia, Hare, M)}$$

To show this, we used an equivalent model of the randomness, where the “levels” are more explicit.

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To show this, we used an equivalent model of the randomness, where the "levels" are more explicit.

The key is that in Hawkes' model the restriction of $\prec_\omega$ to $\{1, 2, \ldots, N\}$ gives each permutation equally likely.

We build the order on $\mathbb{N}$ in stages, randomly "inserting" the "new" elements $\{n+1, n+2, \ldots, n+m\}$ between the already ordered $\{1, 2, \ldots, n\}$.

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2 3 **1**

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2 3 **1**

6 **2** 7 4 **3** **1** 5

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											1		
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11	**6**	**2**	13	8	12	**7**	**4**	10	**3**	9	**1**	15	**5**

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The "places" (between **old ones**) where we insert the new elements turns out to be more important than their order and follow a sequence of independent multinomial random variables.

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If it is a small number of levels (= inserting only a few new points), then extreme things can happen.

If it is a large number of levels (= inserting a very large number of new points), then the behaviour is close to the "average" and thus (roughly) the CLT takes over.

Φ-dimensions – (finally we get there!)

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The idea for $\dim_\Phi F$ is to have fine control of how "deep" you look into the tree.

In particular, we were interested in the "shallow depths" between the Assouad dimension and quasi-Assouad.

This arose for us exactly in this problem of the a.s. dimension of random rearrangements.

It was also suggested by Fraser and co-authors.

Φ-dimensions: "localize" in space and deep enough scale

Choose x and $0 < r < R^{1+\Phi(R)}$

Cover $B(x, R) \cap F$ with $B(x_i, r)$

Compare $N(x, r, R)$ to $(R/r)^d$

Extremize d over r, x, R as $R \to 0$

Φ-dimensions

We call $\Phi : (0,1) \to (0,\infty)$ a *dimension function* if $x^{1+\Phi(x)}$ decreases as $x \searrow 0$.

Examples include $\Phi(x) = \delta$, $\Phi(x) = 1/|\log x|$, and $\Phi(x) = \log|\log(x)|/|\log(x)|$.

Φ-dimensions

Let $N_r(S)$ be the minimal number of balls of radius $r > 0$ which cover S.

Upper Φ-dimension:

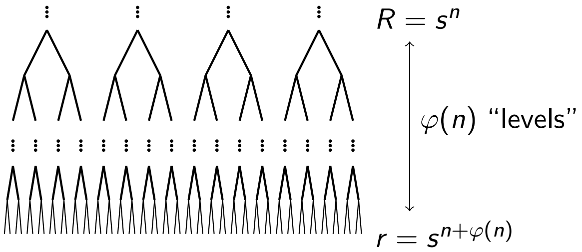
$$\overline{\dim}_\Phi(F) = \inf\{\alpha > 0 : \exists C, \rho > 0, \forall 0 < r < R^{1+\Phi(R)} \leq R \leq \rho, \sup_{x\in F} N_r(B(x,R)\cap F) \leq C\left(\frac{R}{r}\right)^\alpha\}.$$

Lower Φ-dimension:

$$\underline{\dim}_\Phi(F) = \sup\{\alpha > 0 : \exists C, \rho > 0, \forall 0 < r < R^{1+\Phi(R)} \leq R \leq \rho, \inf_{x\in F} N_r(B(x,R)\cap F) \geq C\left(\frac{R}{r}\right)^\alpha\}.$$

"Depth function" for Φ

If $R = s^n$ then $R^{1+\Phi(R)} = s^{n+\varphi(n)}$.



So $\dim_\Phi$ "reaches $\varphi(n)$-deep into the tree" for its comparisons.

Examples of the depth function $\varphi(n)$ for $R = s^n$

$\Phi(x) = \delta$ results in $\varphi(n) \sim \delta n$, like the Assouad spectrum.

$\Phi(x) = c/|\log(x)|$ results in $\varphi(n) \sim C$, like the Assouad dimension.

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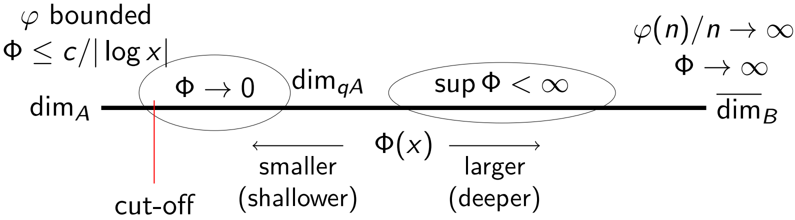
$\Phi(x) = \log|\log(x)|/|\log(x)|$ results in $\varphi(n) \sim c \log(n)$.

(*This is the cut-off depth for a phase change in the behaviour of iid random "1-variable" constructions and for the random rearrangement problem.*)

Comparing dimensions

$$\dim_L E \le \underline{\dim}_\Phi E \le \underline{\dim}_B E \le \overline{\dim}_B E \le \overline{\dim}_\Phi E \le \dim_A E.$$

If $\Phi(x) \le \Psi(x) \implies \overline{\dim}_\Psi E \le \overline{\dim}_\Phi E$ larger Φ are closer to box dim



quasi-Assouad dimension is a Φ-dimension

Given E, there is a Φ, $\Phi(x) \xrightarrow{x \to 0} 0$, with $\dim_{qA} E = \overline{\dim}_{\Phi} E$.

However, Φ depends on E.

Different Φ are different

Suppose $\Phi_1(x) \geq (1+\delta)\Phi_2(x)$ for small x.

Also suppose $\Phi_2(x)|\log(x)| \to \infty$ (so that Φ_2 is "deeper" than the Assouad, in particular that $\varphi(n) \to \infty$).

Then there is a Cantor set F so that $\overline{\dim}_{\Phi_1} F < \overline{\dim}_{\Phi_2} F$.

Central Cantor set with a continuum of different Φ-dimensions

By varying the scaling ratios in the construction of a central Cantor set one can precisely control the Φ-dimensions of the set.

With very careful control, we can specify $\dim_\Phi \mathcal{C}$ for a continuous family of Φ.

Take $d : (0,1) \to [\alpha, \beta] \subset (0,1)$ continuous and decreasing and Φ_p, $p \in (0,1)$, be a "continuous increasing" family.

Then there is a central Cantor set $\mathcal{C}$ with $\overline{\dim}_{\Phi_p} \mathcal{C} = d(p)$ for all $p \in (0,1)$.

Back to random rearrangements

If Φ is a "small" (shallow) dimension function, then almost surely

$$\overline{\dim}_\Phi E = \dim_A \mathcal{D}_a = 1.$$

If Φ is a "large" (deep) dimension function, then almost surely

$$\overline{\dim}_\Phi E = \dim_\Phi \mathcal{C}_a.$$

Thank you for listening!

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Questions?