On support measures and complex dimensions of fractals

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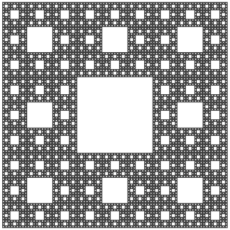
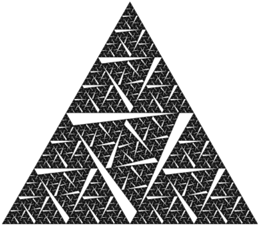
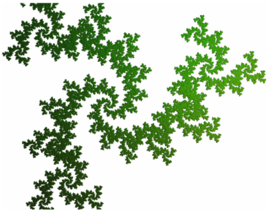
How to distinguish sets with same fractal dimension?

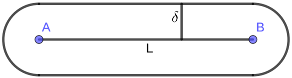
- Possible to find some new parameters which detect/quantify geometric differences between the sets?
- Define a geometric function on the parallel set A_ε and investigate behavior as $\varepsilon \to 0^+$
- For instance: Minkowski dimension and content.
- Also:
  - fractal curvature measures and associated scaling exponents
  - complex dimensions via fractal zeta functions

Approximation by parallel sets

- For $A \subset \mathbb{R}^d$ and $\varepsilon > 0$ let

$$A_\varepsilon := \{x \in \mathbb{R}^n : \operatorname{dist}(x, A) \leq \varepsilon\}$$

be the ε-parallel set of A.



- upper s-dimensional outer Minkowski content of $A \subset \mathbb{R}^d$:

$$\overline{\mathcal{M}}^{\mathrm{out,s}}(A) := \limsup_{\varepsilon \searrow 0} \varepsilon^{s-d} V_d(A_\varepsilon \setminus A)$$

- (outer) Minkowski dimension:

$$\overline{\dim}_M^{\mathrm{out}} A := \inf\{s \geq 0 : \overline{\mathcal{M}}^{\mathrm{out,s}}(A) = 0\}$$
$$= \sup\{s \geq 0 : \overline{\mathcal{M}}^{\mathrm{out,s}}(A) = \infty\}$$

Steiner formulas

- Classical: $\forall$ convex $K \subset \mathbb{R}^d$ $\exists$ $C_0(K), \ldots, C_{d-1}(K)$, s.t. $\forall\, \varepsilon \geq 0$,

$$V_d(K_\varepsilon \setminus K) = \sum_{i=0}^{d-1} \kappa_{d-i} \varepsilon^{d-i} C_i(K).$$

- General Steiner formula **[Hug, Last, Weil 04]**: For any compact $A \subset \mathbb{R}^d$, $\exists$ signed measures $\mu_0(A;\cdot), \ldots, \mu_{d-1}(A;\cdot)$ s.t. $\forall\, \varepsilon > 0$

$$V_d(A_\varepsilon \setminus A) = \sum_{i=0}^{d-1} \frac{\kappa_{d-i}}{d-i} \int_0^\varepsilon t^{d-i-1} \int_{N(A)} \mathbb{1}_{\{t<\delta(A,x,u)\}}\, \mu_i(A; \mathrm{d}(x,u))\mathrm{d}t.$$

- fractal tube formulas **[Lapidus, Radunovic, Zubrinic 17]**: For certain compact sets $A \subset \mathbb{R}^d$

$$V_d(A_\varepsilon \setminus A) = \sum_{w \in \mathcal{P}(\zeta_A)} \frac{\varepsilon^{d-w}}{d-w} \operatorname{res}\left(\zeta_A(s), w\right),$$

where $\mathcal{P}(\zeta_A)$ is the set of complex dimensions of A.

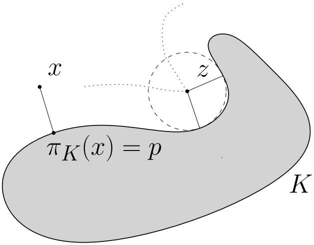
Questions

- Can we compute the Minkowski content using the general Steiner formula?
- Can we obtain more refined information on how the parallel volume of fractals grows from support measures?
- How are support measures related to complex dimensions and fractal tube formulas?
- What is the relation between support measures and fractal curvatures?
  (s-dimensional) k-th fractal curvature of F:

  $$\mathcal{C}_k^s(A) := \operatorname*{esslim}_{\varepsilon \searrow 0} \varepsilon^{s-k} C_k(A_\varepsilon)$$

  where $C_k(F_\varepsilon)$ are the total curvatures of F_ε (additive generalizations of the coefficients in the Steiner formula)
- Most basic question: How are the associated scaling exponents related?

Generalized normal bundle and local reach



- $\pi_K : \mathbb{R}^d \setminus \text{exo}(K) \to K$ metric projection: maps x to its nearest point $p \in K$

- generalized normal bundle

$$N(K) := \Big\{ \underbrace{\Big(\pi_K(x), \tfrac{x - \pi_K(x)}{\text{dist}(x,K)}\Big)}_{=:\Pi_K(x)} \Big\} \subseteq K \times \mathbb{S}^{d-1}$$

- local reach $\delta(K, p, u) := \sup\{t \geq 0 : \pi_K(p + tu) = p\}$

- $\mathrm{reach}(K) := \inf \big\{\delta(K, p, u) : (p, u) \in N(K)\big\}$

- K convex $\Rightarrow \mathrm{reach}(K) = \infty$, C^2-manifold $\mathrm{reach}(K) > 0$

Support measures [Hug, Last, Weil' 03]

- *i-th support measure* of a closed set $A \subset \mathbb{R}^d$ ($\omega_d := \mathrm{Area}(\mathbb{S}^{d-1})$):

$$\mu_i(A;\cdot) = \frac{1}{\omega_{d-i}} \int_{N(A)\cap\cdot} H_{d-1-i}(A,x,u)\mathscr{H}^{d-1}(\mathrm{d}(x,u)),$$

- $\mathscr{H}^{d-1}\ldots$ $(d-1)$-Hausdorff measure on $N(A) \subseteq \mathbb{R}^d \times \mathbb{S}^{d-1}$,
- H_j ... symmetrical functions of generalized principal curvatures:

$$H_j(A,x,u) := \prod_{i=1}^{d-1} \left(1 + k_i(A,x,u)^2\right)^{-1/2} \sum_{|I|=j, I\subseteq\{1,\ldots,d-1\}} \prod_{l\in I} k_l(A,x,u)$$

- relation with curvature measures: $\mu_i(A;\cdot) = C_i(A; N(A)\cap\cdot)$ whenever $C_i(A,\cdot)$ is defined

General Steiner-type formula [Hug, Last, Weil '03]

$$\mathscr{L}^d(A_\varepsilon \setminus A) = \sum_{i=0}^{d-1} \omega_{d-i} \int_0^\varepsilon t^{d-i-1} \underbrace{\int_{N(A)} \mathbb{1}_{\{t<\delta(A,x,u)\}} \mu_i(A; \mathrm{d}(x,u))}_{=:\, \beta_i(A;t)} \, \mathrm{d}t.$$

The support measure $\mu_i(A, \cdot)$ is

- a signed measure (only $\mu_{d-1}(A, \cdot)$ is always nonnegative);
- motion covariant and homogeneous of degree i:

$$\mu_i(\lambda A, \lambda(\cdot)) = \lambda^i \mu_i(A, \cdot),$$

- locally defined: if $A_1 \cap U = A_2 \cap U$ for some U open, then

$$\mu_i(A_1, D) = \mu_i(A_2, D) \text{ for all } D \subset U \times \mathbb{S}^{d-1}.$$

Introducing the basic functions for compact sets

Definition (Basic functions [RadWin24+])

Define the i-th basic function of A as

$$\beta_i(t) := \beta_i(A; t) := \int_{N(A)} \mathbb{1}_{\{t<\delta(A,x,u)\}} \mu_i(A; \mathrm{d}(x,u)),$$

for $i = 0, \ldots, d-1$ and $t > 0$.
We also denote by $\beta_i^{\mathrm{var}}(t)$ the total variation analog of $\beta_i(t)$.

The general Steiner formulas then becomes:

$$\mathscr{L}^d(A_\varepsilon) = \mathscr{L}^d(A) + \sum_{i=0}^{d-1} \omega_{d-i} \int_0^\varepsilon t^{d-i-1} \beta_i(t) \mathrm{d}t$$

Some immediate properties

- $\forall i \in \{0, 1, \ldots, d-1\}$ and $t > 0$, $\beta_i(t)$ and $\beta_i^{\mathrm{var}}(t)$ are finite
- $\beta_i^{\mathrm{var}}(t)$, is nonincreasing and right continuous in $t > 0$
- need not be left continuous
- $\lim\limits_{t \to 0^+} \beta_i^{\mathrm{var}}(t) = \begin{cases} 0, & \Leftrightarrow \beta_i \equiv 0 \\ \mathrm{const} > 0, & \mathrm{or} \\ +\infty & \end{cases}$

▶ Strategy: define new quantities corresponding to β_i^{var} and explore their connection to complex dimensions and fractal curvature measures

Introducing basic contents and exponents

Definition (Basic contents and exponents [RadWin24+])

Let $\emptyset \neq A \subseteq \mathbb{R}^d$ compact and $q \in \mathbb{R}$. We define the (*q*-dimensional) upper *i*-th basic content of *A* by

$$\overline{\mathcal{M}}_i^q(A) := \limsup_{t \to 0^+} t^{q-i} \beta_i(t),$$

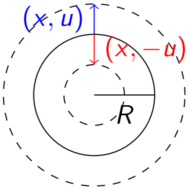
by analogy, also $\underline{\mathcal{M}}_i^q(A)$ and $\underline{\mathcal{M}}_i^{\mathrm{var},q}(A)$.
We also introduce the upper *i*-th basic scaling exponent

$$\begin{aligned}\overline{\mathfrak{m}}_i(A) = \overline{\mathfrak{m}}_i &:= \inf\{q \in \mathbb{R} : \overline{\mathcal{M}}_i^{\mathrm{var},q}(A) = 0\} \\ &= \sup\{q \in \mathbb{R} : \overline{\mathcal{M}}_i^{\mathrm{var},q}(A) = +\infty\},\end{aligned}$$

as well as its lower counterpart $\underline{\mathfrak{m}}_i(A)$.

- contents: homogeneous of degree q and motion invariant
- exponents: scaling invariant and motion invariant

Example: Circle of Radius R in $\mathbb{R}^2$

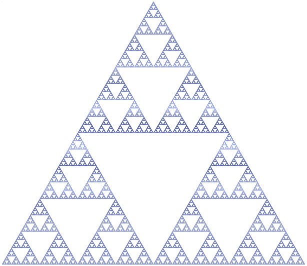


- S ... circle of radius R
- $N(S)^+$ ... outer normals; $k_1 = R^{-1}$
- $N(S)^-$ ... inner normals; $k_1 = -R^{-1}$

$\beta_i(t) = \int_{N(S)} \mathbb{1}_{\{t<\delta(A,x,u)\}}\mu_i(S;\mathrm{d}(x,u))$

- $$\begin{aligned}\beta_0(t) &= \int_{N(S)^+} \mu_0(S,\mathrm{d}(x,u)) + \mathbb{1}_{\{t<R\}} \int_{N(S)^-} \mu_0(S,\mathrm{d}(x,u)) \\ &= 1 - \mathbb{1}_{\{t<R\}}\end{aligned}$$
- $\beta_0^{\mathrm{var}}(t) = 1 + \mathbb{1}_{\{t<R\}}$
  - $\mathcal{M}_0^0(S) = 0,\ \mathcal{M}_0^{\mathrm{var},0}(S) = 2,$ and so $\mathfrak{m}_0 = 0$
- $\beta_1(t) = (1 + \mathbb{1}_{\{t<R\}})R\pi = \beta_1^{\mathrm{var}}(t)$,
  - $\mathcal{M}_1^1(S) = \mathcal{M}_1^{\mathrm{var},1}(S) = 2R\pi$ and so $\mathfrak{m}_1 = 1 = \dim_M S$

Example: Sierpinski gasket



For any $t > 0$, $\beta_0(A; t) = 1$. Hence $\mathfrak{m}_0 = 0$ and $\mathcal{M}_0^0(SG) = 1$.

Set $D := \log_2 3$ and $\nu_k := D + \frac{2\pi \mathbb{i} k}{\log 2}$, $k \in \mathbb{Z}$.

Then, for any $t > 0$ sufficiently small,

$$\beta_1(SG; t) = t^{1-D} \underbrace{\frac{6\sqrt{3}}{2\log 2} \sum_{k \in \mathbb{Z}} \frac{(4\sqrt{3})^{-\nu_k} t^{-\frac{2\pi \mathbb{i} k}{\log 2}}}{\nu_k(\nu_k - 1)}}_{=: G(t)} + \frac{3\sqrt{3}}{2} t,$$

where G is strictly positive, bounded and multipicatively periodic.

Hence $\mathfrak{m}_1 = D = \dim_M SG$ and $\mathcal{M}_1^D(SG)$ does not exist.

(But $\overline{\mathcal{M}}_1^D(SG)$ and $\underline{\mathcal{M}}_1^D(SG)$ can be computed explicitly.)

Properties of the basic exponents

Theorem ([RadWin24+])

Let $\emptyset \neq A \subseteq \mathbb{R}^d$ be a compact set. For each $i \in \{0, \dots, d-1\}$ one of the following is true:

(a) $\mu_i(A; \cdot) \equiv 0$ *(and then we set* $\underline{\mathfrak{m}}_i(A) := \overline{\mathfrak{m}}_i(A) := -\infty$*).*

(b) $i \leq \underline{\mathfrak{m}}_i(A) \leq \overline{\mathfrak{m}}_i(A) \leq \overline{\dim}_M^{\mathrm{out}} A$.

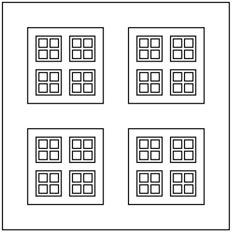
Furthermore, one always has $\mu_0(A; \cdot) \not\equiv 0$*, and*

$$\overline{\dim}_M^{\mathrm{out}} A = \max\{\overline{\mathfrak{m}}_i(A) : i \in \{0, \dots, d-1\}\}. \tag{1}$$

- As a consequence, for all $i > \overline{\dim}_M^{\mathrm{out}} A$, assertion (a) holds
- (a) is possible for each $i \neq 0$, and the bounds in (b) are attained for some sets A.
- Any basic exponent can be the largest and can thus determine $\dim_M A$.

Example: fractal window

- Is it possible to have $\mathfrak{m}_1 \leq \mathfrak{m}_0 = \dim_M A$? Yes!
- "fractal window" with scaling ratio $r \in (0, 1/2)$



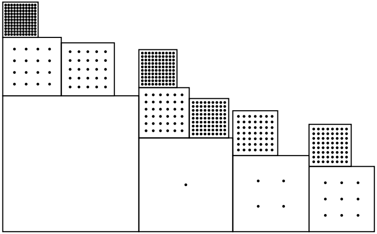
- inhomogeneous self-similar: $A = \bigcup_{i=1}^{4} \Phi_i(A) \bigcup B$, where $B = \square$
- Φ_i are the 4 similarity contraction mappings.
- $\mathfrak{m}_0 = \mathfrak{m}_1 = \dim_M A = \max\{1, \log_{1/r} 4\}$ and can be anything in $[1, 2)$.
- on the Figure: $r = 1/3$, $\dim_M A = \log_3 4$.

Example: Enclosed fractal dust

Is it possible to have

$$\mathfrak{m}_1 < \mathfrak{m}_0 = \dim_M A?$$

Yes!



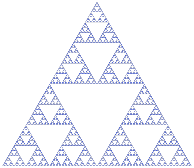
- family of sets with two parameters $\alpha \in (\frac{1}{2}, \frac{2}{3}]$ and $m \in \mathbb{N}$
- sidelengths of squares given by $\ell_j = j^{-\alpha}$, $j \in \mathbb{N}$
- n_j^2 equidistant points inside the j-th square with $n_j := j^m - 1$

Then

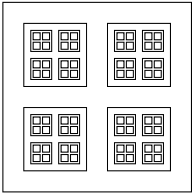
$$1 < \mathfrak{m}_1 = \frac{1+m}{\alpha + m} < \frac{1+2m}{\alpha + m} = \mathfrak{m}_0 = \dim_M A < 2.$$

$\dim_M A = \mathfrak{m}_0$ can be any number $\in [1.8, 2)$

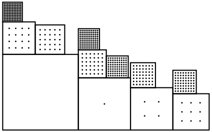
Comparing the three examples



- $0 = \mathfrak{m}_0 < \mathfrak{m}_1 = \log_2 3 = \dim_M(SG)$
- geomatically: 1-dim features dominate the fractal behavior and give raise to the Minkowski dimension



- $\mathfrak{m}_0 = \mathfrak{m}_1 = \dim_M A \in [1, 2)$
- geomatically: 1-dim segments and 0-dim corners contribute to the fractal behavior



- $1 < \mathfrak{m}_1 < \mathfrak{m}_0 = \dim_M(A)$
- geomatically: 0-dim points dominate the fractal behavior but 1-dim segments feature "subdominant" fractality

Distance Zeta Function of a cpt. set [LaRaZu'17]

$$\zeta_A(s) := \int_{A_\varepsilon \setminus A} \mathrm{dist}(x, A)^{s-d}\, dx$$

- $\zeta_A(s)$ is **holomorphic on** $\{\mathrm{Re}\, s > \overline{\dim}_M A\}$
- diverges if $s \in (-\infty, \overline{\dim}_M A)$
- **set of complex dimensions** of A: poles of ζ_A
- generalization of $\zeta_{\mathcal{L}}$ for fractal strings [Lapidus, van Frankenhuijsen, Pomerance, Maier]
- Under additional assumptions, a fractal tube formula holds:

$$V_d(A_\varepsilon \setminus A) = \sum_{w \in \mathcal{P}(\zeta_A, W)} \frac{\varepsilon^{d-w}}{d-w} \mathrm{res}\,(\zeta_A(s), w) + R(\varepsilon).$$

It allows to compute the (upper/lower/average) Minkowski content and to obtain higher order asymptotic terms.
Geometric interpr. of coeffs.? $\mathrm{res}(\zeta_A, D) = (d-D)\mathcal{M}^D(A)$

Basic zeta function decomposition

Theorem (Basic zeta functions for compact sets [RadWin])

Let $\emptyset \neq A \subseteq \mathbb{R}^d$ be compact and fix $\varepsilon > 0$. Then for all $s \in \mathbb{C}$ such that $\operatorname{Re} s > \overline{\dim}_M^{\mathrm{out}} A$ the following functional equation holds:

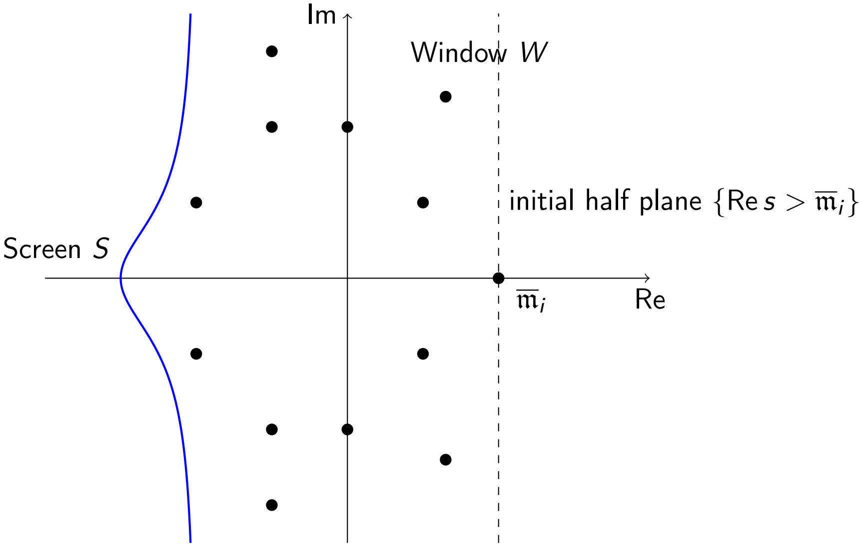
$$\zeta_A(s) = \sum_{i=0}^{d-1} \omega_{d-i} \breve{\zeta}_{A,i}(s),$$

where the i-th basic zeta function of A, $\breve{\zeta}_{A,i}$, for $i \in \{0, \dots, d-1\}$ is defined as

$$\breve{\zeta}_{A,i}(s) = \int_0^{\varepsilon} t^{s-i-1} \beta_i(t) \mathrm{d}t.$$

Furthermore, the integral defining $\breve{\zeta}_{A,i}$ is absolutely convergent, and hence, holomorphic, in the open half-plane $\{\operatorname{Re} s > \overline{\mathfrak{m}}_i\}$.

The Window and the Screen

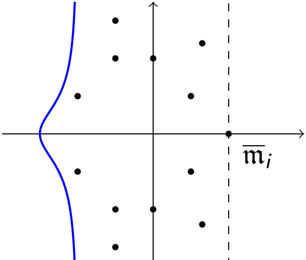


Reconstructing β_i from their basic zeta functions

Theorem (Pointwise formula [RadWin+])

Let $\emptyset \neq A \subseteq \mathbb{R}^d$ be compact and let $\check{\zeta}_{A,i}(\cdot;\varepsilon)$ satisfy appropriate growth conditions on some window $\mathrm{W} \subseteq \mathbb{C}$ with screen S. Then, for every $t \in (0,\varepsilon)$:

$$\hat{\beta}_i(A;t) = \sum_{w \in \mathcal{P}(\check{\zeta}_{A,i}(\cdot;\varepsilon),\mathrm{W})} \operatorname{res}\left(t^{i-s}\check{\zeta}_{A,i}(s;\varepsilon), w\right) + O(t^{i-\sup S}).$$



Refined fractal tube formula

for nice sets this provides a refined tube formula:

$$V_d(A_\varepsilon \setminus A) = \sum_{i=0}^{d-1} \omega_{d-i} \int_0^\varepsilon t^{d-i-1} \beta_i(t) \mathrm{d}t$$

$$= \sum_{i=0}^{d-1} \frac{\omega_{d-i}}{d-i} \sum_{w \in \mathcal{P}(\breve{\zeta}_{A,i}, W)} t^{d-w} \operatorname{res}\left(\breve{\zeta}_{A,i}(s), w\right) + \tilde{R}(t)$$

Hence, it is useful to find functional equations for the basic zeta functions

- advantage: possible reconstruction of (complex) basic exponents from poles of $\breve{\zeta}_{A,i}$ without explicit knowledge of the basic functions β_i
- reconstruction of complex dimensions from poles of $\breve{\zeta}_{A,i}$

Functional equations for basic zeta functions

Theorem (2nd functional equation for basic zeta functions)

$A \subseteq \mathbb{R}^d$ *cpt. and fix* $\varepsilon > 0$*. Then for all* $s \in \mathbb{C}$ *such that* $\operatorname{Re} s > \overline{\mathfrak{m}}_i$*:*

$$\breve{\zeta}_{A,i}(s) = \int_{A_\varepsilon \setminus A} \operatorname{dist}(z, A)^{s-i-1} \breve{K}_i(z) \, \mathrm{d}z,$$

$$\breve{K}_i(z) := \frac{1}{\omega_{d-i}} \prod_{m=1}^{d-1} \frac{1}{1 + \operatorname{dist}(z, A) k_m(A, \Pi_A(z))} \cdot \sum_{\substack{I \subseteq \{1,\ldots,d-1\} \\ |I| = d-1-i}} \prod_{l \in I} k_l(A, \Pi_A(z))$$

- $\Pi_A(z) := \left(\pi_A(z), \frac{z - \pi_A(z)}{\operatorname{dist}(z,A)}\right)$ *... directional metric projection*

Example: going back to the Sierpinski gasket

- $A =$ Sierpinski gasket; B the initial unit triangle; ε fixed.

$$k_1(A, \Pi_A(z)) = \begin{cases} \infty, & \text{if } \pi_A(z) \text{ is a vertex of } B \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

$$\breve{\zeta}_{A,0}(s) = \int_{A_\varepsilon \setminus A} \operatorname{dist}(z, A)^{s-1} \breve{K}_0(z) \mathrm{d}z = \frac{2\pi\varepsilon^s}{\omega_2 s},$$

$$\begin{aligned} \breve{\zeta}_{A,1}(s) &= \int_{A_\varepsilon \setminus A} \operatorname{dist}(z, A)^{s-2} \breve{K}_1(z) \mathrm{d}z \\ &= \frac{3\varepsilon^{s-1}}{\omega_1(s-1)} + \frac{6(\sqrt{3})^{1-s} \cdot 2^{-s}}{\omega_1 s(s-1)(2^s-3)}, \end{aligned}$$

$\Rightarrow \mathfrak{m}_0 = 0, \quad \mathfrak{m}_1 = \log_2 3 = \overline{\dim}_M^{\mathrm{out}} A$

$\Rightarrow \zeta_A(s) = \omega_2 \breve{\zeta}_{A,0}(s) + \omega_1 \breve{\zeta}_{A,1}(s)$

Reconstruction of basic functions for the Sierpinski gasket

- we now have for all $t \in (0, g)$ that

$$\hat{\beta}_0(A; t) = \mathrm{res}(t^{-s}\breve{\zeta}_{A,0}(s), 0) = \frac{2\pi}{\omega_2}$$

and

$$\begin{aligned}\hat{\beta}_1(A; t) &= \sum_{w \in \mathcal{P}(\breve{\zeta}_{A,1}, \mathbb{C})} \mathrm{res}\left(t^{1-s}\breve{\zeta}_{A,1}, w\right) \\ &= t^{1-\log_2 3}\frac{6\sqrt{3}}{\omega_1 \log 2}\underbrace{\sum_{k \in \mathbb{Z}} \frac{(4\sqrt{3})^{-w_k} t^{-\frac{2\pi \mathbb{i} k}{\log 2}}}{w_k(w_k - 1)}}_{G(\log_2 t)} + \frac{3\sqrt{3}}{\omega_1} t,\end{aligned}$$

- $w_k := \log_2 3 + \frac{2\pi \mathbb{i} k}{\log 2}$ for all $k \in \mathbb{Z}$

Further research directions

- Obtain existence criteria for the basic contents in terms of the basic zeta functions
- Applying the theory to problems from dynamical systems and fractal geometry
- What can be said about special kinds of sets? - self-similar, self-affine, etc..
- Further connections to the theory of fractal curvatures via the newly defined support contents and support zeta functions

Some references

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Support functions and scaling exponents

Let $A \subseteq \mathbb{R}^d$ be a compact set and $i \in I_d$. We are interested in

$$\varepsilon \mapsto \mu_i(A_\varepsilon)$$

and its behavior as $\varepsilon \to 0^+$.

Remark ([HugLasWei'04])

$\mu_{d-1}(A_\varepsilon)$ is a positive measure essentially equal to half of the surface area of A_ε, i.e., we have

$$\mu_{d-1}(A_\varepsilon) = \frac{1}{2}\mathscr{H}^{d-1}(\partial A_\varepsilon)$$

where the last equality holds for almost every $\varepsilon > 0$.

- we define now support scaling exponents $\mathfrak{s}_i$ analogously as $\mathfrak{m}_i$.

Support contents and support exponents

Observe that, for any $\varepsilon > 0$, the total mass $\mu_i(A_\varepsilon)$ of $\mu_i(A_\varepsilon, \cdot)$ is finite, similarly the total mass of $|\mu_i|(A_\varepsilon)$ of $|\mu_i|(A_\varepsilon, \cdot)$.

Definition

Let $A \subseteq \mathbb{R}^d$ be a compact set, $q \in \mathbb{R}$ and $i \in I_d$. We define the (*q*-dimensional) upper *i*-th support content of *A* by

$$\overline{\mathcal{S}}_i^q(A) := \limsup_{\varepsilon \to 0^+} \varepsilon^{q-i} \mu_i(A_\varepsilon),$$

as well as its total variation analog

$$\overline{\mathcal{S}}_i^{\mathrm{var},q}(A) := \limsup_{\varepsilon \to 0^+} \varepsilon^{q-i} |\mu_i|(A_\varepsilon).$$

We also introduce the upper/lower *i*-th support scaling exponent of *A* as

$$\overline{\underline{\mathfrak{s}}}_i := \overline{\underline{\mathfrak{s}}}_i(A) := \inf\{q \in \mathbb{R} : \overline{\underline{\mathcal{S}}}_i^{\mathrm{var},q}(A) = 0\},$$

Support vs. basic exponents

Theorem [RW24+]

Let $A \subset \mathbb{R}^d$ be a nonempty compact set. For each $i \in I_d$ and each $\varepsilon > 0$, $\mu_i(A_\varepsilon; \cdot) \not\equiv 0$, and

$$0 \leq \underline{\mathfrak{s}}_i(A) \leq \overline{\mathfrak{s}}_i(A) \leq \overline{\dim}_M^{\mathrm{out}} A.$$

Furthermore,

$$\overline{\mathfrak{s}}_i(A) \leq \max\{\overline{\mathfrak{m}}_j(A) : j \leq i\}$$

and also

$$\overline{\mathfrak{m}}_i(A) \leq \max\{\overline{\mathfrak{s}}_j(A) : j \leq i\}.$$

Moreover,

$$\overline{\dim}_M^{\mathrm{out}}(A) = \max\{\overline{\mathfrak{s}}_i(A) : i < d\}.$$

Support vs. basic contents

Theorem [RW24+]

Let $A \subseteq \mathbb{R}^d$ be a compact set, $q \geq 0$ and $i \in \{0, \ldots, d-1\}$. Assume also that the q-dimensional j-th basic content exists (in $\mathbb{R} \cup \{+\infty\}$) for $j = 0, \ldots, i$. Then,

$$\mathcal{S}_i^q(A) = \sum_{j=0}^{i} c_{i,j} \mathcal{M}_j^q(A).$$

In particular, if $i = d-1$ and $q = \dim_M A = D$, then

$$\tfrac{d-D}{2} \mathcal{M}^D(A) = \mathcal{S}_{d-1}^D(A) = \sum_{j=0}^{d-1} c_{d-1,j} \mathcal{M}_j^D(A).$$

Remarks:

- The last statement is due to the relation $2\mu_{d-1}(A_\varepsilon) = \mathscr{H}^{d-1}(\partial A_\varepsilon)$, which holds for all $\varepsilon > 0$ except countably many and a result in **[Rataj, W. 10]**.
- In general,

$$\tfrac{d-D}{2}\overline{\mathcal{M}}^D(A) \le \overline{\mathcal{S}}^D_{d-1}(A) \le \sum_{j=0}^{d-1} c_{d-1,j}\overline{\mathcal{M}}^D_j(A)$$

  and all inequalities can be strict.

Fractal grills - basic type of direct product formula

Proposition (RadWin)

Let $\emptyset \neq A \subseteq \mathbb{R}^d$ be cpt. Then for any $i \in \{0, \dots, d\}$ we have that

$$\beta_i^{[d+1]}(A \times [0, L]; t) = L \cdot \beta_{i-1}^{[d]}(A; t) + \beta_i^{[d+1]}(A \times \{0\}; t),$$

where we let $\beta_{-1}^{[d]} \equiv 0$ for all $d \geq 0$.

Corollary (RadWin)

Let $A \subseteq \mathbb{R}^d$ be cpt, and $B \subseteq \mathbb{R}^k$ a hyperrectangle of side-lengths $L_1, \dots, L_k$; $K := \{1 \dots, k\}$. Then $\forall i \in \{0, \dots, d+k-1\}$:

$$\beta_i^{[d+k]}(A \times B; t) = \sum_{l=\max\{0,k-i\}}^{k} C_{k-l}(B) \cdot \beta_{i-(k-l)}^{[d+l]}(A \times \{0\}^l; t),$$

where $C_j(B)$ are the Steiner functionals of B.

Embedings into higher dimensional space

Proposition (RadWin)

Let $A \subseteq \mathbb{R}^d$ be compact. Then, for any $i \in \{0, \ldots, d\}$ we have that

$$\beta_i^{[d+1]}(A \times \{0\}; t) = \frac{\omega_{d-i}}{\omega_{d+1-i}} \int_{-1}^{1} \beta_i^{[d]}(A; t\sqrt{1 - v^2})\,\mathrm{d}v, \qquad (3)$$

where we let $\beta_d^{[d]}(A; t) := \mathscr{L}^d(A)$ and $\omega_0 := 1$, i.e.,
$\beta_d^{[d+1]}(A \times \{0\}; t) \equiv \mathscr{L}^d(A)$.

$\Rightarrow$ Basic exponents and basic zeta functions are invariant to the ambient space.

Support zeta functions

Theorem (Support zeta functions for compact sets)

Let $A \subseteq \mathbb{R}^d$ be a compact set, $\varepsilon > 0$ fixed and $i \in \{0, \dots, d-1\}$ fixed. Then,

$$\hat{\zeta}_{A,i}(s) = \int_0^{\varepsilon} t^{s-i-1} \mu_i(A_t) \mathrm{d}t$$

is called the i-th fractal support zeta function and is holomorpic in $\{\mathrm{Re}\, s > \bar{\mathfrak{s}}_i\}$ as an absolutely convergent integral.
Furthermore, the following decomposition into the basic zeta functions $\check{\zeta}_{A,j}$ is valid for $\mathrm{Re}\, s > \max\{\overline{\mathfrak{m}}_j : j = 0, \dots, i\}$:

$$\hat{\zeta}_{A,i}(s) = \sum_{j=0}^{i} c_{i,j} \check{\zeta}_{A,j}(s).$$

Functional equations for support zeta functions

Theorem (Functional equation for support zeta functions [RadWin])

$A \subseteq \mathbb{R}^d$ cpt. such that $\overline{\dim}_M^{\mathrm{out}} A < d$ and fix $\varepsilon > 0$. Then for all $s \in \mathbb{C}$ such that $\operatorname{Re} s > \bar{\mathfrak{s}}_i$:

$$\hat{\zeta}_{A,i}(s) = \int_{A_\varepsilon \setminus A} \operatorname{dist}(z, A)^{s-i-1} \hat{K}_i(z)\, \mathrm{d}z,$$

$$\hat{K}_i(z) := \frac{1}{\omega_{d-i}} \sum_{\substack{I \subseteq \{1,\dots,d-1\} \\ |I| = d-1-i}} \prod_{l \in I} \frac{k_l(A, \Pi_A(z))}{1 + \operatorname{dist}(z, A) k_l(A, \Pi_A(z))}$$

- $\Pi_A(z) := \left(\pi_A(z), \frac{z - \pi_A(z)}{\operatorname{dist}(z,A)}\right)$ ... *directional metric projection*