Dynamical subsets in iterated function systems

Henna Koivusalo (joint with Barany and Troscheit)

University of Bristol

Iterated function systems

Iterated function systems

- $\{f_1, \dots, f_m\}$ similarities (or affine maps)
- $r_1, \dots, r_m$ contraction ratios
- Λ attractor

$$\Lambda = \bigcup_{i=1}^{m} f_i(\Lambda)$$

- assumption: strong separation condition (or open set condition)

Symbolic space

- symbolic space: $\Sigma = \{1, \ldots, m\}^{\mathbb{N}}$
- shift: $\sigma(i_1, i_2, i_3, \ldots) = (i_2, i_3, \ldots)$
- length of a finite word $\mathbf{i}$: $|\mathbf{i}|$

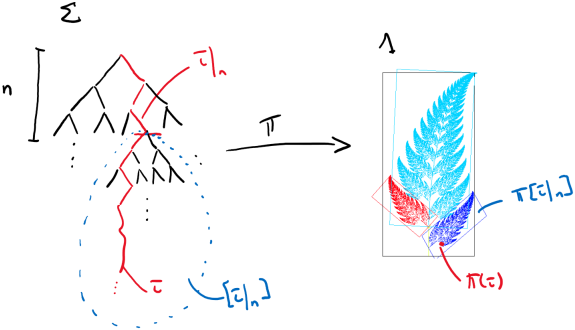
Symbolic space

- $\mathbf{i} \in \Sigma$: $\mathbf{i}|_n$ the first n digits of $\mathbf{i}$
- cylinder:

$$[\mathbf{i}|_n] = \{\mathbf{j} \in \Sigma \mid \mathbf{j}|_n = \mathbf{i}|_n\}$$

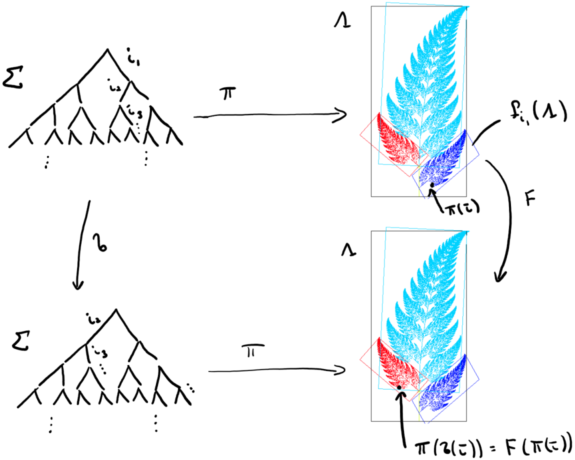
- projection: $\pi : \Sigma \to \Lambda$ (bijection)

$$\pi(\mathbf{i}) = \lim_{n\to\infty} f_{\mathbf{i}|_n}(0)$$

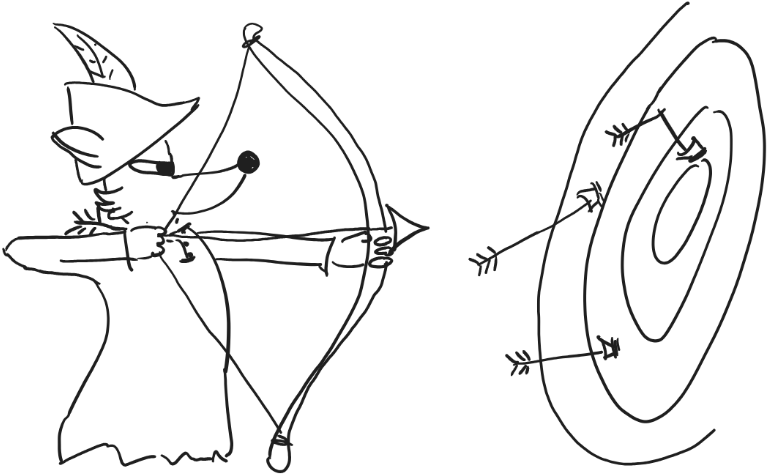


Dynamics on fractals

Expanding dynamics



- $F : \Lambda \to \Lambda : \pi \circ \sigma = F \circ \pi$



Shrinking targets

Dynamical subsets: Shrinking targets

- $A_n \subset X$ and (T, X),

$$\{x \in X \mid T^n(x) \in A_n \text{ for infinitely many } n\}$$

Question

How big is the shrinking target set for a given sequence A_n?

Shrinking cylinder targets in iterated function systems

- symbolic shrinking target set: $\mathbf{j} \in \Sigma$, $\ell_n \to \infty$

$$R^*([\mathbf{j}|_{\ell_n}]) = \{\mathbf{i} \in \Sigma \mid \sigma^n(\mathbf{i}) \in [\mathbf{j}|_{\ell_n}] \text{ for infinitely many } n\}$$

- corresponding subset on Λ: $\pi(\mathbf{j}) = y \in \Lambda$,

$$R(\pi[\pi^{-1}(y)|_{\ell_n}]) = \{x \in \Lambda \mid F^n(x) \in \pi[\mathbf{j}|_{\ell_n}] \text{ for infinitely many } n\}$$

- satisfies:

$$R(\pi[\pi^{-1}(y)|_{\ell_n}]) = \pi(R^*([\mathbf{j}|_{\ell_n}]))$$

Past results on shrinking cylinder targets in affine IFSs

- Affine $f_1, \ldots, f_m$
- Hausdorff dimension for $\pi(R^*(\mathbf{j}|_{\ell(n)})$

What is known

For translation generic affine IFSs, the Hausdorff dimension is given by a pressure formula:

- *K. and Ramírez* $\sim$ *2015*
- *Barany and Rams* $\sim$ *2018*
- *Barany and Troscheit* $\sim$ *2021*
- *K., Liao and Rams* $\sim$ *2022*
- $+$*Morris* $\Rightarrow$ *2024?*

Multiplicativity conditions! Transversality!

Shrinking ball targets in conformal dynamical systems

- geometric shrinking target set: $y \in \Lambda$, $r_n \to 0$,

$$R(B(y, r_n)) = \{x \in \Lambda \mid F^n(x) \in B(y, r_n)\}$$

- conformal F
- Note! the ball $B(y, r_n) \approx$ cylinder

What is known

Hausdorff dimension for $R(B(y, r_n))$

- *Hill and Velani $\sim$ 1998*
- *Li, Wang, Wu, Xu $\sim$ 2013*

Past results on shrinking ball targets in affine IFSs

- Special cases of affine IFSs

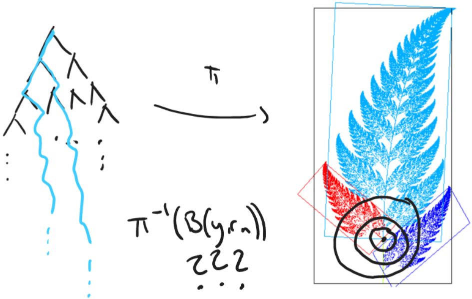
What is known

Hausdorff dimension of $R(B(y, r_n))$

- *Barany and Rams* $\sim$ *2018*
- *Jordan, K.* $\sim$ *one day*

Case by case geometric considerations!

Now balls are not cylinders!



Past measure results on shrinking targets in IFSs

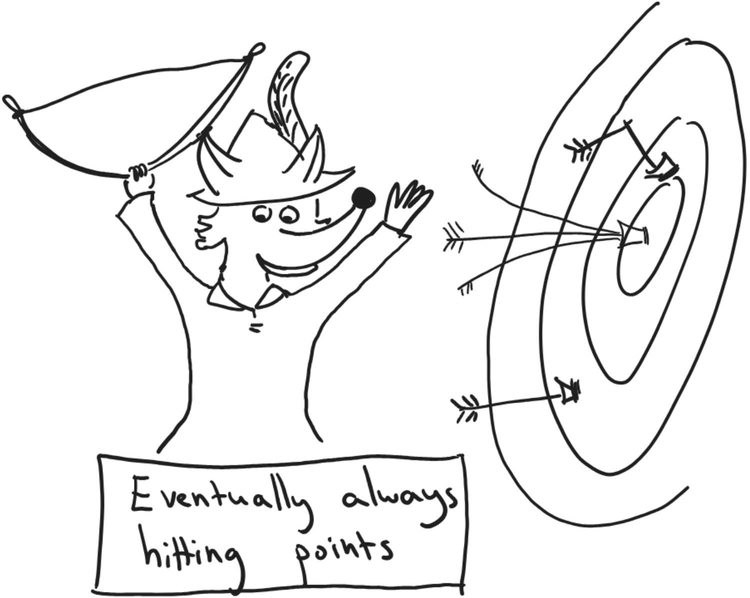
Theorem

How large is the **measure** *of $R(A_n)$, wrt some natural measure on Λ (e.g. Hausdorff, Lebesgue, Gibbs...)?*

What is known

- *K. and Ramírez $\sim$ 2015*
- *Allen and Barany $\sim$ 2020*
- *Baker $\sim$ 2022*
- *Allen, Baker, Barany $\sim$ 2023*
- *Baker and K. $\sim$ 2023*

Dynamical Borel-Cantelli-type results! Independence!


Eventually always hitting points

Dynamical subsets: Eventually always hitting points

- $A_n \subset X$ and (T, X),

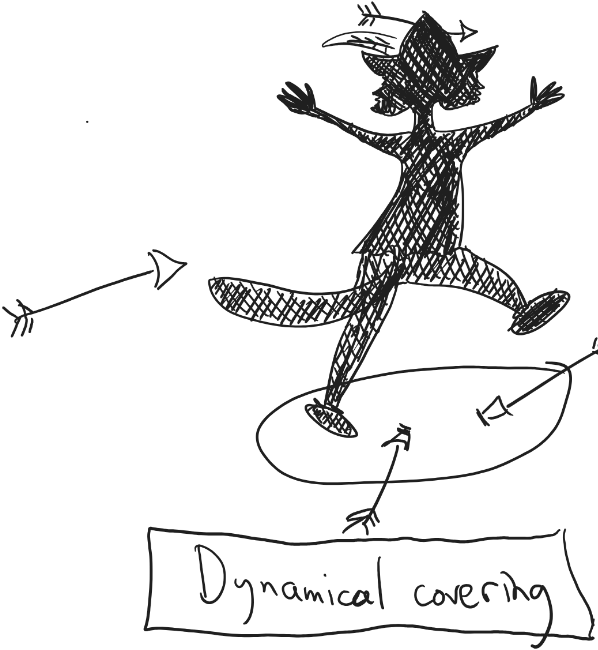
$$\{x \in X \mid \text{there is } N \text{ s.t. for all } n > N \text{ for some } k < n, T^k(x) \in A_k\}$$

What is known

For similarities or conformal $f_1, \ldots, f_m$, targets A_n balls, the Hausdorff dimension

- *Bugeaud and Liao $\sim$ 2015*
- *Zhang $\sim$ 2023*

Liminf techniques very underdeveloped!



Dynamical subsets: Dynamical covering

- symbolic dynamical covering: (σ, Σ), $\mathbf{j} \in \Sigma$, $\ell_n \to \infty$,

$$C^*(\mathbf{j}, \ell_n) = \{\mathbf{i} \in \Sigma \mid \mathbf{i}|_{\ell_n} = \sigma^n(\mathbf{j})|_{\ell_n} \text{ for infinitely many } n\}$$

- corresponding geometric set: $C(\mathbf{j}, \ell_n) = \pi C^*(\mathbf{j}, \ell_n)$

What is known

For similarities $f_1, \ldots, f_m$, the Hausdorff dimension of $C(\mathbf{j}, \ell_n)$

- *Liao and Seuret ∼ 2012*
- *Persson and Rams ∼ 2017*

Statement of results

Statement

Theorem (Barany, K., Troscheit (in progress))

Let $\{f_1, \ldots, f_m\}$ *be an IFS of similarities satisfying the open set condition, and with contraction ratios* $r_1, \ldots, r_m$. *Let* $\alpha > 0$ *and assume that some* $\ell : \mathbb{N} \to \mathbb{N}$ *satisfies*

$$\lim_{n\to\infty} \frac{\ell(n)}{\log n} = \frac{1}{\alpha}.$$

Then denote

$$m(n) = \lfloor \frac{\#\ell^{-1}(n)}{n} \rfloor.$$

Then, with respect to a Bernoulli measure $\mu_{(p_1,\ldots,p_m)}$, *for* $\mu_{(p_1,\ldots,p_m)}$*-almost every* $\mathbf{k} \in \Sigma$, *the Hausdorff dimension of* $C(\mathbf{k}, \ell_n)$ *is given by the solution* $s = s(\alpha)$ *to*

$$\limsup_{n\to\infty} \frac{1}{n} \log \sum_{|\mathbf{j}|=n} r_{\mathbf{j}}^s (1 - (1 - p_{\mathbf{j}})^{m(n)}) = 0.$$

Closed forms

Lemma

$$s(\alpha) = \begin{cases} s_1 : & \sum_{i=1}^m r_i^{s_1} p_i e^{\alpha} = 1, \\ & \quad when\ \alpha < -\sum_{i=1}^m r_i^{s_1} p_i e^{\alpha} \log p_i \\ s_2 : & \inf\{p_\alpha(q) : \sum_{i=1}^m r_i^{p_\alpha(q)} (p_i e^{\alpha})^q = 1, q > 0\} \\ & \quad when\ \ -\sum_{i=1}^m r_i^{s_0} \log p_i > \alpha > -\sum_{i=1}^m r_i^{s_1} p_i e^{\alpha} \log p_i \\ s_0 & \quad otherwise \end{cases}$$

where s_0 is the dimension of Λ.

Justification for the dimension value (upper bound)

Here:

$$C(\mathbf{k}, \ell(n)) = \limsup_{n\to\infty} \pi[\sigma^n(\mathbf{k})|_{\ell(n)}] = \bigcap_{k=1}^{\infty} \bigcup_{n\geq k} \pi[\sigma^n(\mathbf{k})|_{\ell(n)}].$$

Hence for all k,

$$C(\mathbf{k}, \ell(n)) \subset \bigcup_{n\geq k} \pi[\sigma^n(\mathbf{k})|_{\ell(n)}].$$

Justification for the dimension value (lower bound)

Sequence (n_k) sparse. Denote

$$P(n_k) = \{0, n_k, \ldots, qn_k \mid q \in \mathbb{N} \text{ maximal s.t. } qn_k \in \ell^{-1}(n_k)\}$$

Let

$$C_0 = \bigcup_{\{\mathbf{i}:\mathbf{k}_p \ldots \mathbf{k}_{p+n_0} = \mathbf{i} \text{ for some } p \in P(n_0)\}} [\mathbf{i}]$$

and

$$C_{k+1} = \bigcup_{\{\mathbf{i}:\mathbf{k}_p \ldots \mathbf{k}_{p+n_{k+1}} = \mathbf{i} \text{ for some } p \in P(n_k+1) \text{ and } \exists \mathbf{j} \in C_k \text{ s.t. } \mathbf{i}|_{|\mathbf{j}|} = \mathbf{j}\}} [\mathbf{i}].$$

Intersect. Measure...


Q.E.D.