Linear images of self-affine sets

(work in progress, joint with Çağrı Sert)

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Self-affine sets: some fundamentals

- Given a tuple of invertible affine contractions $T_1, \ldots, T_N \colon \mathbb{R}^d \to \mathbb{R}^d$, there exists a unique nonempty compact set $X \subset \mathbb{R}^d$ satisfying $X = \bigcup_{i=1}^N T_i X$. We call $(T_1, \ldots, T_N)$ an *affine IFS* and X a *self-affine set*.

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- There also exists a unique well-defined associated *coding map* $\Pi \colon \{1, \ldots, N\}^{\mathbb{N}} \to X$ which satisfies

$$\Pi[(x_k)_{k=1}^\infty] = \lim_{n\to\infty} T_{x_1} \cdots T_{x_n} v$$

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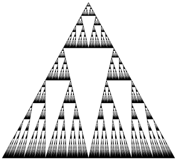

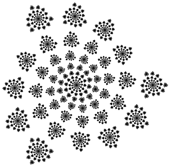

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These issues are relatively well-understood in the case where every T_i is conformal, and much less so otherwise.

Standard dimension bounds

If $V \subset \mathbb{R}^d$ is a closed ball such that $X \subseteq V$, then clearly

$$X = \bigcup_{i_1,\ldots,i_n=1}^{N} T_{i_1} \cdots T_{i_n} X \subseteq \bigcup_{i_1,\ldots,i_n=1}^{N} T_{i_1} \cdots T_{i_n} V$$

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To cover X efficiently, and hence bound $\dim X$, we could try to efficiently cover the sets $T_{i_1} \cdots T_{i_n} V$.

If T is affine with $Tx \equiv Ax + v$, and V is a Euclidean ball, then TV is an ellipsoid with semiaxes $\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_d(A)$, say.

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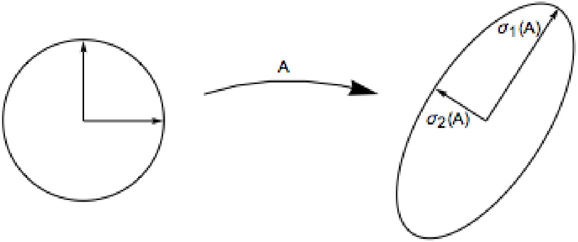

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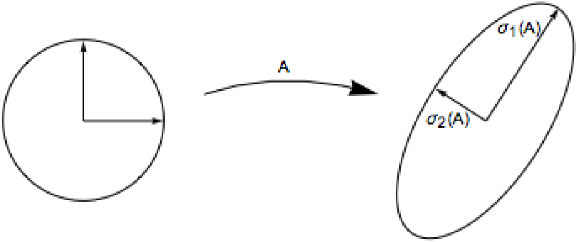

We can exploit this to cover TV with cubes in a way which provides an efficient estimate of its s-dimensional volume.

Define for each $s \in [0, d]$

$$\varphi^s(A) := \sigma_1(A) \cdots \sigma_{\lfloor s \rfloor}(A) \sigma_{\lceil s \rceil}(A)^{s - \lfloor s \rfloor}.$$

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Let X be the attractor of an affine IFS with linearisation $(A_1, \ldots, A_N)$. One may show that if

$$\sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n=1}^{N} \varphi^s(A_{i_1} \cdots A_{i_n}) < \infty$$

then $\dim_{\mathrm{H}} X \leq s$.

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$$\dim_{\mathrm{aff}} \mathbf{A} = \inf \left\{ s > 0 \colon \sum_{n=1}^{\infty} \sum_{i_1, \ldots, i_n = 1}^{N} \varphi^s(A_{i_1} \cdots A_{i_n}) < \infty \right\}$$

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$\ldots$ and we also define the *Lyapunov dimension* of μ as

$$\dim_{\mathrm{Lyap}}(\mu,\mathbf{A}) = \inf\left\{ s>0\colon \lim_{n\to\infty}\log\left(\frac{\varphi^s(A_{i_1}\cdots A_{i_n})}{\mu([i_1\cdots i_n])}\right)<0 \ \mu\text{-a.e.}\right\}$$

which is an upper bound for the Hausdorff dimension of $\Pi_*\mu$.

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2. Käenmäki '04: under the same hypotheses, if μ is an ergodic shift-invariant measure on $\{1, \ldots, N\}^{\mathbb{N}}$, then $\dim_{\mathrm{H}} \Pi_* \mu = \dim_{\mathrm{Lyap}}(\mathbf{A}, \mu)$ for Lebesgue almost every $(v_1, \ldots, v_N)$.

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3. Bárány-Hochman-Rapaport '19: if X is the attractor of an affine IFS on $\mathbb{R}^2$ which satisfies the SSC and is strongly irreducible, then $\dim_{\mathrm{H}} X = \dim_{\mathrm{aff}} \mathbf{A}$.

4. Feng '23: if μ is an ergodic shift-invariant measure on $\{1,\ldots,N\}^{\mathbb{N}}$ and $(T_1,\ldots,T_N)$ is an affine IFS with coding map Π, then $\Pi_*\mu$ is exact-dimensional.

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6. Pyörälä '24: if X and Y are the attractors of affine IFS's on $\mathbb{R}^2$ which satisfy the SSC, proximality and strong irreducibility, then $\dim_{\mathrm{H}}(X+Y) = \min\{2, \dim_{\mathrm{H}} X + \dim_{\mathrm{H}} Y\}$.

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I will describe some complications which occur in dimension 4 and higher.

Example 1: abundance of exceptional projections

Let $n, m \geq 2$. Then there exists a self-affine set $X \subset \mathbb{R}^{nm}$, defined by an affine IFS with linearisation $\mathbf{A}$, such that:

- For every $u \in S^{n-1}$, if Q is given by orthogonal projection onto the subspace $\{(a_1 \cdot u, a_2 \cdot u, \ldots, a_m \cdot u) \colon a \in \mathbb{R}^m\}$, then

$$\dim_{\mathrm{H}} QX < \dim_{\mathrm{H}} X = \dim_{\mathrm{aff}} \mathbf{A}\,.$$

In particular, X has an $(n-1)$-dimensional algebraic variety of exceptional orthogonal projections in the sense of Marstrand's theorem.

Example 2: failure of exact-dimensionality for projections

Let $n \geq 2$. Then there exists a self-affine set $X \subset \mathbb{R}^{2n}$, defined by an affine IFS with linearisation $\mathbf{A}$, such that:

- There exists a unique ergodic shift-invariant measure μ on $\{1, \ldots, N\}^{\mathbb{N}}$ such that

$$\dim_{\mathrm{H}} \Pi_* \mu = \dim_{\mathrm{H}} X = \dim_{\mathrm{aff}} \mathbf{A}\,.$$

- For every $U \in \mathrm{SO}(n)$, if Q_U denotes the orthogonal projection Q onto the subspace $\{(v, Uv) \colon u \in \mathbb{R}^n\}$, then $Q_* \Pi_* \mu$ is *not* exact-dimensional.

Example 3: dimension defects for sumsets

Let $n \geq 2$. Then there exists a self-affine set $X \subset \mathbb{R}^{2n}$, defined by an affine IFS with linearisation $\mathbf{A}$, such that:

- If n is odd:

$$\dim_{\mathsf{H}}(X+X) < 2\dim_{\mathsf{H}} X = 2\dim_{\mathsf{aff}} \mathbf{A} \leq 2n.$$

- If n is even:

$$\dim_{\mathsf{H}}(X+RX) < 2\dim_{\mathsf{H}} X = 2\dim_{\mathsf{aff}} \mathbf{A} \leq 2n,$$

  where R denotes reflection in the first co-ordinate in $\mathbb{R}^{2n}$.

Example 4: failure of exact-dimensionality for convolutions

Let $n \geq 3$ be odd. Then there exist self-affine sets $X, Y \subset \mathbb{R}^{2n}$, each defined by an affine IFS with the same linearisation $\mathbf{A}$, such that:

- There exists a unique ergodic shift-invariant measure on $\{1, \ldots, N\}^{\mathbb{N}}$ such that
$$\dim_{\mathrm{H}} \Pi_*^X \mu = \dim_{\mathrm{H}} X = \dim_{\mathrm{aff}} \mathbf{A}$$
and this measure is also the unique ergodic shift-invariant measure on $\{1, \ldots, N\}^{\mathbb{N}}$ such that
$$\dim_{\mathrm{H}} \Pi_*^Y \mu = \dim_{\mathrm{H}} Y = \dim_{\mathrm{aff}} \mathbf{A}\,.$$
- The convolution
$$\Pi_*^X \mu * \Pi_*^Y \mu$$
is *not* exact-dimensional.

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For each of Examples 1–4:

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- The results hold for almost all choices of translation component (in the sense of Falconer '88).
- The strong separation condition can be assumed to hold.
- The IFS does *not* act strongly irreducibly on every exterior power of $\mathbb{R}^d$.

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- A similar remark applies to convolutions of measures on self-affine sets.
- Thus, all four examples listed above are all derived from a new analysis of projections of self-affine sets.

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and it follows in the same manner that

$$\dim_{\mathrm{H}} QX \leq \inf \left\{ s > 0 \colon \sum_{i_1,\ldots,i_n=1}^{N} \varphi^s(QA_{i_1} \cdots A_{i_n}) < \infty \right\}.$$

Similarly, if Π is the associated coding map and μ is an ergodic shift-invariant measure on $\{1,\ldots,N\}^{\mathbb{N}}$, we may bound the *upper* Hausdorff dimension of $Q_*\Pi_*\mu$ by

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These bounds may be unequal.

Çağrı Sert and I generalise the existing thermodynamic formalism of affine IFS so as to encompass pressure functions of the form

$$P_Q(\mathbf{A}, s) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{i_1,\ldots,i_n=1}^{N} \varphi^s\left(QA_{i_1}\cdots A_{i_n}\right).$$

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- Dependence on Q: sub-level sets with respect to Q are algebraic varieties!

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- Strong irreducibility of exterior powers fails in many natural circumstances, e.g. for affine IFS with holomorphic structure, symplectic structure, or Cartesian product structure (in particular, when investigating sumsets and convolutions).

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- Strong irreducibility of exterior powers fails in many natural circumstances, e.g. for affine IFS with holomorphic structure, symplectic structure, or Cartesian product structure (in particular, when investigating sumsets and convolutions).
- The results on projections of sets and measures are currently being written up (70 pages and counting. . .).

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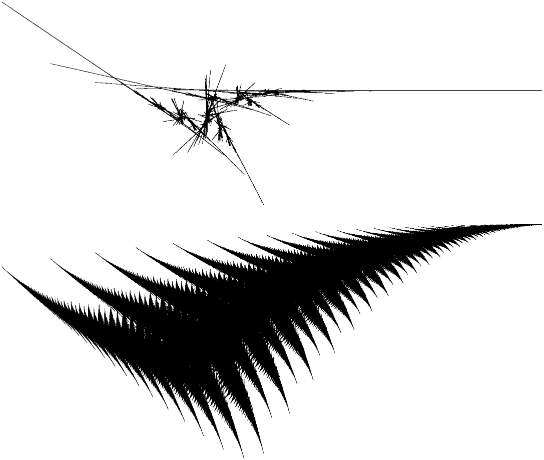
- A sequel article will prove an almost sure dimension formula for sumsets and convolutions of self-affine sets and their natural measures.



Thanks for listening!