Dimension interpolation on planar carpets

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joint works with A. Banaji, J. M. Fraser and A. Rutar

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Different notions of dimension

For any bounded set $F \subset \mathbb{R}^d$,

$$\dim_{\mathrm{H}} F \leqslant \underline{\dim}_{\mathrm{B}} F \leqslant \overline{\dim}_{\mathrm{B}} F \leqslant \dim_{\mathrm{A}} F,$$

where $H =$'Hausdorff', $B =$'box' and $A =$'Assouad'.

In some "typical" sense there is equality between them, for the talk interested more in the "exceptional" case when there is strict inequality.

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Examples

- $F = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ has $\dim_{\mathrm{H}} F = 0 < \dim_{\mathrm{B}} F = 1/2 < \dim_{\mathrm{A}} F = 1$
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Let's interpolate!

Intermediate dimensions (Falconer, Fraser, Kempton '19)

For a non-empty bounded set F and $\theta \in (0,1)$, define

$$\underline{\dim}_\theta F = \inf\Big\{ s \geq 0 : \forall \varepsilon > 0, \exists\, \delta_n \to 0 \text{ and a cover } \{U_i^{(n)}\} \text{ of } F \text{ s.t. } \delta_n^{1/\theta} \leq |U_i^{(n)}| \leq \delta_n \text{ and } \sum |U_i^{(n)}|^s \leq \varepsilon \Big\}.$$

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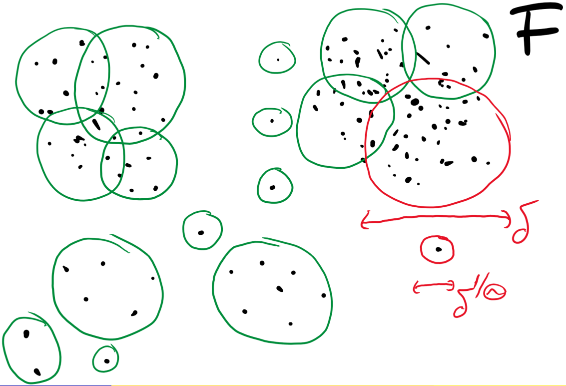

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Define $\overline{\dim}_\theta F$ similarly and $\dim_\theta F$ if the two are equal.

$$\dim_{\mathrm{H}} F = \inf\Big\{ s \geq 0 : \forall \varepsilon > 0, \exists \text{ a cover } \{U_i\} \text{ of } F \text{ s.t. } \sum |U_i|^s \leq \varepsilon \Big\},$$

$$\underline{\dim}_{\mathrm{B}} F = \inf\Big\{ s \geq 0 : \forall \varepsilon > 0, \exists \text{ a cover } \{U_i\} \text{ of } F \text{ s.t. } |U_i| = |U_j| \text{ for all } i,j \text{ and } \sum |U_i|^s \leq \varepsilon \Big\},$$

$\underline{\dim}_0 F = \overline{\dim}_0 F = \dim_H F$; $\underline{\dim}_B F = \underline{\dim}_1 F$; Continuous for $\theta \in (0,1]$.
Monotonically increasing; bi-Lipschitz stable; characterisation of form.

The Assouad spectrum (Fraser, Yu '18)

For a non-empty bounded set F and $\theta \in (0,1)$, define

$$\dim_{\mathrm{A}}^{\theta} F = \inf\Big\{ \alpha : (\exists C>0)(\exists \rho>0)(\forall 0<R\leqslant\rho)(\forall x\in F)$$
$$N\big(B(x,R)\cap F, R^{1/\theta}\big) \leqslant C\,\big(R^{1-1/\theta}\big)^{\alpha} \Big\}.$$

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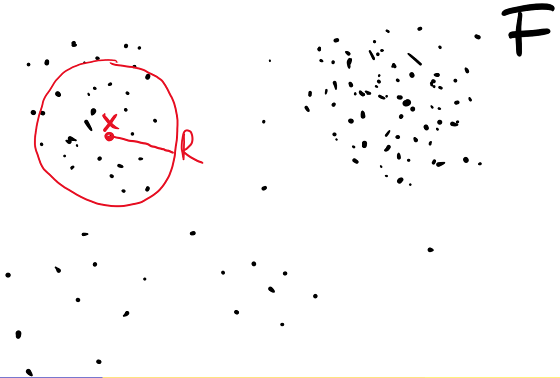

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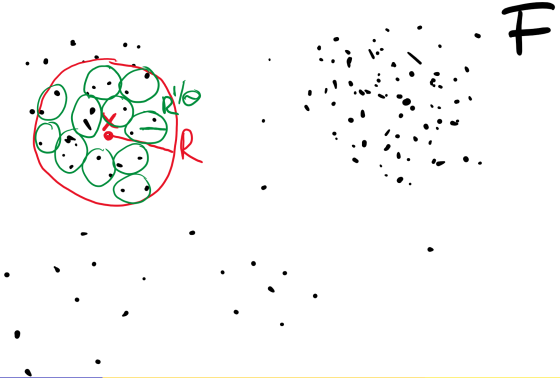

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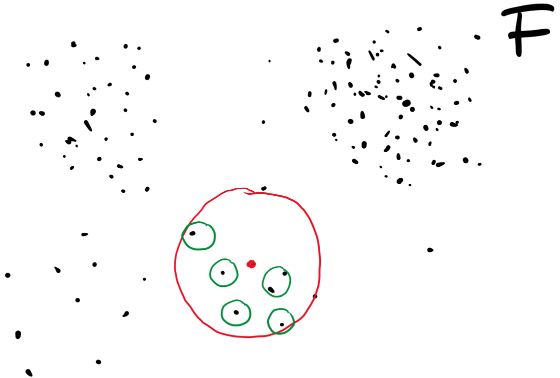

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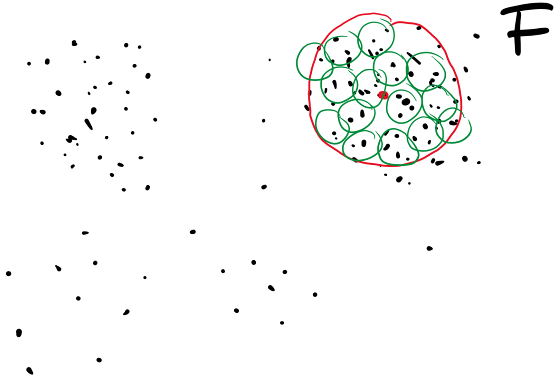

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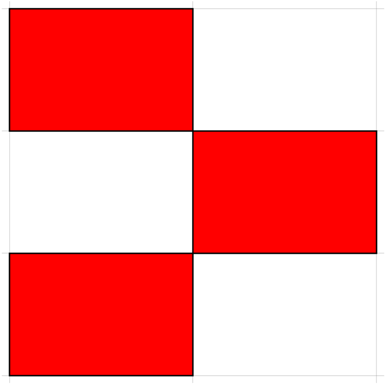
$$\dim_{\mathrm{A}} F = \inf \Big\{ \alpha : (\exists C > 0)(\forall 0 < r < R)(\forall x \in F) \\ N\big(B(x,R) \cap F, r\big) \leqslant C \big(R/r\big)^{\alpha} \Big\},$$

where $N(F,r) =$ smallest number of closed balls of radius r that cover F.
For $\overline{\dim}_{\mathrm{B}} F$ 'replace' R with the fixed radius 1.

$\dim_{\mathrm{A}}^{\theta} F \to \overline{\dim}_{\mathrm{B}} F$ as $\theta \to 0$ and $\dim_{\mathrm{A}}^{\theta} F \to \dim_{\mathrm{qA}} F$ as $\theta \to 1$; Cont.
Can be non-monotonic; bi-Lipschitz stable; characterisation of form.

Bedford–McMullen carpets (Bedford '84, McMullen '84)



- $m \times n$ grid ($m < n$)
- IFS $\mathcal{F} = \{f_i\}_{i=1}^{N}$, where

$$f_i(\underline{x}) = \begin{pmatrix} 1/m & 0 \\ 0 & 1/n \end{pmatrix} \underline{x} + \underline{t}_i.$$

- Attractor $F = \bigcup_i f_i(F)$.

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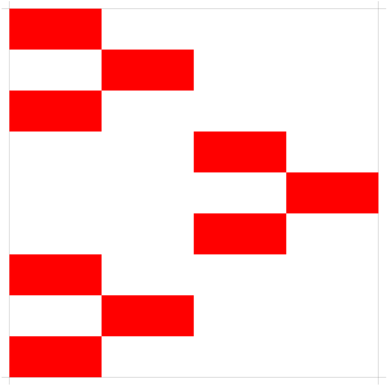

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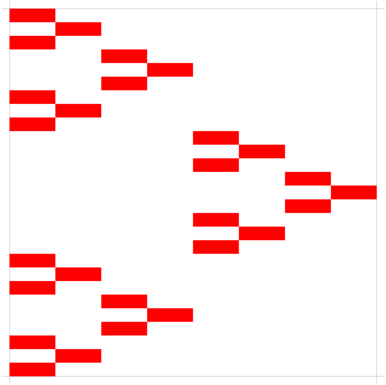

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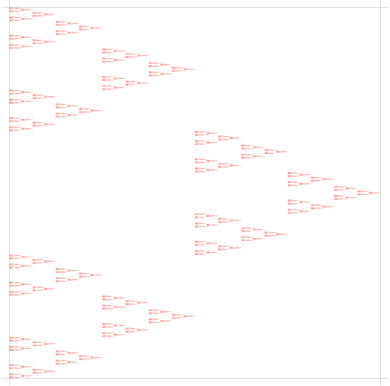

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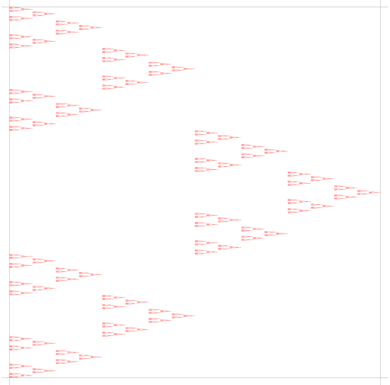

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- Attractor $F = \bigcup_i f_i(F)$.
- $M = \#$non-empty columns,
$N_k = \#$maps in column k.

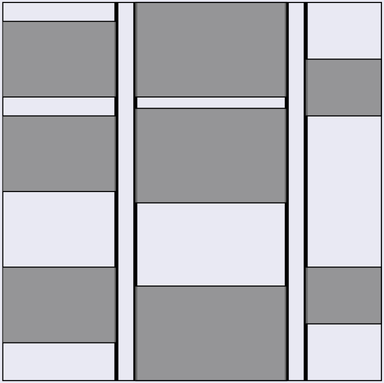
Explicit formula for $\dim_{\mathrm{H}} F, \dim_{\mathrm{B}} F$ and $\dim_{\mathrm{A}} F$ in terms of parameters all have the form 'dim of projection' + 'dim of fibre'.

In particular, either all are different or all are equal (uniform fibres).

Gatzouras–Lalley carpets ('92)

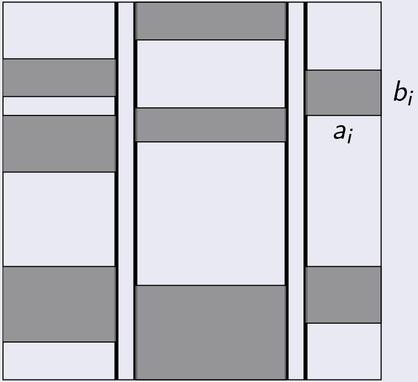
a_i = width of rectangle $f_i([0,1]^2)$ and b_i = height.

Homogeneous case



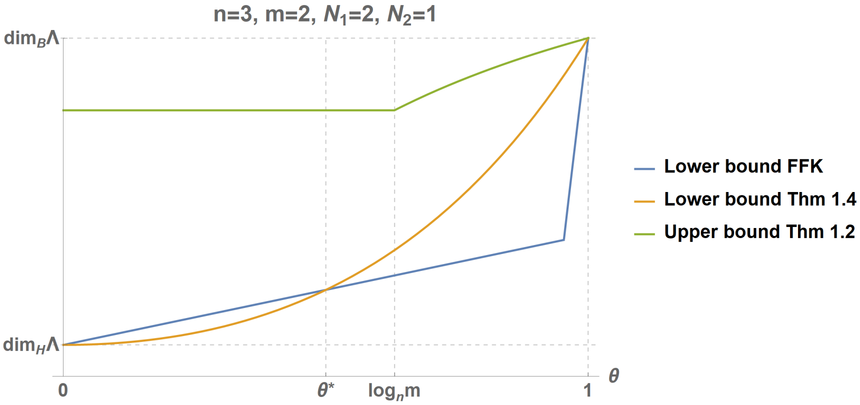
Contractions within columns (in y-direction) is the same.

General case

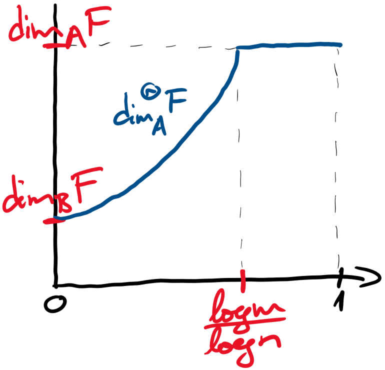


No condition on contractions except that $b_i < a_i$.

Bounds on the intermediate dimensions of BM carpets



- F-F-K: small θ; continuity and positive derivative at $\theta = 0$.
- K: large θ; positive derivative at $\theta = 1$, not necessarily concave.



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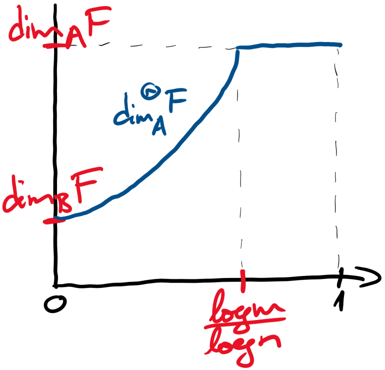
Theorem (Fraser–Yu '18)

$$\dim_{\mathrm{B}} + \frac{\theta}{1-\theta}\left(\frac{\log n}{\log m} - 1\right)(\dim_{\mathrm{A}} - \dim_{\mathrm{B}})$$

Assouad spectrum of Bedford–McMullen carpets



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Question: Does the three parameter form hold for more general carpets?
If not, then what kind of new phase transitions can we witness?

Auxiliary functions for intermediate dimensions

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. r.v.-s with distribution

$$\mathbb{P}(X_1 = \log N_i) = \frac{1}{M} \cdot \#\{j \in \{1, \ldots, M\} : N_j = N_i\}.$$

Then $\mathbb{E}X_1 = \frac{1}{M}\sum_{i=1}^{M} \log N_i$ and the large deviations rate function is

$$I(t) = \sup_{\lambda \in \mathbb{R}} \left\{ \lambda t - \log\left(\frac{1}{M} \sum_{j=1}^{M} N_j^{\lambda} \right) \right\},$$

i.e. the Legendre-transform of $\log \mathbb{E} e^{\lambda X_1}$.

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i.e. the Legendre-transform of $\log \mathbb{E} e^{\lambda X_1}$. Let $\gamma = \frac{\log n}{\log m}$. Also introduce

$$T_s(t) := \left(s - \frac{\log M}{\log m} \right) \log n + \gamma I(t) \quad \text{and}$$

$$t_\ell(s) := T_s^{\ell - 1}\left(\left(s - \frac{\log M}{\log m} \right) \log n \right).$$

Main theorem for intermediate dimensions (B-K '24)

For all $\theta \in (0,1)$, $\dim_\theta \Lambda = s(\theta) = s$ is the unique solution to

$$\gamma^L\theta \log N - (\gamma^L\theta - 1)\, t_L(s) + \gamma(1-\gamma^{L-1}\theta)\big(\log M - I(t_L(s))\big) - s\log n = 0,$$

where $L \in \mathbb{N}$ satisfies $\gamma^{-L} < \theta \le \gamma^{-(L-1)}$.

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where $L \in \mathbb{N}$ satisfies $\gamma^{-L} < \theta \leq \gamma^{-(L-1)}$.

For $\theta \geq \log m / \log n$,

$$s = \dim_{\mathrm{B}} \Lambda - \frac{1}{\log n}\left(\frac{1}{\theta} - 1\right) I\left(t_1(s)\right),$$

and for $\theta = \gamma^{-(L-1)}$ with $L \geq 2$,

$$s = \dim_{\mathrm{B}} \Lambda - \frac{1}{\log m}\left(1 - \frac{1}{\gamma}\right) I\left(t_{L-1}(s)\right).$$

Explicit cover using scales $\delta, \delta^\gamma, \delta^{\gamma^2}, \ldots, \delta^{\gamma^{L-1}}$ $\delta^{1/\theta}, \delta^{1/(\gamma\theta)}, \ldots, \delta^{1/(\gamma^{L-1}\theta)}$.

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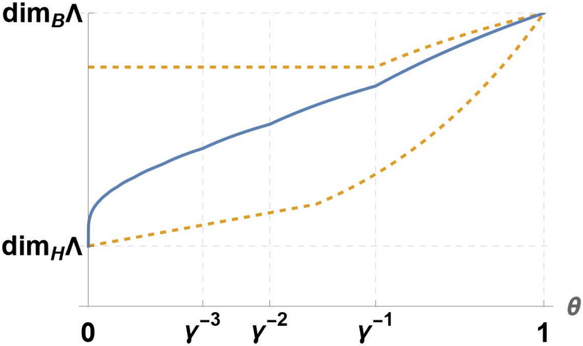


Connection to multifractal analysis

Two BM carpets with non-uniform fibres can be realised on the same grid if and only if

$$\frac{\log n_1}{\log n_2} = \frac{\log m_1}{\log m_2} \in \mathbb{Q}.$$

The following are equivalent for two BM carpets with non-uniform fibres:

1. $\dim_\theta \Lambda = \dim_\theta \Lambda'$ fore every $\theta \in [0,1]$;
2. $f_\nu(\alpha) = f_{\nu'}(\alpha)$ for all $\alpha \in (\alpha_{\min}, \alpha_{\max})$,

where $f_\nu(\alpha)$ is the Hausdorff multifractal spectra of the uniform self-affine measure. ($f_\nu(\alpha) := \dim_{\mathrm{H}} \{x \in \operatorname{supp} \nu : \dim_{\mathrm{loc}}(\nu, x) = \alpha\}$)

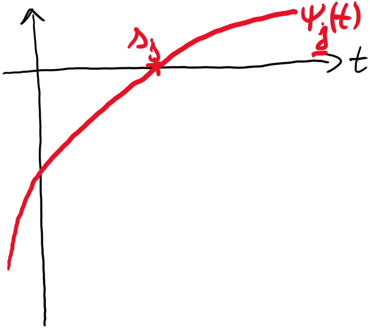
Corollary

Two Bedford–McMullen carpets with non-uniform fibres which are bi-Lipschitz equivalent must satisfy $f_\nu(\alpha) = f_{\nu'}(\alpha)$ for all $\alpha \in (\alpha_{\min}, \alpha_{\max})$.

Auxiliary function for Assouad spectrum

For $\underline{j} \in \eta(\mathcal{I})$ and $t \in \mathbb{R}$ define

$$\psi_{\underline{j}}(t) = \frac{\log \sum_{i \in \eta^{-1}(\underline{j})} b_i^t}{\log a_{\underline{j}}}.$$



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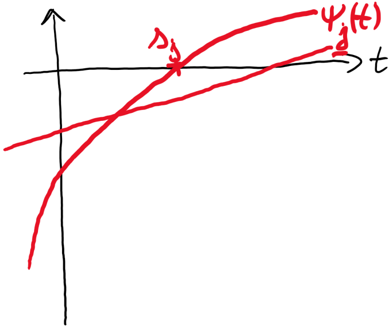

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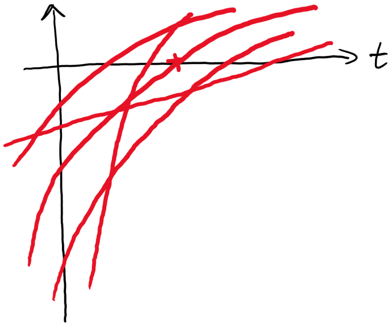

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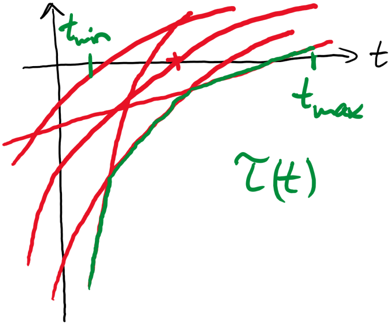
$$\tau(t) = \begin{cases} \min_{\underline{j} \in \eta(\mathcal{I})} \psi_{\underline{j}}(t) & : t \in [t_{\min}, t_{\max} \\ -\infty & : \text{ otherwise} \end{cases}$$



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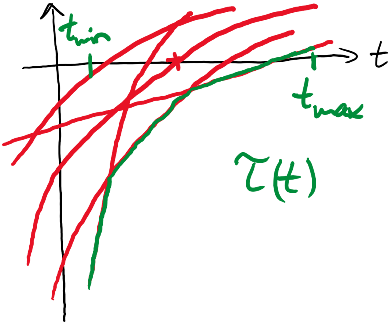

τ is strictly increasing, concave. Its concave conjugate is

$$\tau^*(\alpha) = \inf_{t \in \mathbb{R}} (t\alpha - \tau(t))$$

Main theorem for Assouad spectrum (B-F-K-R '24+)

For any Gatzouras–Lalley carpet K and all $\theta \in (0,1)$,

$$\dim_{\mathrm{A}}^{\theta} K = \dim_{\mathrm{B}} \eta(K) + \frac{\tau^*(\phi(\theta))}{\phi(\theta)},$$

where

$$\phi(\theta) = \frac{1/\theta - 1}{1 - 1/\kappa_{\max}} \quad \text{and} \quad \kappa_{\max} = \max_{i \in \mathcal{I}} \frac{\log b_i}{\log a_i}.$$

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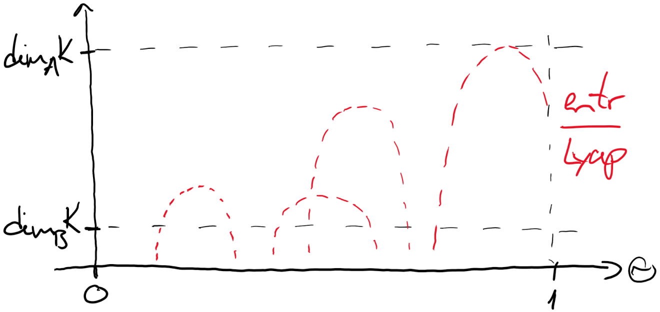

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For any Gatzouras–Lalley carpet K and all $\theta \in (0,1)$,

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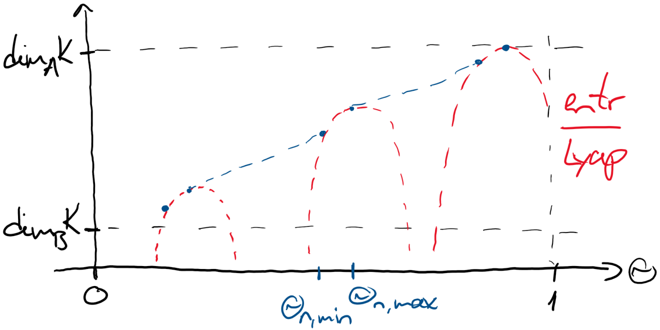

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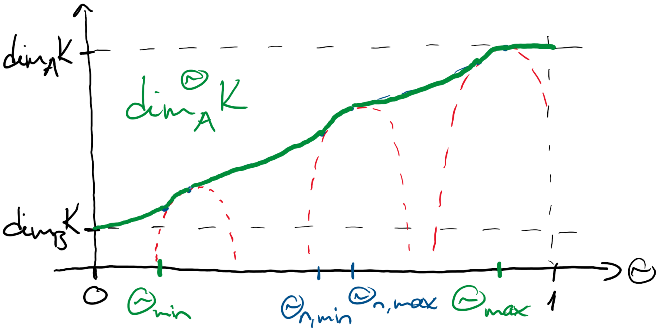

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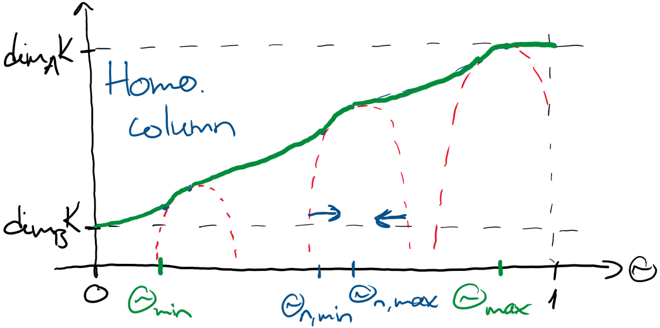

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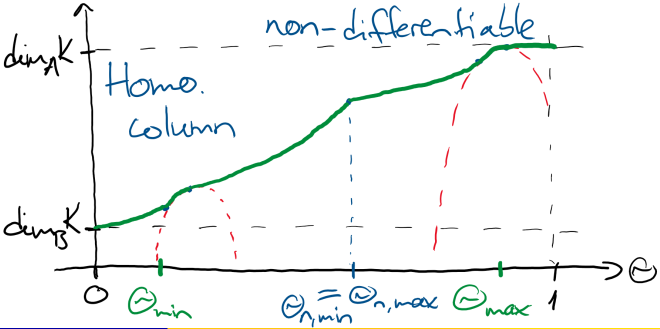

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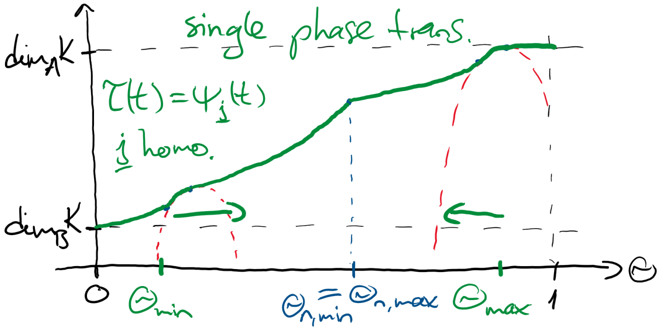

Form of the spectrum

Phase transitions

- $\theta \mapsto \dim_{\mathrm{A}}^{\theta} F$ is piecewise analytic and the set of points where the function is not analytic is given precisely by

$$H = \{\theta : \theta = \theta_{n,\min} \text{ or } \theta = \theta_{n,\max} \quad \text{for some} \quad n = 1, \ldots, m\} .$$

- First order $H_1 := \{\theta : \theta = \theta_{n,\min} = \theta_{n,\max}$ for some $n = 1, \ldots, m\}$. Differentiable if and only if each I_n is inhomogeneous.
- Higher order odd $H_{\text{higher}} := \{\theta : \theta = \theta_{n,\max} = \theta_{n+1,\min}$ for some $n = 1, \ldots, m-1\} \setminus H_1$. Even can only be second.

Convexity and concavity

Non-trivial interval of convexity $[0, \theta_{\min}]$, and if every column is inhomogeneous, then $\dim_{\mathrm{A}}^{\theta} F$ also contains a non-trivial interval of concavity (close to $\theta_{\max}$).

A variational formula

For any Gatzouras–Lalley carpet F and all $\theta \in (0,1)$,

$$\dim_{\mathrm{A}}^{\theta} F = \max_{(\boldsymbol{v},\boldsymbol{w})\in\mathcal{P}\times\mathcal{P}} f(\theta, \boldsymbol{v}, \boldsymbol{w}),$$

where $\mathcal{P} = \mathcal{P}(\mathcal{I}) = \left\{ (p_i)_{i\in\mathcal{I}} : p_i \geq 0, \sum_{i\in\mathcal{I}} p_i = 1 \right\}$ and

$$f(\theta, \boldsymbol{v}, \boldsymbol{w}) = \begin{cases} f_{\text{thin}}\,(\theta, \boldsymbol{v}, \boldsymbol{w}) & : (\boldsymbol{v}, \boldsymbol{w}) \in \Delta_{\text{thin}}\,(\theta); \\ f_{\text{thick}}\,(\theta, \boldsymbol{v}, \boldsymbol{w}) & : (\boldsymbol{v}, \boldsymbol{w}) \in \Delta_{\text{thick}}\,(\theta). \end{cases}$$

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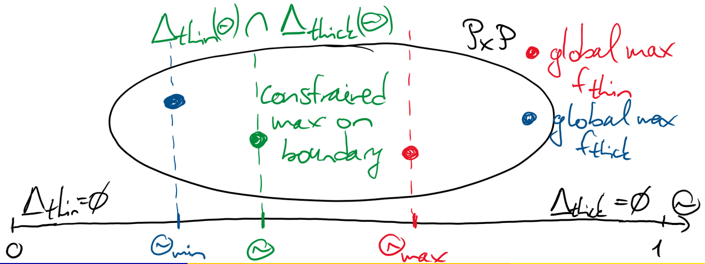

Open problems

- Bounds on number of phase transitions in general case.
- Assouad spectrum of Barański carpets and higher dimensional sponges.
- Lower spectrum ('dual' to the Assouad spectrum).
- Conditions for bi-Lipschitz equivalence. If the spectrum partition of one carpet contains a homogeneous column while the other does not, then the two carpets cannot be bi-Lipschitz equivalent.
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Thank you for your attention!