Randomised mixed labyrinth fractals

Janett Prehl[1], Daniel Dick[1,2], and Ligia L. Cristea[3]

[1] Institute of Physics, Chemnitz University of Technology, Chemnitz
[2] Fraunhofer Insitute for Electronic Nano Systems (ENAS), Chemnitz
[3] Freelancer (lecturer at different academic institutions in Graz/Kapfenber)

Fractal Geometry and Stochastics 7
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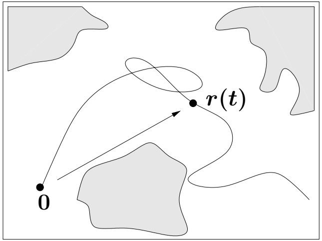
Diffusion in porous materials

Dynamical behavior of particles

- inside a bulk
- on the surface (deposit)



How to model porous (“real”) materials?

- Holes, barriers and connections on all length scales
- Self similar over certain length scales
- At larger length scales → rather homogeneous
- Smallest length scale → given by material

Usage of fractal sets:Sieprinski carpets

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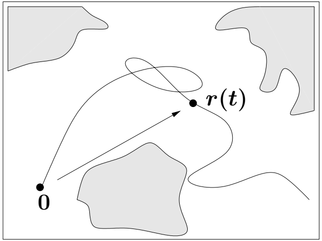

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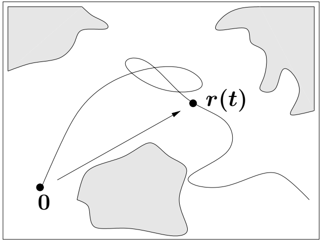

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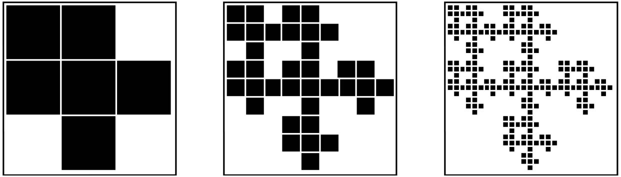
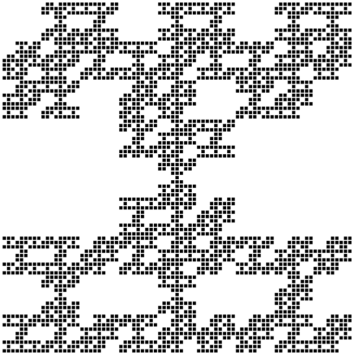
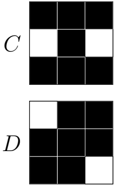
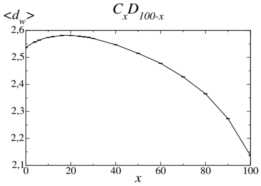


Diffusion in porous materials

How does randomness of structure influence dynamics?







DHN Anh, et al.; Europhys Lett, 70:109, 2005;
DHN Anh, et al.; J Phys A: Math Theor, 40:11453, 2007

Main results:

- Dynamical behavior depends on underlying structure
- More “randomness” could increase dynamics
- Major influence factors: connection points, active sites, shortest path lengths

What happens if do not look on the dynamics, but the structure itself?

From dynamics to structure

What happens if do not look on the dynamics, but the structure itself?

- Reduce complexity of structure $\rightarrow$ dentridic structures $\rightarrow$ Labyrinth fractals
- Mixing different pattern with a given selection probability

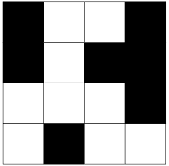
Our question of interest:

- Does randomness alters structural properties of labyrinth fractals?
- If yes, how does it alters?
- Can we predict the changes from the underlying pattern?

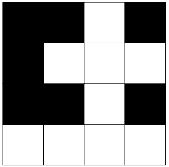
Labyrinth fractals

- Special class of Sierpinski carpets
  LL Cristea, B Steinsky, Proc Edinburgh Math Soc 54:329,2011, DHN Anh et al, J Phys A: Math Gen, 40:11453, 20007
- Generator pattern $\mathcal{A}$, $n = 1$: size $m \times m$ with w white squares
  - Tree property: $\mathcal{G}(\mathcal{A})$ is a tree, where $\mathcal{G}(\mathcal{A})$ is a graph associated to any m-pattern $\mathcal{A}$.
    $\Rightarrow$ No loops
  - Exits property: $\mathcal{A}$ has exactly one vertical exit pair, and exactly one horizontal exit pair.
    $\Rightarrow$ Finitely ramified structure
  - Totally- or non-blocked pattern
  - Corner property: If there is a white square in $\mathcal{A}$ at a corner of $\mathcal{A}$, then there is no white square in $\mathcal{A}$ at the diagonally opposite corner of $\mathcal{A}$.

Pattern $\mathcal{A}$
Totally blocked



Pattern $\mathcal{B}$
Non-blocked



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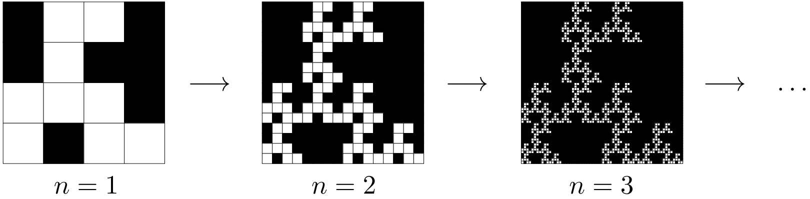
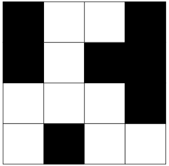
- Generator pattern $\mathcal{A}$, $n = 1$: size $m \times m$ with w white squares
- Labyrinth set of iteration level n: Sequence $\{A_n\}_{n>0}$

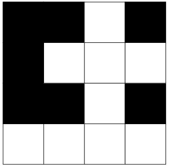


- Labyrinth fractals for $n \to \infty$
- Fractal (box-counting) dimension $d_f = \lim\limits_{n\to\infty} \frac{\log w(n)}{\log m(n)} = \frac{\log w}{\log m}$

Pattern $\mathcal{A}$
Totally blocked

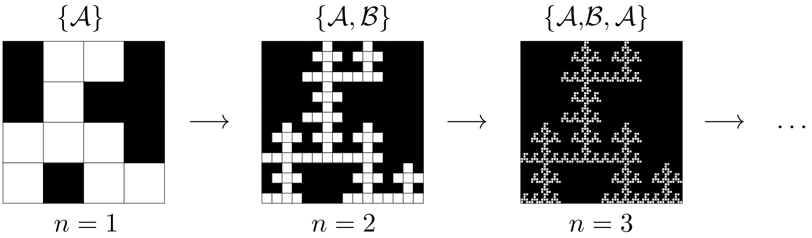


Pattern $\mathcal{B}$
Non-blocked

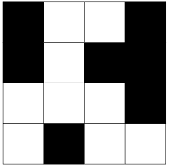


Randomised mixed labyrinth fractals

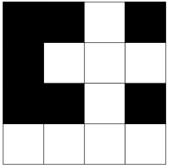
- Given set of (at least) two patterns $\{\mathcal{A}, \mathcal{B}\} \rightarrow$ here two patterns
- Mixed labyrinth (ML) set: Randomly chosen pattern for each iteration level n (LL Cristea, B Steinsky, Topol Appl 229:112, 2017
- Randomised mixed labyrinth (RML) set of iteration level n: Choose a pattern with selection probabilities $\{p, 1-p\}$ for each iteration level n



Pattern $\mathcal{A}$
Totally blocked

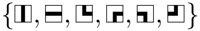


Pattern $\mathcal{B}$
Non-blocked

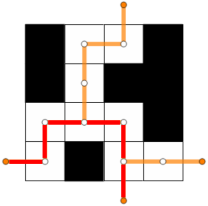


Structural properties

Labyrinth pattern

- Path a connects two exits: $a =$ 
- Path length determination via path matrix M of pattern $\mathcal{A}_n$

pattern $\mathcal{A}$

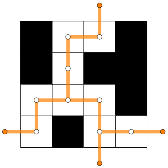


$$\rightarrow \quad M = \begin{pmatrix} 2 & 0 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 2 & 0 & 3 \end{pmatrix}$$

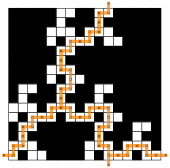
- Lengths $\vec{\ell}(n)$ of all paths a of (RM) labyrinth set

$$\vec{\ell}(n) = M(n) = \prod_{k=1}^{n} M_k \cdot (1,1,1,1,1,1)^T$$

$\{\mathcal{A}\}$



$\{\mathcal{A}, \mathcal{A}\}$



Structural properties

Path properties for ML fractals

- Scaling λ of path lengths $l(a,n)\,\forall\, a \leftrightarrow$ Restoration of isotropy
  MT Barlow et al PRL 75:3042,1995; A Franz, et al. SIGSAM Bulletin, 36:18,2002

$$\lambda = \lim_{n\to\infty} \lambda(a,n) = \lim_{n\to\infty} \frac{\ell(a,n)}{\ell(a,n-1)}$$

- Shortest path dimension d_{min} (for const. m)

$$d_{\text{min}} = \lim_{n\to\infty} \frac{\log \ell_{\text{min}}}{\log m(n)} = \lim_{n\to\infty} \frac{\log \lambda}{\log m}$$

- Arc dimension $\text{dim}_{\text{B}}(a)$ can be determined by spectral radius r of M for (mixed) labyrinth fractals
  LL Cristea, B Steinsky, Proc Edinburgh Math Soc 54:329,2011; Topol Appl 229:112, 2017, LL Cristea, G Leonbacher, Monatsh Math 185:575, 2018

$$\text{dim}_{\text{B}}(a) = \lim_{n\to\infty} \frac{\log \ell(n,a)}{\log m(n)} = \begin{cases} 1 & \text{non-blocked} \\ \frac{\log r}{\log m} > 1 & \text{totally blocked} \end{cases}$$

Structural properties of RML sets

Application to RML sets:

1. Direct connection between $\text{dim}_\text{B}(a)$ and d_min if m is const.

$$\text{dim}_\text{B}(a) = \lim_{n\to\infty} \frac{\log \ell(n,a)}{\log m(n)} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\log \frac{\ell(a,n)}{\ell(a,n-1)}}{n \log m} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{\log \lambda_k(a)}{\log m} \approx \frac{1}{n} \sum_{k=1}^{n} d_\text{min}$$

2. Generalization of path matrix approach for RML sets
   Approximated path matrix $\rightarrow$ approximated arc dimension

$$\widetilde{M} = p\widetilde{M}_\mathcal{A} + (1-p)\,\widetilde{M}_\mathcal{B} \quad \rightarrow \quad \text{Spectral radius } \tilde{r} \quad \rightarrow \quad \widetilde{\text{dim}_\text{B}}(a) = \frac{\log \tilde{r}}{\log m}$$

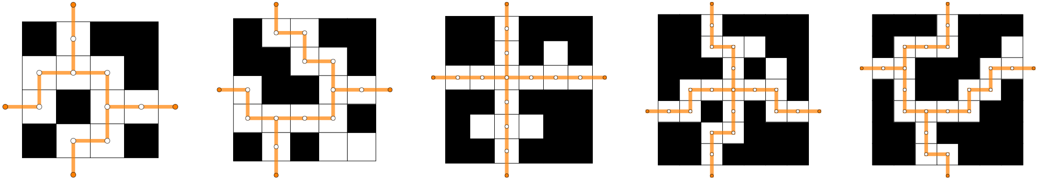
What are we interested?

Our question of interest:

- Does randomness alters structural properties of labyrinth fractals?
- If yes, how does it alters?
- **Can we predict the changes from the underlying pattern?**
  - Single pattern geometry of $\mathcal{A}$ and $\mathcal{B}$ and
  - Selection probability p

Analysed labyrinth pattern

- Combinations of two pattern of same m
- d_{f} const. for given m, but different d_{min}
- Patterns of different width $m = \{4, 5, 6, 7\}$
- Totally-, non-blocked pattern and rotated versions of them
- Different path length properties (sym., asym., ...)



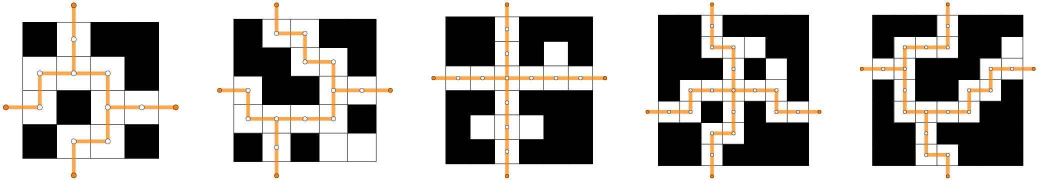
Some of used patterns

Analysed labyrinth pattern

Numerical details:

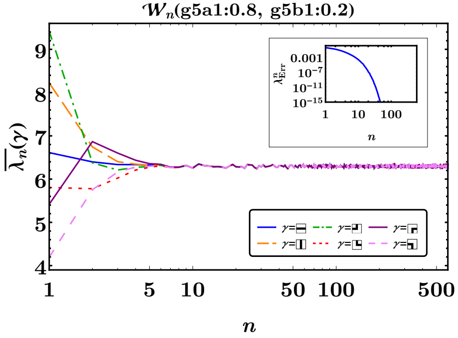
- Iteration level $n = 1000$
- Ensemble averages over 2000 realizations $\rightarrow \langle d_{\text{min}} \rangle$
- Min-max-error and standard deviation
- Determination of $\widetilde{\dim_B}(a)$

$\Rightarrow$ Investigation and comparison of statistical obtained $\langle d_{\text{min}} \rangle$ and $\widetilde{\dim_B}(a)$ for different p.

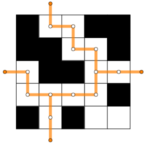


Some of used patterns

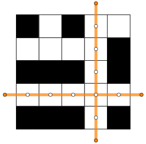
Results - Restoration of Isotropy



pattern g5a1
with $p = 0.8$



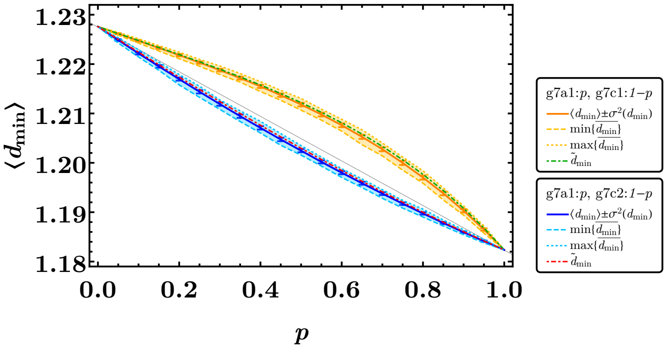
pattern g5b1
with $p = 0.2$

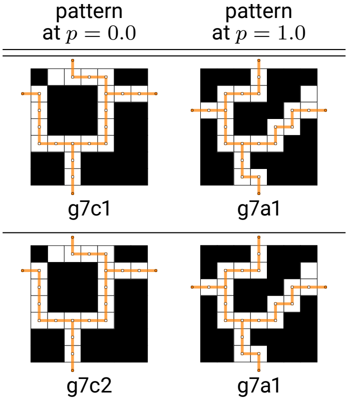


- All paths scaling factors $\lambda_n = l(a,n)/l(a,n-1)$ reach same value (for all pair combinations)
- Restoration of isotropy

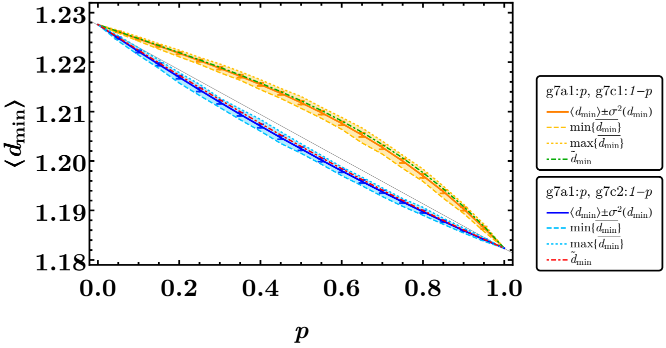
Results - Monotonic behaviour

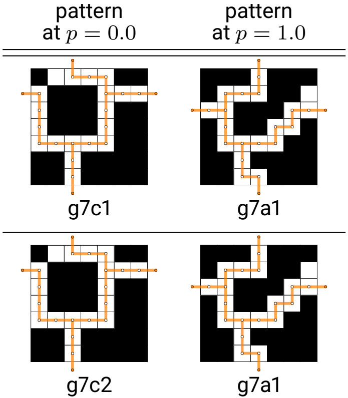
$\langle d_{\text{min}} \rangle$ and $\widetilde{\text{dim}_\text{B}}(a)$ over p for pairs of labyrinth pattern





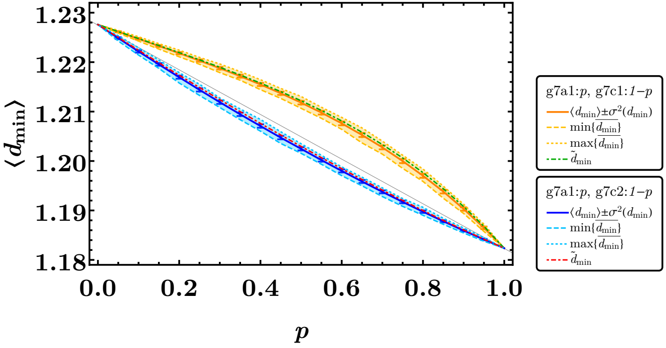
Results - Monotonic behaviour

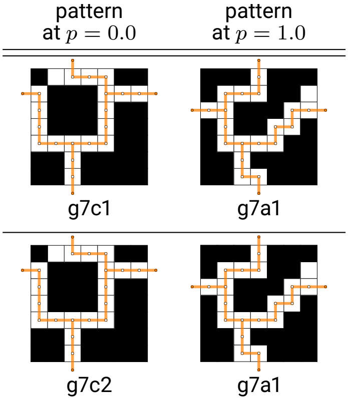




- $\widetilde{\dim_B}(a)$ is within min-max-error of $\langle d_{\min} \rangle$
- Monotonic behavior: Two totally blocked pattern, where path lengths of $\mathcal{A}$ are **nearly** all larger then $\mathcal{B}$.

Results - Monotonic behaviour





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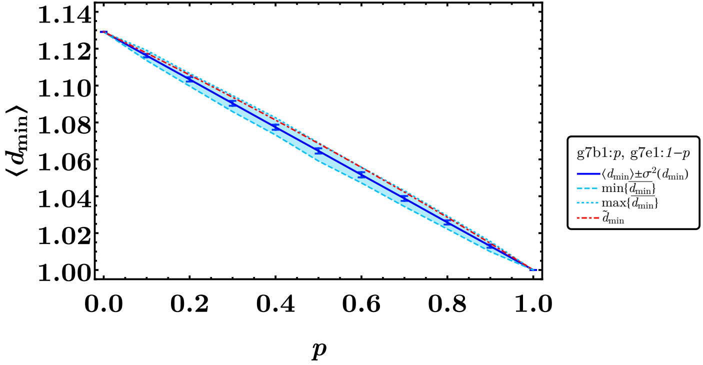
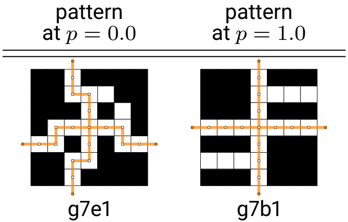
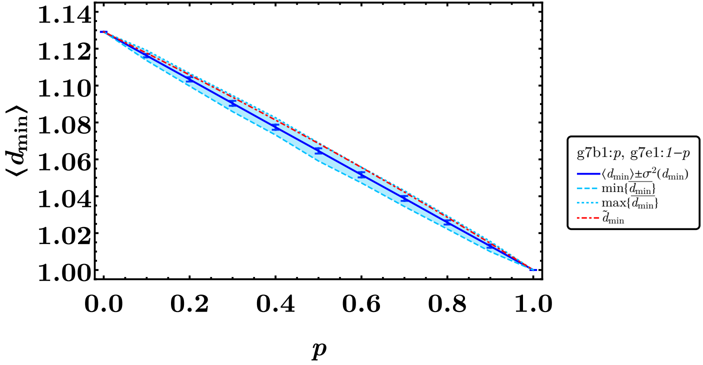
Results - Special case: Linear behaviour

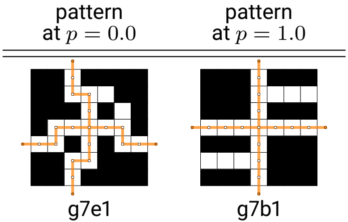




- $\widetilde{\dim}_B(a)$ is within min-max-error of $\langle d_{\min} \rangle$
- Linear behavior for combination of two totally blocked pattern, where path lengths of $\mathcal{A}$ are **all** larger then $\mathcal{B}$.

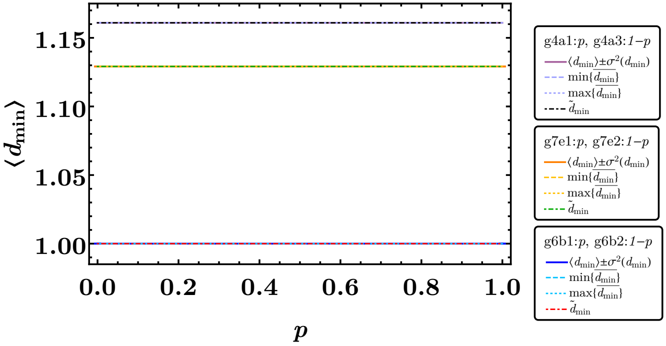
Results - Special case: Linear behaviour

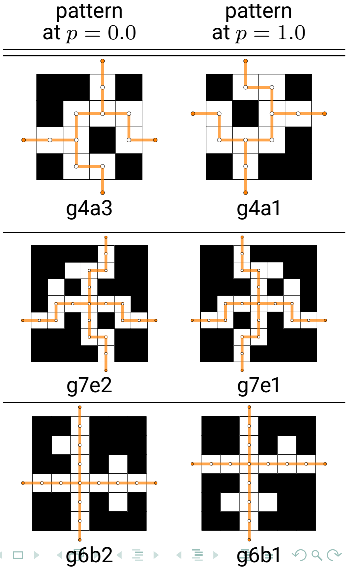




- $\widetilde{\dim}_B(a)$ is within min-max-error of $\langle d_{\min}\rangle$
- Linear behavior for combination of two totally blocked pattern, where path lengths of $\mathcal{A}$ are **all** larger then $\mathcal{B}$.

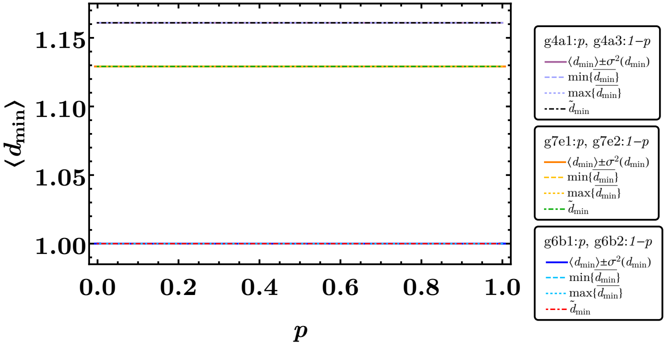
Results - Special case: Constant behaviour



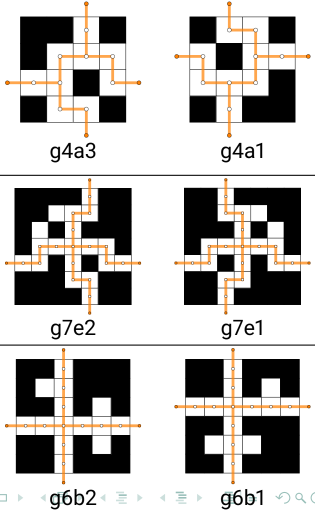


- $\widetilde{\dim_B}(a)$ is within min-max-error of $\langle d_{\min} \rangle$
- Constant behavior: Both patterns are non-blocked or totally blocked, with identical/mirror symmetric path lengths

Results - Special case: Constant behaviour



pattern at $p = 0.0$	pattern at $p = 1.0$



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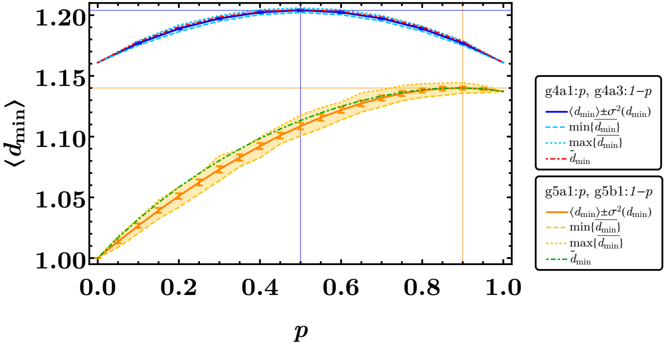
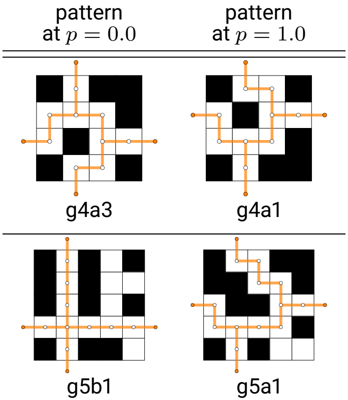
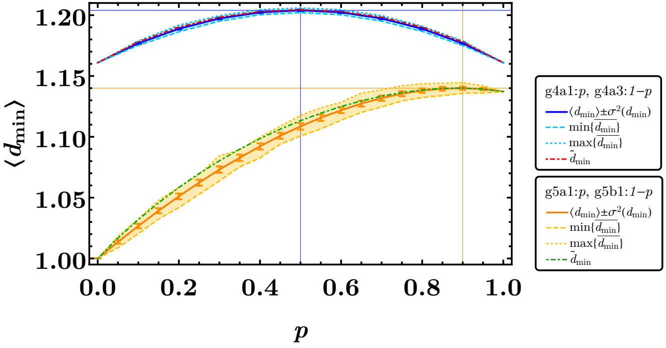
Results - Maximum behaviour

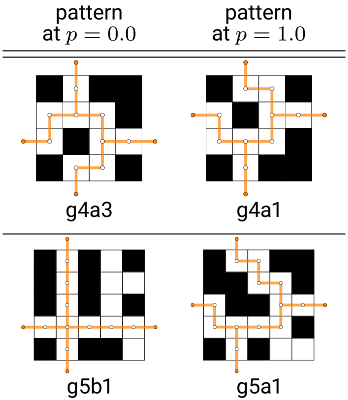




- $\widetilde{\dim}_B(a)$ is within min-max-error of $\langle d_{\min} \rangle$
- Some combinations of two totally blocked patterns, that are rotated versions of each other by $90°$ or $270°$
- Some combinations of patterns, where one “short” path “replaces” a “longer”one

Results - Maximum behaviour



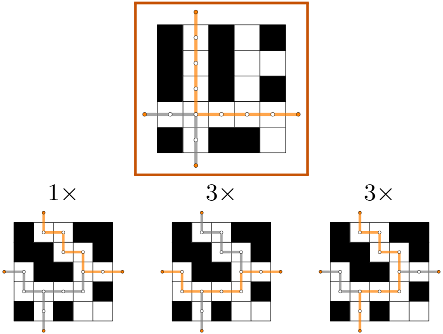


- $\widetilde{\dim}_B(a)$ is within min-max-error of $\langle d_{min} \rangle$
- Some combinations of two totally blocked patterns, that are rotated versions of each other by $90°$ or $270°$
- Some combinations of patterns, where one “short” path “replaces” a “longer”one

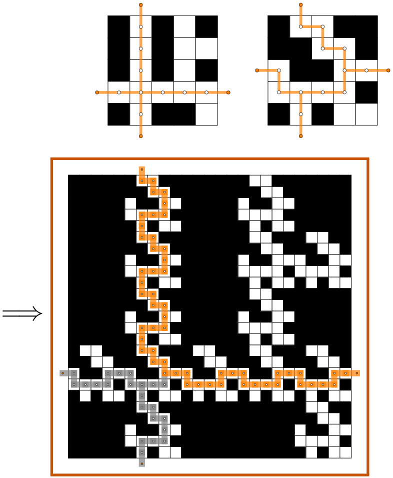
Results - Maximum behaviour

How can one understand the maximum?

- Analysis of iteration process via $\langle d_{\text{min}} \rangle$:



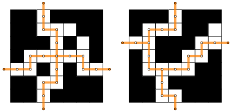
"Shortcut" "Detour" "Detour"



Overall "detour"

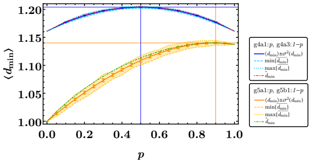
- Sometimes "shortcuts" get "detours", leading to larger $\langle d_{\text{min}} \rangle$ due to iteration

Summary

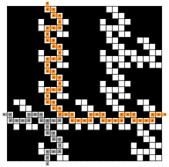


- Mixing pairs of labyrinth patterns with given selection probability

- Direct connection between $\dim_{\mathrm{B}}(a)$ and d_{min} for const. m
- Approximated path matrix $\widetilde{M} \rightarrow \widetilde{\dim_{\mathrm{B}}}(a)$



- Restoration of isotropy
- Four types of behavior of $\widetilde{\dim_{\mathrm{B}}}(a)$ and $\langle d_{\mathrm{min}} \rangle$:
  - Predictable: constant and linear behaviour
  - Somehow predictable: monotonic behaviour
  - Unpredictable: maximum behaviour



- In all cases fits $\widetilde{\dim_{\mathrm{B}}}(a)$ the ensemble average $\langle d_{\mathrm{min}} \rangle$ with the min-max-error.

$\Rightarrow$ Approximation of path matrix seems a good tool for prediction

Summary

Appendix: Labyrinth pattern properties

Property

[Tree Property] $\mathcal{G}(\mathcal{A})$ is a tree.

To any m-pattern $\mathcal{A}$ we associate a graph $\mathcal{G}(\mathcal{A})$:

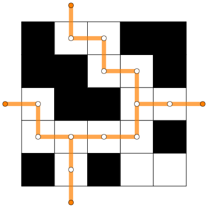
- Each vertex in the graph corresponds to a white square in the pattern..
- Two distinct vertices in the graph are connected by an edge if the corresponding squares share a common edge.
- A tree is a connected graph that contains no loops.

Appendix: Labyrinth pattern properties

Property

[Exits Property] $\mathcal{A}$ has exactly one vertical exit pair, and exactly one horizontal exit pair.

- The horizontal exit pair in a pattern is a pair of white squares that lie in the same row of the pattern. (leftmost = *left exit*, rightmost = *right exit*)
- The vertical exit pair consists of the *top exit* and the *bottom exit*, which lie in the same column of the pattern, in the top and, respectively, bottom row.



Appendix: Labyrinth pattern properties

Property

[Corner Property] If there is a white square in $\mathcal{A}$ at a corner of $\mathcal{A}$, then there is no white square in $\mathcal{A}$ at the diagonally opposite corner of $\mathcal{A}$.

Appendix: Definition fractal dimensions

Box counting dimension $\dim_{\mathrm{B}}(F)$

$\dim_{\mathrm{B}}(F)$ of a set F is given as

$$\dim_{\mathrm{B}}(F) = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}, \qquad (1)$$

where $N_\delta(F)$ is the number of δ-mesh squares that cover F. (see K Falconer 2014)

Fractal dimension d_{f}

$w \sim \ell^{d_{\mathrm{f}}}$ is given as

$$d_{\mathrm{f}} = \lim_{n \to \infty} \frac{\log\left(w(n)\right)}{\log\left(\ell(n)\right)} = \frac{\log w}{\log \ell} \;. \qquad (2)$$

(equals Hausdorff dimension for self-similar case)

Appendix: Definition fractal dimensions

Shortest path dimension d_{min}

$\ell_{\text{min}} \sim \ell^{d_{\text{min}}}$ is given as

$$d_{\text{min}} = \lim_{n\to\infty} \frac{\log(\ell_{\text{min}}(n))}{\log(\ell(n))} = \frac{\log \ell_{\text{min}}}{\log \ell} \; . \tag{3}$$

Chemical dimension d_{chem}

$$d_{\text{chem}} = \frac{d_{\text{f}}}{d_{\text{min}}} = \frac{\log w}{\log \ell_{\text{min}}} \; , \tag{4}$$

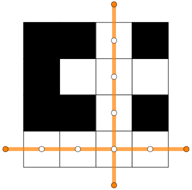
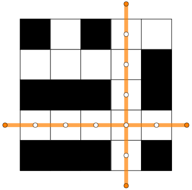
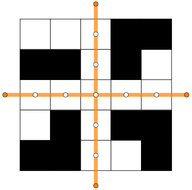
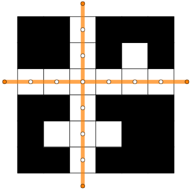
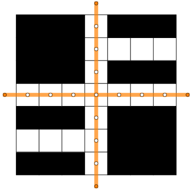
Appendix: Connection arc to shortest path dimension

$$\text{dim}_B(a) = \lim_{n\to\infty} \frac{\log \ell(a,n)}{\log m(n)} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\log \frac{\ell(a,n)}{\ell(a,n-1)}}{n \log m}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{\log \lambda_k(a)}{\log m}$$

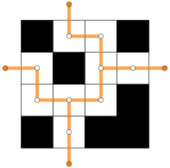
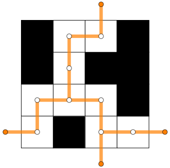
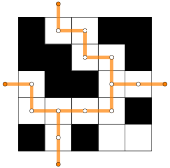
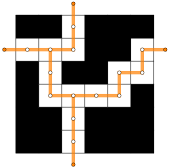
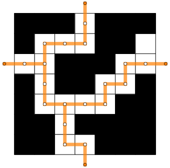
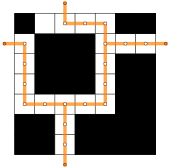
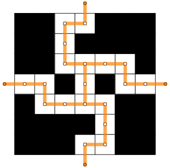
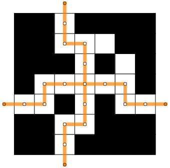
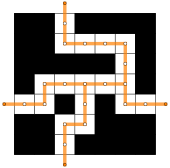
$$\approx \frac{1}{n} \sum_{k=1}^{n} d_{\text{min}}$$

Appendix: Analysed labyrinth patterns

g4b1	g5b1	g5c1	g6b1	g7b1
				

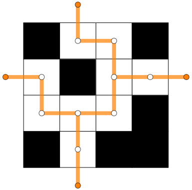
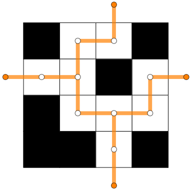
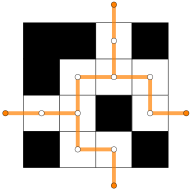
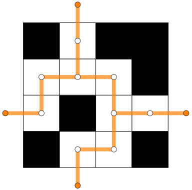
Analysed non-blocked labyrinth patterns.

Appendix: Analysed labyrinth patterns

g4a1	g4c1	g5a1	g6a1	
				
g7a1	g7c1	g7d1	g7e1	g7f1
				

The analysed totally blocked labyrinth patterns.

Appendix: Analysed labyrinth patterns

g4a1	g4a2	g4a3	g4a4
			
$\circlearrowright$ 0°	$\circlearrowright$ 90°	$\circlearrowright$ 180°	$\circlearrowright$ 270°

Example (g4a) for notation of labyrinth patterns obtained by rotation.