Rate of memory loss in nonstationary dynamical systems with some hyperbolicity

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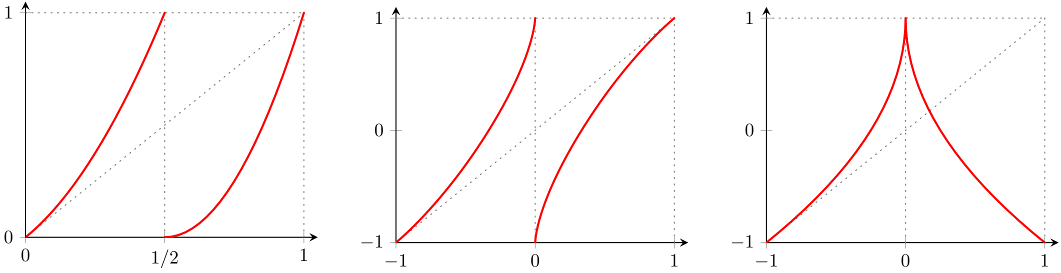
Interval maps with neutral fixed points

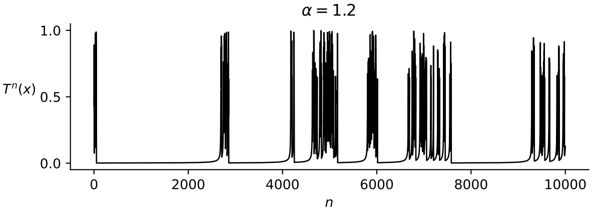


Numerical studies by physicists:

- Pomeau, Manneville '80: intermittent transition to turbulence in convective fluids
- Grossmann, Horner '85: slowly (polynomially) decaying correlations, infinite static susceptibility
- Pikovsky '91: dynamically generated anomalous diffusion

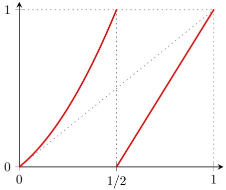
Intermittent dynamics

Trajectories alternate between a laminar phase, caused by neutral fixed points, and a chaotic phase

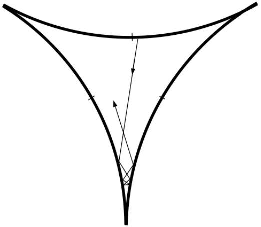


$$T_\alpha : [0,1] \to [0,1], \quad \alpha > 0$$

$$T_\alpha(x) = \begin{cases} x(1 + 2^\alpha x^\alpha), & x \le \frac{1}{2}, \\ 2x - 1, & x > \frac{1}{2} \end{cases}$$

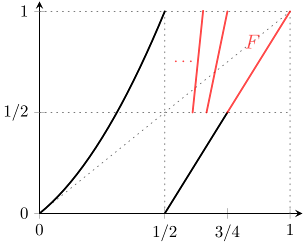


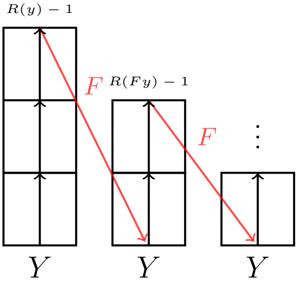
Similar phenomena occurs in more complex systems, e.g. dispersing billiards with cusps: laminar phase occurs (for the collision map) as the billiard particle becomes trapped deep in a cusp



Inducing schemes

$$Y := [1/2, 1] \quad R(y) := \inf\{n \geq 1 \,:\, T_\alpha^n(y) \in Y\}, \quad F(y) := T_\alpha^{R(y)}(y)$$



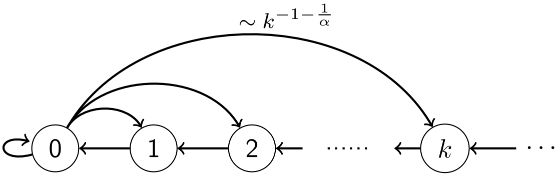


Young '99: ∃! absolutely continuous invariant probability measure (a.c.i.p.) μ_α iff $\int_Y R(y)\,dy < \infty$.
The rate of mixing is determined by the decay rate of $\{R > n\}$: for Hölder φ, ψ,

$$\text{Leb}(\{R > n\}) = O(n^{-1/\alpha}) \implies \int \varphi \circ T_\alpha^n \cdot \psi \, d\mu_\alpha - \int \varphi \, d\mu_\alpha \int \psi \, d\mu_\alpha = O(n^{1-1/\alpha})$$

Remark: $\int_Y R(y)\,dy < \infty \iff \alpha \in (0,1)$

Memory loss in Markov chains (Lindvall '79)



- $\exists!$ probability measure π invariant under Perron-Frobenius transfer operator $\mathcal{P}$
- If μ_k is concentrated at k, then $|\mathcal{P}^n(\pi - \mu_k)| \lesssim n^{1-\frac{1}{\alpha}}$
- But $|\mathcal{P}^n(\mu_k - \mu_j)| \lesssim n^{-\frac{1}{\alpha}}$

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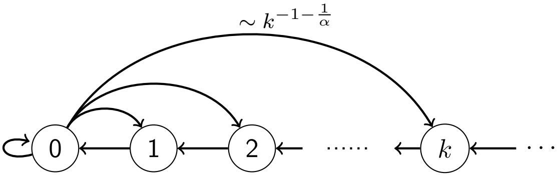

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Memory loss for intermittent maps

Classic (Young '99, Hu '04, Gouëzel '04): $|(T_\alpha^n)_*\mu - (T_\alpha^n)_*\mu_\alpha| \lesssim n^{1-\frac{1}{\alpha}}$ for "regular" measures μ

Unexpected (Gouëzel '04): $|(T_\alpha^n)_*\nu - (T_\alpha^n)_*\nu'| \lesssim n^{-\frac{1}{\alpha}}$ for "special" measures ν, ν' (including those with Hölder densities)

Proof: operator renewal theory + Wiener lemma

Memory loss for nonstationary intermittent dynamical systems

Nonstationary dynamical systems: description of nonequilibrium systems whose laws vary with time under external influence (noise, fluctuating environment, control-signals, etc.)

Consider a sequence $T_1, T_2, \ldots$ of intermittent maps with corresponding parameters $\alpha_1, \alpha_2, \ldots$.

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Thm (Korepanov, L. '21)

Suppose that $\sup_n \alpha_n \le \alpha_* < 1$. Then, for measures ν, ν' with Hölder densities:

a) $|(T_{1,n})_* \mu_\beta - (T_{1,n})_* \nu| \lesssim n^{1 - \frac{1}{\alpha_*}}$ $\qquad (\beta \le \alpha_*)$

b) $|(T_{1,n})_* \nu - (T_{1,n})_* \nu'| \lesssim n^{-\frac{1}{\alpha_*}}$

Here, $T_{1,n} := T_n \circ \cdots \circ T_1$ and $|\cdot|$ denotes the total variation distance.

Related: Aimino, Hu, Nicol, Török, Vaienti '15 obtained $\lesssim (\log n)^{1/\alpha_*} n^{1-1/\alpha_*}$ in a)

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Question

How are the rates of convergence in a) and b) affected by subsequences (T_{n_k})?

Condition on the density of "good" maps

- (T_n): sequence of intermittent maps with parameters $0 < \alpha_n \leq \alpha_* < 1$
- ν, ν': probability measures with Hölder densities
- μ_β: a.c.i.p. of T_β

Condition

Let $\gamma \in (0, \alpha_*]$. Suppose that for a sufficiently small $\varepsilon > 0$ there exists $N \geq 1$ such that

$$\frac{\#\{1 \leq k \leq n \; : \; \alpha_k \leq \gamma\}}{n} \in [b - \varepsilon, b + \varepsilon] \quad \forall n \geq N. \tag{$*$}$$

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Thm (Korepanov, L. '24+)

Assume $(*)$. Then:

a) $|(T_{1,n})_* \mu_\beta - (T_{1,n})_* \nu| \lesssim n^{1-\frac{1}{\gamma}} \qquad (\beta \le \gamma)$

b) $|(T_{1,n})_* \nu - (T_{1,n})_* \nu'| \lesssim n^{-\frac{1}{\gamma}}$

RDS with an ergodic driving system

Suppose that a sequence $\omega = (\omega_n)$ of parameters is sampled randomly from a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ where $\Omega = (0, \alpha_*]^{\mathbb{N}}$ and $\mathcal{F} =$ Borel(Ω).

Assumptions

- the shift map $\sigma : \Omega \to \Omega$, $(\sigma\omega)_n = \omega_{n+1}$, preserves $\mathbf{P}$,
- $(\sigma, \mathbf{P})$ is ergodic,
- $\mathbf{P}(\omega_1 \leq \gamma) > 0$

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Let $\pi_1 : \Omega \to (0, \alpha_*]$ project onto the first coordinate. By Birkhoff's ergodic theorem, for $\mathbf{P}$-a.e. $\omega \in \Omega$,

$$\frac{|\{1 \leq k \leq n \ : \ \omega_k \leq \gamma\}|}{n} = n^{-1} \sum_{k=0}^{n-1} \mathbf{1}_{(0,\gamma]} \circ \pi_1(\sigma^k \omega) \stackrel{n\to\infty}{\to} \mathbf{P}(\omega_1 \leq \gamma) > 0.$$

Thus, we obtain estimates on the rate of quenched memory loss for ergodic compositions.

Related: Bahsoun, Bose, Ruziboev '18 obtained (classic) rates of mixing for i.i.d. compositions.

Application of memory loss: concentration inequalities

Let $S_n = \sum_{k=0}^{n-1} v_k \circ T_{1,k}$ where $v_k : [0,1] \to \mathbb{R}$ are Hölder with $\lambda(v_k) = 0$. We consider the sums S_n as random variables/process on $([0,1], \text{Borel}, \lambda)$ where $\lambda = \text{Leb}|_{[0,1]}$

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Question

How fast does S_n grow?

If $\mathbf{E}(S_n^p) \le M_n$, then $\mathbf{P}(S_n \ge t) \le M_n / t^p$ (the smaller M_n and larger p, the better)

Notation:

$$\|X\|_p = \mathbf{E}(|X|^p)^{1/p}, \quad \|X\|_{p,\infty} = \left[\sup_{t>0} t^p \mathbf{P}(|X| \ge t)\right]^{1/p}$$

Martingale approach

Definitions: for a martingale (M_n) w.r.t. a filtration $(\mathcal{F}_n)$, define

$$[M]_n = \sum_{k \le n} |M_k - M_{k-1}|^2 \quad \text{(quadratic variation)}$$

$$\sigma_n = \sum_{k \le n} \mathbf{E}(|M_k - M_{k-1}|^2 | \mathcal{F}_{k-1}) \quad \text{(conditioned quadratic variation)}$$

$$M_n^* = \sup_{k \le n} M_k \quad \text{(record process)}$$

Thm (Burkholder)

For every $p \ge 1$,

$$\|M_n^*\|_p \sim \|[M]_n^{1/2}\|_p \qquad \text{(same estimate holds for } \|\cdot\|_{p,\infty}\text{)}$$

$$\|M_n^*\|_p \lesssim \|\sigma_n^{1/2}\|_p + \|\max_{j \le n} |M_j - M_{j-1}|\|_p.$$

Constructing the martingale

Symbolic coding: recall that $Y = [1/2, 1]$, and let

$$a_n = a_n(x) = \begin{cases} 0, & T_{1,n}(x) \notin Y, \\ 1, & T_{1,n}(x) \in Y \end{cases}$$

Filtration: $\mathcal{F}_n$ is generated by $a_0, \ldots, a_{n-1}$

Doob martingale: fix $N > 0$ and let

$$M_n = \mathbf{E}(S_N | \mathcal{F}_n).$$

Then, $M_n \to S_N$ pointwise as $n \to \infty$.

Increments $M_k - M_{k-1}$

Consider

$$M_1 - M_0 = \mathbf{E}(S_n|\mathcal{F}_1) - \mathbf{E}(S_n) = \mathbf{E}(S_n|\mathcal{F}_1).$$

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Let $m = \text{Leb}|_Y/\text{normalization}$. We have

$$\mathbf{E}(S_n|a_0 = 1) = \int \sum_{k=0}^{n-1} v_k \circ T_{1,k}\,(d\lambda - dm) = \sum_{k=0}^{n-1} \int v_k d(T_{1,k})_*(\lambda - m).$$

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Thus, $\mathbf{E}(S_n|a_0 = 1)$ is bounded if

$$\sum_{k=1}^{\infty} |(T_{1,k})_*(\lambda - m)| < \infty.$$

Further, we use the exact rate $|(T_{1,n})_*(\lambda - m)| = O(n^{-1/\gamma})$ to control $M_k - M_{k-1}$ for $k > 1$.

Quenched moment bounds

$(\Omega, \mathcal{F}, \mathbf{P}) = ((0, \alpha_*]^{\mathbb{N}}, \text{Borel}, \mathbf{P})$, $\sigma_* \mathbf{P} = \mathbf{P}$, $(\sigma, \mathbf{P})$ ergodic

Thm (Korepanov, L. '21)

For $\mathbf{P}$-a.e. $\omega \in \Omega$:

a) If $0 < \alpha_* < 1/2$ then $\|S_n\|_{2(\frac{1}{\alpha_*}-1)} \lesssim n^{\frac{1}{2}}$

b) If $\alpha_* = 1/2$ then $\|S_n\|_2 \lesssim \sqrt{n \log n}$

c) If $1/2 < \alpha_* < 1$ then $\|S_n\|_{\frac{1}{\alpha_*},\infty} \lesssim n^{\alpha_*}$

Thm (Korepanov, L. '24+)

If $\mathbf{P}(\omega_1 \leq \gamma) > 0$, then $\uparrow$ hold with γ in place of α_*.

Related: Gouëzel, Melbourne '14 and Dedecker, Merlevede '15 obtained similar bounds for stationary dynamics (iterations of a single map T_α)

Structure of proof

For any "regular" measure μ on $[0,1]$, construct measures μ_n and numbers $\kappa_n \in [0,1]$ such that:

- $\mu = \sum_{n \geq 1} \kappa_n \mu_n$
- $(T_{1,n})_* \mu_n = m$

The sequence (κ_n) along with the decay rate of $\sum_{j \geq n} \kappa_j$ as $n \to \infty$ are determined by dynamical constants and tail bounds $r, (h^k)_{k \geq 1}$:

$$\mu(R_1 \geq \ell) \leq r(\ell) \quad \text{and} \quad m(R_k \geq \ell) \leq h^k(\ell),$$

where

$$R_k(x) = \inf\{\ell \geq 1 \,:\, T_{k+\ell-1} \circ \cdots \circ T_k(x) \in Y\}.$$

Hence, for two "regular" measures μ, μ' with the same tail bounds $r, (h^k)_{k \geq 1}$:

$$|(T_{1,n})_* \mu - (T_{1,n})_* \mu'| \leq 2 \sum_{j > n} \kappa_j.$$