Heat kernel gradient bounds on the Vicsek set

Fractal Geometry and Stochastics 7, Chemnitz

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Motivation from smooth settings

Since the 1980s, gradient estimates for heat kernels and their connections with functional inequalities have been extensively studied in smooth structures satisfying Gaussian heat kernel bounds:

$$p_t(x, y) \simeq \frac{C}{\mathrm{Vol}(B(x, \sqrt{t}))} \exp\left(-c\frac{d(x, y^2)}{t}\right).$$

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- Pointwise gradient bound for the heat kernel

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- L^p $(p \ge 1)$ Poincaré inequality

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A. Grigor'yan, L. Saloff-Coste, K.-T. Sturm: Gaussian bounds are equivalent to the volume doubling property and L^2-Poincaré inequality.

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Among many applications, gradient estimates for heat kernels and Poincaré inequalities play an important role in the comparison

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On Euclidean spaces, the Riesz transform is a Calderón-Zygmund type singular integral. From the viewpoint in operator theory,

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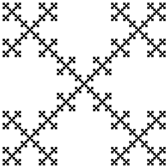
Study of Riesz transforms in smooth settings: D. Barkry 1987, Coulhon-Duong 1999, Auscher-Coulhon-Duong-Hofmann 2004, Coulhon-Jiang-Koskela-Sikora 2020, etc.

Our goal

In this talk, our main goal is to understand heat kernel "gradient" bounds, Sobolev spaces, Poincaré inequalities and the Riesz transform on the Vicsek set.



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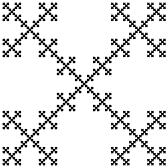


Figure 1: Vicsek set

The Vicsek set has both fractal and tree structure. Whereas neither analogue of curvature nor obvious differential structure exist.

Some relevant results

Chen-Coulhon-Feneuil-Russ 2017.

On the Vicsek manifold or graph, (R_p) holds if and only if $1 < p \leq 2$, (RR_p) holds if and only if $2 \leq p < \infty$.

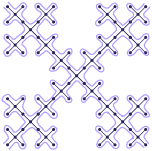
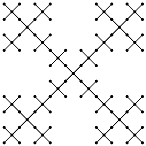


Figure 2: Vicsek manifold



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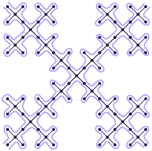

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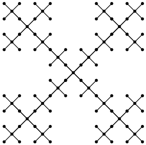

Figure 3: Vicsek cable system

Devyver-Russ-Yang 2022, Devyver-Russ 2024.

For the Vicsek cable system, the reverse quasi Riesz inequality $\|\Delta^\gamma e^{-\Delta} f\|_p \lesssim \| \, |\nabla f| \, \|_p$ holds whenever $\gamma \in [1/2, 1)$ and $p > p^*$, and is false whenever $\gamma \in (0, 1)$ and $p < p^*$.

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- From 1990s, detailed information about solutions of the heat equation on regular fractals (e.g., Sierpiński gasket/carpet) and related analysis. (M. Barlow, R. Bass, A. Grigor'yan, W. Hebisch, J. Kigami, N. Kajino, Janna Lierl, T. Kumagai, M. Murugan, L. Saloff-Coste, R. Strichartz, A. Teplyaev et al)

Vicsek set

Let $\{q_i\}_{1\leq i\leq 5}$ be the center and 4 corners of a unit square. Define

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- The compact Vicsek set can be blown up to unbounded Vicsek sets. If $\phi_1(x) = x/3$, then $X = \bigcup_{n=1}^{\infty} 3^n K$.
- The Vicsek set has both the fractal and tree structure, which makes the analysis more accessible and would open the door to study more general trees and fractals.

Vicsek metric graphs (or cable systems)

$W_0 = \{q_i\}_{1 \le i \le 5}$ and $W_{n+1} = \Psi(W_n), n \ge 1$

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Vicsek metric graphs: $\bar{V}_n = \bigcup_{m \ge 1} 3^{m-n} \bar{W}_m$, with vertices $V_n = \bigcup_{m \ge 1} 3^{m-n} W_m$.

Skeleton of X: $\mathcal{S} = \bigcup_{n \ge 0} \bar{V}_n$, dense in X.

Reference measure ν on $\mathcal{S}$: Length measure, σ-finite.

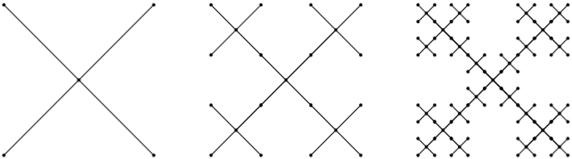

Figure 4: Parts of Vicsek metric graphs $\bar{V}_0, \bar{V}_1, \bar{V}_2$

Volume, heat kernel and Poincaré inequality on the Vicsek set

μ: Hausdorff measure; d: Euclidean distance.

Δ: Kigami Laplacian; $p_t(x, y)$: the associated heat kernel.

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- Sub-Gaussian heat kernel estimate:

$$p_t(x, y) \simeq Ct^{-\frac{d_h}{d_w}} \exp\left(-c\left(\frac{d(x, y)^{d_w}}{t}\right)^{\frac{1}{d_w - 1}}\right),$$

where $d_w = \frac{\log 15}{\log 3} = d_h + 1$ is the walk dimension.

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- L^2-Poincaré inequality $\mathrm{PI}(d_w)$:

$$\int_{B(x,r)} |u - u_{B(x,r)}|^2 \, d\mu \le Cr^{d_w} \int_{B(x,r)} d\Gamma(u,u).$$

Different perspectives towards Sobolev spaces on fractals

To understand the L^p energy measure and Sobolev spaces on fractals, one may adapt concepts of Sobolev spaces on metric measure spaces such as

- Korevaar-Schoen spaces (1993);
- Hajłasz–Sobolev spaces (1996);
- Poincaré-Sobolev spaces (Hajłasz-Koskela 2000).

See e.g. the work of J.X. Hu 2003, Pietruska-Pałuba and Stós 2013.

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In recent years, other perspectives have been taken through

- Heat kernel based Besov spaces (Alonso-Ruiz-Baudoin-C.-Rogers-Shanmugalingan-Teplyaev 2020, 2021);
- Approximation by discrete p-energies (Herman-Peirone-Strichartz 2004; after 2020: Kigami, Murugan-Shimizu, Cao-Qiu, Kajino-Shimizu).

Korevaar-Schoen-Sobolev spaces

For any $p > 1$ and $\alpha > 0$, define

$$E_{p,\alpha}(f,r) := \frac{1}{r^\alpha}\left(\int_X \int_{B(x,r)} \frac{|f(y)-f(x)|^p}{\mu(B(x,r))} d\mu(y) d\mu(x)\right)^{\frac{1}{p}}.$$

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Korevaar-Schoen-Sobolev space: let $\alpha = 1 + \frac{d_h - 1}{p}$,

$$KS^{1,p}(X) = \left\{ f \in L^p(X,\mu) : \limsup_{r\to 0^+} E_{p,\alpha}(f,r) < +\infty \right\}.$$

Korevaar-Schoen semi-norm: $\|f\|_{KS^{1,p}(X)} = \limsup_{r\to 0^+} E_{p,\alpha}(f,r)$.

Sobolev spaces via a sequence of discrete energies

Discrete energies on Vicsek graphs

$$\mathcal{E}^n_{A,p}(f) := \frac{1}{2} 3^{(p-1)n} \sum_{x,y \in V_n \cap A, x \sim y} |f(x) - f(y)|^p, \quad 1 \leq p < \infty.$$

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p-**energy via approximation**:

$$\mathcal{E}_{A,p}(f) := \lim_{m \to \infty} \mathcal{E}^m_{A,p}(f).$$

Sobolev space via *p*-**energy**:

$$\mathcal{F}_p(X) = \left\{ f \in C(X) : \sup_{n \ge 0} \mathcal{E}^n_p(f) < +\infty \right\}.$$

Seminorm: $\|f\|_{\mathcal{F}_p(X)} = \mathcal{E}_p(f)^{1/p}$.

Sobolev spaces via weak gradients

For any $n \geq 0$, let $\mathbf{e}(u,v)$ be the edge of neighboring vertices $u, v \in V_n$.

Weak gradient of $f \in C(X)$: $\partial f \in L^1_{\mathrm{loc}}(\mathcal{S}, \nu)$ such that

$$f(v) - f(u) = \int_{\mathbf{e}(u,v)} \partial f \, d\nu.$$

Sobolev spaces via weak gradient:

$$W^{1,p}(X) = \left\{ f \in C(X) : \|\partial f\|_{L^p(\mathcal{S},\nu)} < +\infty \right\}.$$

Equivalence of different Sobolev characterizations

Theorem (Baudoin-C. 2022)

Let $1 < p < +\infty$. For $f \in C(X)$ the following are equivalent:

- $f \in KS^{1,p}(X)$;
- $f \in \mathcal{F}_p(X)$;
- $f \in W^{1,p}(X)$.

Moreover, one has $\|f\|_{KS^{1,p}(X)} \simeq \|\partial f\|_{L^p(\mathcal{S},\nu)} \simeq \|f\|_{\mathcal{F}_p(X)}$.

Ideas for the proof

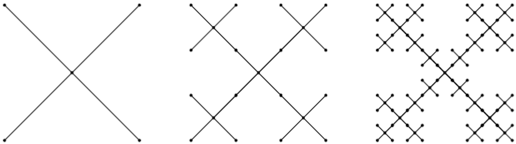

Figure 5: Parts of Vicsek graphes $\bar{V}_0$, $\bar{V}_1$, $\bar{V}_2$

n-piecewise affine functions: linear between the vertices of $\bar{V}_n$ and constant on any connected component of $\bar{V}_m \setminus \bar{V}_n$ for every $m > n$.

- Let $\Phi : X \to \mathbb{R}$ be n-piecewise affine. Then $\mathcal{E}_p^n(\Phi) = \mathcal{E}_p^m(\Phi)$, $\forall m \geq n$.

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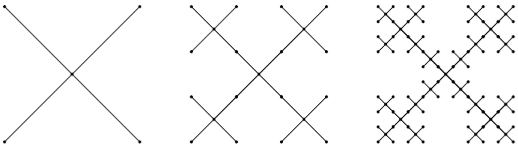

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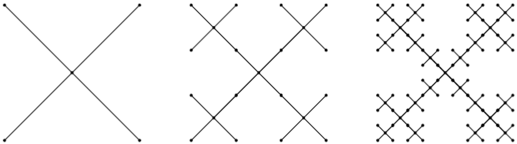

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- Any $f \in C(X)$ can be approximated by a sequence of n-piecewise affine functions $\{H_n f\}_n$ which coincide with f on V_n.
- The set of compactly supported piecewise affine functions is the core of Dirichlet form $(\mathcal{E}, W^{1,2}(X))$.

Morrey type estimates and Poincaré inequalities

Another key ingredient in the proof is the Morrey type estimate. That is, for a closed connected set A,

$$|f(x) - f(y)|^p \leq d(x, y)^{p-1} \mathcal{E}_{A,p}(f), \quad x, y \in A.$$

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As a consequence, we easily obtain the L^p-Poincaré inequality: for any $f \in \mathcal{F}_p$ there holds

$$\int_{B(x_0,r)} \left| f(x) - \fint_{B(x_0,r)} f d\mu \right|^p d\mu(x) \leq C r^{p-1+d_h} \mathcal{E}_{B(x_0,r),p}(f).$$

Gradient estimates for the heat kernel

Theorem (Baudoin-C. 2023)

- **Pointwise gradient bound**: *for every* $t > 0$, $y \in X$ *and* ν *a.e.* $x \in \mathcal{S}$

$$|\partial_x p_t(x,y)| \le \frac{C}{t^{1/d_w}} p_{ct}(x,y).$$

- $L^p(X,\mu) \to W^{1,q}(\mathcal{S},\nu)$ **continuity of** P_t: *for* $1 \le p \le q \le \infty$

$$\|\partial P_t f\|_{L^q(\mathcal{S},\nu)} \le \frac{C}{t^{(1-\frac{1}{p}-\frac{1}{q})\frac{1}{d_w}+\frac{1}{p}}} \|f\|_{L^p(X,\mu)}.$$

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In particular, $P_t : L^p(X,\mu) \to W^{1,p}(\mathcal{S},\nu)$ is bounded for $p \geq 1$ with

$$\|\partial P_t f\|_{L^p(\mathcal{S},\nu)} \leq \frac{C}{t^{\alpha_p}} \|f\|_{L^p(X,\mu)}, \qquad \text{with } \alpha_p = \left(1-\frac{2}{p}\right)\frac{1}{d_w} + \frac{1}{p}.$$

From Lipschitz continuity to gradient bound

Follow the approach of M. Barlow 1995 through probabilistic potential theory, then for the resolvent kernel $g_\lambda(x,y) = \int_0^\infty e^{-\lambda t} p_t(x,y)dt$,

$$|g_\lambda(z,x) - g_\lambda(z,y)| \leq C\lambda^{1-\frac{d_h}{d_w}} \frac{d(x,y)}{t^{1/d_w}}(g_\lambda(z,x) + g_\lambda(z,y)).$$

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J. Gao and M. Yang 2024: alternative analytical approach for the Hölder/ Lipschitz regularity of heat kernels on general metric measure spaces, among many other interesting results.

From gradient bound to $L^p(\mu) \to W^{1,q}(\nu)$ continuity of P_t

To obtain the $L^p(\mu) \to W^{1,q}(\nu)$ continuity of P_t, it suffices to prove

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The idea is to introduce a co-differential operator $\partial^* : L^2(\mathcal{S},\nu) \to L^2(X,\mu)$ as the L^2 adjoint of ∂ so

$$\operatorname{dom}\Delta = \{u \in \operatorname{dom}\mathcal{E} : \partial u \in \operatorname{dom}\partial^*\}, \quad \Delta u = \partial^*\partial u.$$

From gradient bound to $L^p(\mu) \to W^{1,q}(\nu)$ continuity of P_t

To obtain the $L^p(\mu) \to W^{1,q}(\nu)$ continuity of P_t, it suffices to prove

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Thanks to the existence of nice cutoff functions on the Vicsek set, we have

$$\int_{\mathcal{S}} |\eta| d\nu \leq \int_X d(x_0,x)|\partial^*\eta|(x) d\mu(x).$$

Applications: Sobolev and Nash inequalities

Recall that $\alpha_p = (1 - \frac{2}{p})\frac{1}{d_w} + \frac{1}{p}$.

L^p **pseudo-Poincaré inequality for** P_t: for $p \geq 1$

$$\|f - P_t f\|_{L^p(X,\mu)} \leq C t^{\alpha_p} \|\partial f\|_{L^p(\mathcal{S},\nu)}.$$

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The Sobolev norm behaves nicely under cutoffs:

- $\int_{\mathcal{S}} |\partial((f-t)_+ \wedge s)|^p d\nu \leq \int_{\mathcal{S}} |\partial f|^p d\nu$;
- $\sum_{k\in\mathbb{Z}} \int_{\mathcal{S}} |\partial((f-\rho^k)_+ \wedge \rho^k(\rho-1))|^p d\nu \leq \int_{\mathcal{S}} |\partial f|^p d\nu$.

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> **Nash inequality**: for $p > 1$ and $\theta = \frac{(p-1)d_h}{p-1+pd_h}$
>
> $$\|f\|_{L^p(X,\mu)} \leq C \|\partial f\|^{\theta}_{L^p(\mathcal{S},\nu)} \|f\|^{1-\theta}_{L^1(X,\mu)}.$$

Applications: Sobolev and Nash inequalities

Recall that $\alpha_p = (1 - \frac{2}{p})\frac{1}{d_w} + \frac{1}{p}$.

For $p \geq 1$, $0 < s < \alpha_p$

$$\|\Delta^s f\|_{L^p(X,\mu)} \leq C\|f\|_{L^p(X,\mu)}^{1-\frac{s}{\alpha_p}} \|\partial f\|_{L^p(\mathcal{S},\nu)}^{\frac{s}{\alpha_p}}.$$

For $p \geq 1$, $s > \alpha_p$

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Combining the above two inequalities, it is natural to conjecture

Conjecture (Boundedness of the Riesz transform)

$$c\|\partial f\|_{L^p(\mathcal{S},\nu)} \leq \|\Delta^{\alpha_p} f\|_{L^p(X,\mu)} \leq C\|\partial f\|_{L^p(\mathcal{S},\nu)}.$$

Thank you for your attention !