Lipschitz-Killing curvatures for different classes of fractals

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International Conference Fractal Geometry and Stochastics 7,
Chemnitz, September 23-27, 2023

1.Minkowski and S-contents, extension to curvatures

For a compact set $K \subset \mathbb{R}^d$ and $r > 0$ the r-parallel set is given by

$$K(r) := \{x \in \mathbb{R}^d : \min_{y \in K} |x - y| \leq r\},$$

and for $0 \leq D \leq d$, the D-dimensional Minkowski content of K by

$$\mathcal{M}^D(K) := \lim_{\varepsilon \to 0} \frac{\mathcal{L}^d(K(\varepsilon))}{\varepsilon^{d-D}},$$

for $0 \leq D < d$ the D-dimensional S-content (surface area based) by

$$\mathcal{S}^D(K) := \lim_{\varepsilon \to 0} \frac{\mathcal{H}^{d-1}(\partial K(\varepsilon))}{(d-D)\varepsilon^{d-1-D}},$$

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Average versions:

$$\widetilde{\mathcal{M}}^D(K) := \lim_{\delta \to 0} \frac{1}{|\ln \delta|} \int_\delta^1 \frac{\mathcal{L}^d(K(\varepsilon))}{\varepsilon^{d-D}} \frac{1}{\varepsilon} \, d\varepsilon$$

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$\mathcal{L}^d(K(\varepsilon))$ and $\mathcal{H}^{d-1}(\partial K(\varepsilon))$ are completed by the Lipschitz-Killing curvatures $C_k(K(\varepsilon))$, $k = 0, \ldots, d-2$, under some additional conditions of regularity of the boundary $\partial K(\varepsilon)$.

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For all $k = 0, \ldots, d$ and compact sets $\mathcal{K}$ such that $\mathcal{K}$ or $\overline{\mathcal{K}^c}$ has positive reach, there exist **measure variants** $C_k(\mathcal{K}, \cdot)$ with $C_k(\mathcal{K}, \mathbb{R}^d) = C_k(\mathcal{K})$.
(By a result of Fu, 1985, in space dimensions less than 4 the condition is fulfilled for $\mathcal{K} = K(\varepsilon)$ for any compact set K and Lebesgue almost all ε.)

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Essential properties for our purposes: They are

- invariant under Euclidean motions,
- scaling: $C_k(\lambda\mathcal{K}, \lambda(\cdot)) = \lambda^k C_k(\mathcal{K}, \cdot)$, $\lambda > 0$,
- locally determined: $C_k(\mathcal{K}', (\cdot) \cap G) = C_k(\mathcal{K}, (\cdot) \cap G)$, if $\mathcal{K}' \cap G = \mathcal{K} \cap G$ for an open set G.

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(For certain classes of sets the total values $C_k(\mathcal{K})$, $k = 0, 1, \ldots, d$, form a complete system of related invariants under Euclidean motions.)

Question: For which classes of fractal sets K (or domains with fractal boundaries) we can extend the notions of (average) Minkowski- and s-contents to curvatures in the following sense?

$$C_k^{frac}(K) := \lim_{\varepsilon \to 0} \frac{C_k(K(\varepsilon))}{\varepsilon^{k-D}} \, , \, or$$

$$\widetilde{C}_k^{frac}(K) := \lim_{\delta \to 0} \frac{1}{|\ln \delta|} \int_\delta^1 \frac{C_k(K(\varepsilon))}{\varepsilon^{k-D}} \frac{1}{\varepsilon} \, d\varepsilon$$

up to some normalizing constants.

Survey on some related literature

- **Relationships to spectral analysis, certain Zeta functions and fractal drums**
  For the Minkowski content see the books of Lapidus and van Frankenhuijsen 2006, Lapidus, Radunovic and Zubrinic 2017, and the references therein.
- **Self-similar sets**
  Falconer 1995 under (SSC), Gatzouras 2000 under (OSC): existence and integral representation for $\mathcal{M}^D$, resp. $\widetilde{\mathcal{M}}^D$.

  Winter 2008: C_k^{frac} for the case of polyconvex parallel sets.

  Rataj and Winter 2010: for $\mathcal{S}^D$, resp. $\widetilde{\mathcal{S}}^D$, $\mathcal{M}^D = \mathcal{S}^D$, $\widetilde{\mathcal{M}}^D = \widetilde{\mathcal{S}}^D$, if $D < d$, known before: $\mathrm{dim_H = dim_M = D}$ for such sets.

  Z. 2011: C_k^{frac} for a more general case.

  Rataj and Z. 2012: average fractal curvature densities.

  Winter and Z. 2013, Z. 2013: fractal curvature measures.

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  Winter and Z. 2013, Z. 2013: fractal curvature measures.
  Lapidus, Pearse and Winter 2011-2013, Winter 2015: relationships to fractal tilings.

- **Extensions to some classes of self-conformal sets**
  Kombrink 2011, Bohl 2012, Kesseböhmer and Kombrink 2012, Freiberg and Kombrink 2012.
- **Stochastically self-similar sets (random recursive constructions)**
  Gatzouras 2000: existence and integral representation for $\mathcal{M}^D$, resp. $\widetilde{\mathcal{M}}^D$, almost surely and in the mean sense, equality.

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- **Homogeneous random fractals**
  Hambly 1992 (special case), Troscheit 2017: $\dim_{\mathrm{H}} = \dim_{\mathrm{M}} = \mathrm{D}'$ a.s..
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  Z. 2020: existence and integral representation for $\mathbb{E}\mathcal{M}^D$, resp. $\mathbb{E}\widetilde{\mathcal{M}}^D$.
  Rataj, Winter and Z. 2023: extension to $\mathbb{E}C_k^{frac}$, resp. $\mathbb{E}\widetilde{C}_k^{frac}$.

- V-**variable random fractals**
  Z. 2023: existence and integral representation for $\mathbb{E}\mathcal{M}^D$, $\mathbb{E}\mathcal{S}^D$, resp. $\mathbb{E}\widetilde{\mathcal{M}}^D$, $\mathbb{E}\widetilde{\mathcal{S}}^D$, and
  $$\mathbb{E}\mathcal{M}^D = \mathbb{E}\mathcal{S}^D\,,\ \ \mathbb{E}\widetilde{\mathcal{M}}^D = \mathbb{E}\widetilde{\mathcal{S}}^D\,, \text{if } D < d.$$
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  in preparation (joint work with A. Rozanova-Pierrat, A. Teplyaev, and S. Winter in connection with analysis and curvatures).

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Below we will present an extension of the mean value results to a large class of self-similar random code-tree fractals with more dependencies, which contains all former random cases mentioned above (joint work with J. Rataj and S. Winter).

Random labeled code trees

$(f_1, \ldots, f_N)$ random number of random contracting similarities (RIFS) with random contraction ratios $(r_1, \ldots, r_N)$ such that

1. $0 < r_{min} \leq r_i \leq r_{max} < 1$ w.p.1, for deterministic values r_{min}, r_{max}.
2. $1 < \mathbb{E}N < \infty$ (supercritical case)
3. $\bigcup_{i=1}^{N} f_i(O) \subset O$ and $f_i(O) \cap f_j(O) = \emptyset\,,\ i \neq j$, w.p.1,

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Recursive random code tree construction:

$\Sigma_0 := \{i : 1 \le i \le N\}$

$\Sigma_n := \{\sigma = \sigma_1 \ldots \sigma_n : \sigma_1 \ldots \sigma_{n-1} \in \Sigma_{n-1}, 1 \le \sigma_n \le N_{\sigma_1 \ldots \sigma_{n-1}}\}$

(codes at levels n) for random numbers $N_{\sigma_1 \ldots \sigma_{n-1}}$, $n \ge 1$

$$\Sigma_* := \bigcup_{n=1}^{\infty} \Sigma_n \quad \text{(random code tree)}$$

RIFS's as above are chosen as labels at the nodes $\sigma \in \Sigma_*$:

$$\mathcal{F}_\sigma := (f_{\sigma 1}, \ldots, f_{\sigma N_\sigma}) \,.$$

Random labeled code tree:

$$\mathcal{T} := \{(\sigma, \mathcal{F}_\sigma) : \sigma \in \Sigma_*\} \,.$$

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$$\mathcal{T} := \{(\sigma, \mathcal{F}_\sigma) : \sigma \in \Sigma_*\} .$$

For $\sigma \in \Sigma_*$ the subtree rooted at the node σ is given by

$$\mathcal{T}_\sigma := \{(\tau, \mathcal{F}_{\sigma\tau}) : \sigma\tau \in \Sigma_*\}$$

Assumption 1 (back path condition): For each i, under the condition that $i \leq N$, the labeled subtree $\mathcal{T}_i$ is independent of the preceding mapping f_i and has the same distribution as $\mathcal{T}$.
Essential consequence: For all steps $n \in N$, under the condition that $\sigma = \sigma_1 \ldots \sigma_n \in \Sigma_n$, the random labeled subtree $\mathcal{T}_\sigma$ rooted at σ is independent of the corresponding random mappings along the path $(f_{\sigma_1}, f_{\sigma_1\sigma_2}, \ldots, f_{\sigma_1 \ldots \sigma_n})$ and has the same distribution as the primary tree $\mathcal{T}$.

New: Allows much more dependencies between the different paths than in the former models.

Iteration of the random function systems along the paths of the random tree: For $n \geq 1$ and $\sigma = \sigma_1 \ldots \sigma_n \in \Sigma_n$ denote

$$f^\sigma = f_{\sigma_1} \circ f_{\sigma_1 \sigma_2} \circ \cdots \circ f_{\sigma_1 \ldots \sigma_n}$$

Associated self-similar random code-tree fractal:

$$F := \bigcap_{n=1}^{\infty} \bigcup_{\sigma \in \Sigma_n} f^\sigma(\overline{O})$$

is a.s. determined

Assumption 2 (USOSC): Uniform Strong Open Set Condition for some bounded open set O in $\mathbb{R}^d$, i.e., UOSC as above for O and

$$\mathbb{P}(F \cap O \neq \emptyset) > 0\,.$$

Mean Minkowski and S-contents

Let $(r_1, r_2, \ldots, r_N)$ be a random vector with random N having the same distribution as the contraction ratios of the above RIFSs. Let D be determined by

$$\mathbb{E} \sum_{i=1}^{N} (r_i)^D = 1 \quad \text{(mean dimension equation)}\,.$$

Associated probability distribution of the logarithmic contraction ratios:

$$\mu := \mathbb{E}\left(\sum_{i=1}^{N} \mathbf{1}_{(\cdot)}(|\ln(r_i)|)(r_i)^D \right)$$

Corresponding mean value:

$$\eta := \mathbb{E}\left(\sum_{i=1}^{N} |\ln(r_i)|(r_i)^D \right)$$

Denote $\varphi_d(\varepsilon) := \mathbb{E}\,\mathcal{L}^d(F(\varepsilon))$. Under the above conditions we obtain the following:

Theorem 1 [(average) mean Minkowski content]

(i) If the measure μ is non-lattice, then

$$\lim_{\varepsilon\to 0} \frac{\mathbb{E}\,\mathcal{L}^d(F(\varepsilon))}{\varepsilon^{d-D}} = \frac{1}{\eta}\int_0^1 \varepsilon^{D-d} R_d(\varepsilon)\,\frac{1}{\varepsilon} d\varepsilon =: M_F^D\,,$$

where the function $R_d(\varepsilon)$ is given by

$$R_d(\varepsilon) = \mathbf{1}_{(0,1]}(\varepsilon)\varphi_d(\varepsilon) - \mathbb{E}\sum_{i=1}^N \mathbf{1}_{(0,r_i]}(\varepsilon)\, r_i^d\, \varphi_d(r_i^{-1}\varepsilon)\,.$$

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(ii) For general μ we get for the average limit

$$\lim_{\delta\to 0}\frac{1}{|\ln\delta|}\int_\delta^1 \frac{\mathbb{E}\,\mathcal{L}^d(F(\varepsilon))}{(d-D)\varepsilon^{d-D}}\,\frac{1}{\varepsilon}d\varepsilon = M_F^D\,.$$

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(iii) The limit value M_F^D does not vanish.

Theorem 2 [(average) mean S-content]

Items (i) and (ii) are analogous for the surface area based version S_F^D.
Moreover,

$$S_F^D = M_F^D \quad \text{for } D < d\,.$$

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Remark: Barnsley, Hutchinson and Stenflo 2012: determined the almost sure Hausdorff dimension $\mathrm{dim_H F}$ via flow matrices for the V-variable case, Troscheit 2017: $\mathrm{dim_H F} = \mathrm{dim_M F} = \mathrm{D_H}$, given by $\mathbb{E} \ln\big(\sum_{\sigma \in \Sigma_{neck}} (r^\sigma)^{D_H} \big) = 0$.

D_H is, in general, less than the above Minkowski dimension D in the mean sense given by $\mathbb{E} \sum_{i=1}^N (r_i)^D = 1$.

Extensions to fractal Lipschitz-Killing curvatures

Choose an arbitrary constant $R > \sqrt{2}|O|$.

Assumption 3 (regularity of F): w.p.1 for Lebesgue almost all $r < R$ the sets $\overline{F(r)^c}$ have positive reach.

(Remark: It is known that, in general for a.a. $r \geq R$, and in space dimensions $d \leq 3$ for a.a. $r > 0$ this is always fulfilled.)

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Notation (*Markov stop in the code space*):

$$\Sigma(r) := \{\sigma \in \Sigma_* : Rr^{\sigma} \leq r < Rr^{\sigma||\sigma|-1}\}, 0 < r < R,$$

where $\sigma||\sigma|-1$ means $\sigma_1 \dots \sigma_{n-1}$ if $\sigma = \sigma_1 \dots \sigma_n$.

Theorem 3 [(average) mean Lipschitz-Killing curvatures]

Let $k \in \{0, 1, \ldots, d\}$ and let F be a random self-similar code tree fractal in $\mathbb{R}^d$ as above with independence along back paths (Assumption 1) satisfying the Uniform Strong Open Set Condition with basic set $O \subset \mathbb{R}^d$ (Assumption 2). For $k \leq d-2$ we additionally suppose the following:

(i) if $d \geq 4$, then F is regular (Assumption 3),

(ii) there exists a constant $c_k > 0$ such that w.p.1,

$$C_k^{\mathrm{var}}\left(F(r), f^{\sigma}(O)(r) \cap f^{\tau}(O)(r)\right) \leq c_k r^k$$

for Lebesgue a.a. $r < R$ and all $\sigma, \tau \in \Sigma(r)$ with $\sigma \neq \tau$.

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Set for a.a. $r > 0$,

$$R_k(r) := \mathbb{E} C_k(F(r)) - \mathbb{E} \sum_{i=1}^{N} \mathbf{1}_{(0, R r_i]}(r)\, C_k\big(F_i(r)\big)\,.$$

Then we get the following:

(I) If the measure μ is non-lattice, then

$$\lim_{\varepsilon\to 0} \varepsilon^{D-k}\mathbb{E}\, C_k(F(\varepsilon)) = \frac{1}{\eta}\int_0^R r^{D-k-1}R_k(r)\, dr\,.$$

(II) If the measure μ is lattice with constant c, then for almost all $s \in [0, c)$

$$\lim_{n\to\infty} e^{(k-D)(s+nc)}\mathbb{E}\, C_k\big(F(e^{-(s+nc)})\big) = \frac{1}{\eta}\sum_{m=0}^{\infty} e^{(k-D)(s+mc)}R_k\big(e^{-(s+mc)}\big).$$

(III) In general,

$$\lim_{\delta\to 0}\frac{1}{|\ln\delta|}\int_\delta^1 \varepsilon^{D-k}\mathbb{E}\, C_k(F(\varepsilon))\ \varepsilon^{-1}d\varepsilon = \frac{1}{\eta}\int_0^R r^{D-k-1}R_k(r)\, dr.$$

Special cases and an example

Assumption 1 (back path condition) is satisfied for the random recursive case and the V-variable random fractals.
Our extension: certain dependencies between different paths are allowed.

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Assumption 2 (USOSC) is always required.

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Our extension: certain dependencies between different paths are allowed.

There is an intersection of our model with the graph directed random code tree fractals studied by S. Troscheit (2017, 2021).

Assumption 2 (USOSC) is always required.

Assumption 3 (geometric regularity condition) is always fulfilled in $\mathbb{R}^d$ with $d \leq 3$.

Boundedness condition (ii) can be checked for classical examples of random Sierpinski-type fractals (gaskets or carpets)

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THANK YOU !