Korevaar
Schoen '92 (M, d_g, v_g)

$$\mathcal{E}_p(f) = \sup_{\substack{0 \le \varphi \le 1 \\ \varphi \in C_c(M)}} \liminf_{r \to 0} \frac{1}{r^p} \int_M \int_{B(x,r)} \varphi(x) \frac{d_g(f(x), f(y))^p}{r^n} \, dv_g(y) \, dv_g(x)$$

Cheeger (X, d, m) ← volume doubling

$$\int_X |g_f|^p \, dx$$ ← weak upper grad → Bjorn-Bjorn

(p,p)-PI: $\int_{B(x,R)} |f - f_{B(x,R)}|^p \, dm \lesssim R^p \int_{B(x,\Lambda R)} |\mathrm{Lip} f|^p \, dx$

$$\int_X |\Gamma(f)|^{p/2} \, dx$$ ← Carré du Champ → Beznea??-Rock..., Kuwae

$$\mathcal{E}_p(f) = \Gamma\text{-}\lim_{n\to\infty} \boxed{\frac{1}{r_n^p} \int_X ⨍_{B(x,r_n)} |f(x)-f(y)|^p \, dm(y)\, dm(x)}$$

$$= E_{p,r_n}(f)$$

(*)

Pf steps:

separable

⊛ $\{E_{p,r_n} : L^p(X,m) \to \mathbb{R}\}_{n\geq 1} \Rightarrow \exists$ Γ-conv. $\{E_{p,r_{n_k}}\}_{k\geq 1}$ (**)

⊛ $\mathcal{E}_p(f) := \Gamma\text{-}\lim_{n\to\infty} E_{p,r_n}(f)$ $\quad \mathcal{F}_p = \{ f \in L^p(X,m) : \sup_{r>0} E_{p,r}(f) < \infty \}$

⊛ $\mathcal{E}_p(f,g) = \frac{1}{p} \lim_{t\to 0} \frac{1}{t}\left[\mathcal{E}_p(f+tg) - \mathcal{E}_p(f)\right]$ need a pf for $\exists$ of this

⊛ $\mathcal{E}_p(f,g) = 0$ if $g|_{(\operatorname{supp} f)_\varepsilon} \equiv \text{const}$.

⊛ Lemma: $\mathcal{E}_p(f,g) = \lim_{n\to\infty} \frac{1}{r_n^p} \int_X ⨍_{B(x,r_n)} |f_n(x)-f_n(y)|^{p-2} \times (f_n(x)-f_n(y))(g_n(x)-g_n(y))\, dm(y)\, dm(x)$

$\circledast$ $\quad \mathrm{Lip}_{loc}(X) \cap C_c(M) \underset{\text{dense}}{\subseteq} \mathcal{F}_p$ w.r.t. $\left(\mathcal{E}_p(\cdot) + \|\cdot\|_{L^p}^p \right)^{1/p}$

$\nearrow$ weak monotonicity: Let $r_n \downarrow 0$. $\exists\, C > 0$ s.t. $\forall f_n \xrightarrow{s} f$

$$\sup_{r>0} E_{p,r}(f) \leq C \liminf_{n\to\infty} E_{p,r_n}(f_n)$$

$$\leq C \limsup E_{p,r_n}(f_n)$$

$$\leq C\, \mathcal{E}_p(f) \qquad \square$$

<u>Thm</u>: $\exists$ p-energy measure

s.t. $\Gamma_p(f) \ll m \quad \forall f \in \mathcal{F}_p$