Hausdorff dimension of stable Lévy processes with measurable drift

Peter Kern

joint work with Leonard Pleschberger

Heinrich Heine Universität Düsseldorf

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Selfsimilar stochastic processes

Let $X = (X_t)_{t\geq 0}$ be a stochastic process in $\mathbb{R}^d$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The process X is called **selfsimilar** if for all $c > 0$

$$(X_{ct})_{t\geq 0} \stackrel{\text{fd}}{=} (c^H X_t)_{t\geq 0},$$

where $H > 0$ is called the **Hurst index**. This means

$$\mathbb{P}\big((X_{ct_1}, \ldots, X_{ct_n}) \in A\big) = \mathbb{P}\big(c^H (X_{t_1}, \ldots, X_{t_n}) \in A\big)$$

for all $n \in \mathbb{N}$, $0 \leq t_1 < \cdots < t_n$ and $A \in \mathcal{B}(\mathbb{R}^{nd})$.

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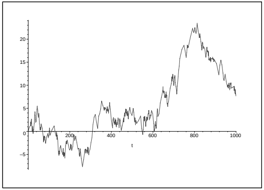

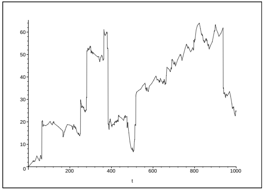

Left: Sample path of a Brownian motion with $H = 1/2$.
Right: Sample path of a stable Lévy process with $H = 1/\alpha$ for $\alpha \in (0, 2)$.

Additionally, add a measurable function $f : \mathbb{R}_+ \to \mathbb{R}^d$ to get a selfsimilar stochastic process with deterministic drift function

$$X + f = (X_t + f(t))_{t \geq 0}.$$

For a restricted time domain $T \in \mathcal{B}(\mathbb{R}_+)$ we consider the random sets

- graph: $\mathcal{G}_T(X + f) = \{(t, X_t + f(t)) : t \in T\}$
- range: $\mathcal{R}_T(X + f) = \{X_t + f(t) : t \in T\}$

Selfsimilar stochastic processes with drift

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Question: What is the interplay between X and f that determines the Hausdorff dimension of $X + f$?

Hausdorff dimension of fractional Bm with drift

Let $B^H = (B_t^H)_{t \geq 0}$ be a fractional Brownian motion with Hurst index $H \in (0,1]$, that is a centered Gaussian process with stationary increments and covariance function

$$\mathbb{E}[B_t^H \cdot B_s^H] = \tfrac{1}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big), \quad s,t \geq 0.$$

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Theorem 1 (Peres & Sousi, 2016)

Let $\varphi_{1/H} := \mathcal{P}^{1/H} - \dim \mathcal{G}_T(f)$, *then for all* $H \in (0,1]$ *we* $\mathbb{P}$*-almost surely have*

$$\dim \mathcal{G}_T(B^H + f) = \begin{cases} \varphi_{1/H} & , \varphi_{1/H} \leq d \\ H \cdot \varphi_{1/H} + (1-H)\cdot d & , \varphi_{1/H} \geq d \end{cases}$$

$$\dim \mathcal{R}_T(B^H + f) = \begin{cases} \varphi_{1/H} & , \varphi_{1/H} \leq d \\ d & , \varphi_{1/H} \geq d \end{cases}$$

Isotropic stable Lévy processes

Let $X = (X_t)_{t \geq 0}$ be an isotropic α-stable Lévy process with $\alpha \in (0,2]$. This is a selfsimilar process with Hurst index $H = 1/\alpha$, starting in $X_0 = 0$, having independent and stationary increments: For all $0 \leq t_0 < \cdots < t_n$

- $X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent
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Obstacles:

- pure jump process for $\alpha \in (0,2)$
- process with power law tails for $\alpha \in (0,2)$
- extension of Hurst parameter region for $\alpha \in (0,1)$

Parabolic Hausdorff dimension

The covering sets are **parabolic cylinders**. For $\alpha > 0$ define

$$\mathcal{P}^\alpha := \left\{ [t, t+c] \times \prod_{j=1}^{d} [x_j, x_j + c^{1/\alpha}] : t \geq 0,\ x_j \in \mathbb{R},\ c \in (0,1] \right\}$$

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and for $\beta > 0$ and $A \subseteq \mathbb{R}_+ \times \mathbb{R}^d$ the α-**parabolic** β-**Hausdorff measure** is given by

$$\mathcal{P}^\alpha\text{–}\mathcal{H}^\beta(A) := \lim_{\delta \downarrow 0} \inf \left\{ \sum_{k=1}^{\infty} |P_k|^\beta : A \subseteq \bigcup_{k=1}^{\infty} P_k,\ P_k \in \mathcal{P}^\alpha, |P_k| \leq \delta \right\}$$

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which determines the α-**parabolic Hausdorff dimension**

$$\begin{aligned} \mathcal{P}^\alpha - \dim A &:= \inf\{\beta > 0 : \mathcal{P}^\alpha - \mathcal{H}^\beta(A) = 0\} \\ &= \sup\{\beta > 0 : \mathcal{P}^\alpha - \mathcal{H}^\beta(A) = \infty\}. \end{aligned}$$

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- $\alpha = 2$: Taylor & Watson (1985)
- $\alpha = 1$: Leads to genuine Hausdorff dimension
- $\alpha \in [1, \infty)$: Peres & Sousi (2012, 2016) $c^{1/H}$ instead of c

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Abuse of notation:

- $\alpha > 2$: hyperbolic
- $\alpha = 2$: parabolic
- $\alpha \in (1, 2)$: elliptic
- $\alpha = 1$: linear
- $\alpha \in (0, 1)$: sublinear

Trivial estimates

For $A \subseteq \mathbb{R}_+ \times \mathbb{R}^d$ and $\varphi_\alpha := \mathcal{P}^\alpha - \dim A$ we have

$$\dim A \leq \begin{cases} \varphi_\alpha \ \wedge \ (\alpha \cdot \varphi_\alpha + 1 - \alpha), & \alpha \in (0,1], \\ \varphi_\alpha \ \wedge \ \left(\frac{1}{\alpha} \cdot \varphi_\alpha + \left(1 - \frac{1}{\alpha}\right) \cdot d\right), & \alpha \in [1,\infty) \end{cases}$$

and

$$\dim A \geq \begin{cases} \varphi_\alpha + \left(1 - \frac{1}{\alpha}\right) \cdot d, & \alpha \in (0,1], \\ \varphi_\alpha + 1 - \alpha, & \alpha \in [1,\infty). \end{cases}$$

Main result

Let $X = (X_t)_{t\geq 0}$ be an isotropic α-stable Lévy process with $\alpha \in (0,2]$ and $\varphi_\alpha := \mathcal{P}^\alpha - \dim \mathcal{G}_T(f)$.

Theorem 2 (K. & Pleschberger, 2024)

We $\mathbb{P}$-almost surely have

$$\dim \mathcal{G}_T(X+f) = \begin{cases} \varphi_1 = \dim \mathcal{G}_T(f) & , \alpha \in (0,1] \\ \varphi_\alpha \wedge \left(\frac{1}{\alpha} \cdot \varphi_\alpha + (1 - \frac{1}{\alpha}) \cdot d\right) & , \alpha \in [1,2) \end{cases}$$

$$\dim \mathcal{R}_T(X+f) = \begin{cases} \alpha \cdot \varphi_\alpha \wedge d & , \alpha \in (0,1] \\ \varphi_\alpha \wedge d & , \alpha \in [1,2) \end{cases}$$

Case of constant drift

For $T \in \mathcal{B}(\mathbb{R}_+)$ and the constant function $f : T \to \{0\}$ we have

$$\varphi_\alpha = \mathcal{P}^\alpha - \dim \mathcal{G}_T(f) = (\alpha \vee 1) \cdot \dim T.$$

In this case we recover the classical results:

Theorem 3 (Blumenthal & Getoor, 1960, 1962)

We $\mathbb{P}$-almost surely have

$$\dim \mathcal{G}_T(X) = \begin{cases} \dim T + 1 - \frac{1}{\alpha} & , \alpha \cdot \dim T \geq 1 \\ (\alpha \vee 1) \cdot \dim T & , \textit{else} \end{cases}$$

$$\dim \mathcal{R}_T(X) = (\alpha \cdot \dim T) \wedge d.$$

Proof, upper bound for the graph

- For $\beta > \varphi_\alpha = \mathcal{P}^\alpha - \dim \mathcal{G}_T(f)$ take a cover of $\mathcal{G}_T(f)$ such that $\mathcal{P}^\alpha - \mathcal{H}^\beta(\mathcal{G}_T(f))$ is (arbitrary) small.
- Combine it with a random cover of $\mathcal{G}_T(X)$ with cubes of comparable size.
- Use the covering lemma of Pruitt & Taylor (1969) to estimate the expected number of covering cubes for $\mathcal{G}_T(X+f)$.
- Show that $\mathbb{E}[\mathcal{H}^\beta(\mathcal{G}_T(X+f))] < \infty$.

Proof, lower bound for the graph

- Use a parabolic version of Frostman's lemma:
  For $\beta < \varphi_\alpha = \mathcal{P}^\alpha - \dim \mathcal{G}_T(f)$ there exists a probability measure μ supported on $\mathcal{G}_T(f)$ such that

$$\mu\left([t,t+c]\times\prod_{j=1}^{d}[x_j,x_j+c^{1/\alpha}]\right)\lesssim\begin{cases}c^{\beta} & ,\ \alpha\in(0,1]\\ c^{\beta/\alpha} & ,\ \alpha\in[1,\infty)\end{cases}$$

- Use this measure to show that expected energy integrals

$$\int_{\mathcal{G}_T(f)}\int_{\mathcal{G}_T(f)}\mathbb{E}\left[\|(t-s,X_{|t-s|}+x-y)\|^{-\beta}\right]\,d\mu(t,x)\,d\mu(s,y)$$

  are finite.

- For this we need sharp upper bounds of the kernel function

$$K^{\beta}(\tau,\delta)=\mathbb{E}\left[\|(\tau,X_{|\tau|}+\delta)\|^{-\beta}\right]$$

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