Self-similar measures

Péter P. Varjú

University of Cambridge

23. September 2024.

Self-similar sets and measures

Iterated function system (IFS)

A finite collection of contractive similarities on $\mathbf{R}$:

$$\Phi = \{\varphi_1, \ldots, \varphi_m\},$$

$$\varphi_j(x) = \lambda_j x + t_j.$$

The IFS is homogeneous if $\lambda_1 = \ldots = \lambda_m$.

Self-similar set

Given an IFS Φ, there is a unique compact $K \subset \mathbf{R}$ such that

$$K = \varphi_1(K) \cup \cdots \cup \varphi_m(K).$$

This is also called the attractor.

$$\Phi = \{\varphi_1, \ldots, \varphi_m\}, \quad \varphi_j(x) = \lambda_j x + t_j$$

Self-similar measure

Given an IFS Φ and a probability vector $p_1, \ldots, p_m$, there is a unique probability measure μ on $\mathbf{R}$ such that

$$\mu = p_1\varphi_1(\mu) + \ldots + p_m\varphi_m(\mu),$$

where $\varphi_j(\mu)$ denotes the push-forward.

Where Φ is homogeneous, μ is an infinite convolution. Indeed, let $X_1, X_2, \ldots$ be a sequence of independent random variables, each taking the value t_j with probability p_j for all j.
Then μ is the law of

$$\sum_{n=0}^{\infty} X_n \lambda^n,$$

where $\lambda = \lambda_1 = \ldots = \lambda_m$.

Questions of Interest

- What are the dimensions of self-similar sets and measures?
- Which self-similar measures are a.c. with respect to the Lebesgue measure?
- Fourier decay.
- Diophantine approximation.
- ...

Dimension

Exact overlaps

The IFS Φ has exact overlaps, if there are $l \in \mathbf{Z}_{\geq 0}$ and indices

$$(i_1, \ldots, i_l) \neq (j_1, \ldots, j_l) \in \{1, \ldots, m\}^l$$

such that

$$\varphi_{i_1} \circ \ldots \circ \varphi_{i_l} = \varphi_{j_1} \circ \ldots \circ \varphi_{j_l},$$

that is, if the semi-group generated by Φ is not free.

Conjecture

If Φ has no exact overlaps, then the dimension of the self-similar measure is

$$\dim \mu = \min \left(1, \frac{p_1 \log p_1^{-1} + \ldots + p_m \log p_m^{-1}}{p_1 \log \lambda_1^{-1} + \ldots + p_m \log \lambda_m^{-1}} \right).$$

Conjecture

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Remarks

- There is also a conjecture for sets, and it follows from this.
- The inequality $\leq$ is known and easy.
- If the IFS has good separation properties (e.g. open set condition), this is a classical result. (Moran 1946)
- In the presence of exact overlaps, the numerator should be replaced by the random walk entropy rate, defined later.

Theorem (Hochman '14)

The conjecture holds if the IFS satisfies the exponential separation condition. In particular, it holds for most IFS in a very strong sense. It also holds if all the λ_j and t_j parameters are algebraic numbers.

Now the challenge is to prove the conjecture for all IFS in families. So far, this has been achieved in the following cases.

- $\{\lambda x, \lambda x+1, \lambda x+t\}$, λ is algebraic. (Hochman '14 +Shmerkin-Solomyak)
- All λ_j are algebraic, t_j are arbitrary. (Rapaport '22)
- $\{\lambda x, \lambda x+1\}$ (V '19)
- Homogeneous IFS, all t_j rational. (Rapaport, V '24)
- Homogeneous IFS, all t_j algebraic. (Feng, Feng '24+)
- $\{\lambda x, \lambda x+1, \lambda x+t\}$, $\lambda \in (2^{-2/3}, 1)$, uniform weights.(Rapaport,V'24)
- $\{\lambda x, \lambda x+1, \lambda x+t, \lambda x+1+t\}$, $\lambda \in (2^{-3/2}, 1)$ uniform weights. (Streck '24+)

Entropy rates (Garisa entropy)

Let $\Phi = \{\varphi_1, \ldots, \varphi_m\}$ be an IFS.

Let $\xi_0, \xi_1, \ldots$ be a sequence of i.i.d. random elements of Φ distributed according to the probability weights $p_1, \ldots, p_m$.

Then the entropy rate is

$$h(\Phi) = \lim_{n\to\infty} \frac{1}{n} H(\xi_0 \circ \xi_1 \circ \cdots \circ \xi_{n-1}),$$

where H stands for Shannon entropy.

In the homogeneous case

$$h(\Phi) = \lim_{n\to\infty} \frac{1}{n} H\Big(\sum_{j=0}^{n-1} X_j \lambda^j\Big),$$

where $X_1, X_2, \ldots$ are i.i.d. in $\{t_1, \ldots, t_m\}$.

$$\Phi(\lambda) = \{\lambda x, \lambda x + 1\} \qquad \text{(V '19)}$$
$$h(\lambda) = h(\Phi(\lambda))$$

From now on: weights are always uniform.

Theorem (Breuillard, V '20)

For all $\varepsilon > 0$, there is C such that

$$h(\lambda) \leq \log(2) - \varepsilon \quad \Rightarrow \quad M(\lambda) < C.$$

Mahler measure

Let λ be an algebraic number with minimal polynomial

$$a_d(x - \lambda_1) \cdots (x - \lambda_d).$$

Then

$$M(\lambda) = |a_d| \prod_{j=1}^{d} \max(1, |\lambda_d|).$$

Mahler measure

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$$M(\lambda) = |a_d| \prod_{j=1}^{d} \max(1, |\lambda_d|).$$

If λ is transcendental, we define $M(\lambda) = \infty$.
If $\lambda = p/q \in \mathbf{Q}$, $M(\lambda) = \max(|p|, |q|)$.

small $M(\lambda)$	$\longleftrightarrow$	large $M(\lambda)$
small $h(\lambda)$	$\longleftrightarrow$	large $h(\lambda)$
good separation	$\longleftrightarrow$	bad separation

$$\Phi(\lambda) = \{\lambda x, \lambda x + 1\} \qquad \text{(V '19)}$$
$$h(\lambda) = h(\Phi(\lambda))$$
$$\Phi(\lambda, t) = \{\lambda x, \lambda x + 1, \lambda x + t\} \qquad \text{(Rapaport,V'24)}$$
$$h(\lambda, t) = h(\Phi(\lambda, t))$$

Theorem (Breuillard, V '20)

For all $\varepsilon > 0$, there is C such that

$$h(\lambda) \leq \log(2) - \varepsilon \quad \Rightarrow \quad M(\lambda) < C.$$

Problem

$h(\lambda, t)$ may be small even if λ is transcendental.
Example: If $t = 1$, then

$$h(\lambda, t) \leq \frac{1}{3}\log 3 + \frac{2}{3}\log(3/2) = \log 3 - \frac{2}{3}\log 2.$$

Entropy rate for sets of parameters

Fix some $A \subset (0,1) \times \mathbf{R}$.
Now consider the IFS $\Psi(A) = \{\lambda x, \lambda x + 1, \lambda x + t\}$ as maps

$$A \times \mathbf{R} \to \mathbf{R}.$$

For $\varphi_1, \varphi_2 : A \times \mathbf{R} \to \mathbf{R}$, consider the composition operation

$$(\varphi_1 \circ \varphi_2)(\lambda, t, x) = \varphi_1(\lambda, t, \varphi_2(\lambda, t, (x))) : A \times \mathbf{R} \to \mathbf{R}.$$

Let $\xi_0, \xi_1, \ldots$ be an i.i.d sequence of random elements of $\Psi(A)$, and define

$$h(A) = h(\Psi(A)) = \lim \frac{1}{n} H(\xi_0 \circ \cdots \circ \xi_{n-1}).$$

Observation

If $A_1 \subset A_2$, then $h(A_1) \leq h(A_2)$.
In particular, $h(\lambda, t) \leq h(A)$ for $(\lambda, t) \in A$.

Question (Rapaport, V '24)

Is there a C for all $\varepsilon > 0$ such that the following holds?
Let λ, t be such that

$$h(\lambda, t) < \log 3 - \varepsilon$$

and for all curves γ passing through (λ, t), we have

$$h(\gamma) > \log 3 - C^{-1}.$$

Then $M(\lambda) < C$.

Theorem (Rapaport, V '24)

If YES, then the conjecture about dimension holds for the IFS

$$\{\lambda x, \lambda x + 1, \lambda x + t\}.$$

Theorem (Rapaport, V '24)

For all $\varepsilon > 0$, there is C such that

$$h(\lambda, t) < \frac{2}{3} \log 2 - \varepsilon \quad \Rightarrow \quad M(\lambda) \leq C.$$

Theorem (Rapaport, V '24)

For all $\varepsilon > 0$, there is C such that

$$h(\lambda, t) < \frac{2}{3}\log 2 - \varepsilon \quad \Rightarrow \quad M(\lambda) \leq C.$$

Observations

$$\frac{2}{3}\log 2 < \log 3.$$

Even

$$\frac{2}{3}\log 2 < \log 3 - \frac{2}{3}\log 2.$$

In fact, there are no curves γ with

$$h(\gamma) \leq \frac{2}{3}\log 2.$$

The number $(2/3)\log 2$ can probably be improved both in the above observation and in the theorem.

Change the setting to

$$\Phi = \{\lambda x, \lambda x + 1, \lambda x + t, \lambda x + 1 + t\}.$$

Question (Rapaport, V '24)

Is there a C for all $\varepsilon > 0$ such that the following holds?
Let λ, t be such that

$$h(\lambda, t) < \log 4 - \varepsilon$$

and for all curves γ passing through (λ, t), we have

$$h(\gamma) > \log 4 - C^{-1}.$$

Then $M(\lambda) < C$.

Theorem (Streck '24+)

For all $\varepsilon > 0$, there is C such that

$$h(\lambda, t) < \frac{3}{4} \log 4 - \varepsilon \quad \Rightarrow \quad M(\lambda) \leq C \text{ or } t = 0.$$

Theorem (Streck '24+)

For all $\varepsilon > 0$, there is C such that

$$h(\lambda, t) < \frac{3}{4}\log 4 - \varepsilon \quad \Rightarrow \quad M(\lambda) \le C \text{ or } t = 0.$$

Remarks

- Implies the conjecture about dimension if $\lambda > 4^{-3/4}$.
- With $\log 2$ in place of $(3/4)\log 4$, the theorem follows from Breuillard-V.
- $h(\{t = 0\}) = \log 2$.
- $h(\{t = 1\}) = (3/4)\log 4$.

Absolute continuity – results for typical parameters

Theorem (Solomyak 1995) – Transversality

The self-similar measure for

$$\{\lambda x, \lambda x + 1\}$$

is a.c. for almost all $\lambda \in (1/2, 1)$.

Theorem (Shmerkin '14)

The set of exceptional $\lambda \in (1/2, 1)$ is 0 dimensional.

- The method combines Hochman's result about dimension 1 with power Fourier decay (Erdős, Kahanne).
- It was extended to non-homogeneous IFS's and to dimension 2 (Saglietti, Shmerkin, Solomyak '18), (Solomyak, Śpiewak '23+).

Absolute continuity – explicit examples

Theorem (Garsia 1962)

Let $\Phi = \{\lambda x, \lambda x + 1\}$ with uniform weights.
If $M(\lambda) = 2$ the self-similar measure is a.c.

Theorem (Dai, Feng, Wang '07)

Let $\Phi = \{\lambda x, \lambda x + 1, \ldots, \lambda x + M - 1\}$ with uniform weights.
If $M(\lambda) = M$ and all Galois conjugates are < 1 in modulus, the self-similar measure is a.c.

Theorem (Streck '23+)

Let Φ be homogeneous with common contraction λ and integer translations.
If the weights are such that $h(\Phi) = M(\lambda)$, and λ has no conjugates on the unit circle, then the self-similar measure is a.c.

Theorems (V '19), (Kittle '24)

Let $\Phi = \{\lambda x, \lambda x + 1\}$ with uniform weights.
Suppose $h(\lambda) = \log 2$, that is, no exact overlaps.
Then there is an explicit monotone increasing function F on $(1/2, 1)$ such that the self-similar measure is a.c. provided $M(\lambda) < F(\lambda)$.

Remarks

- The result in (V '19) is applicable only when λ is near 1, and $\lim_{\lambda \to 1} F(\lambda) = \infty$. It gives examples like $\lambda = 1 - 10^{-50}$.
- In (Kittle '24), $F(\lambda) > 2$ for all $\lambda \in (1/2, 1)$. It gives examples like $\lambda = 0.78207\ldots$
- The condition $h(\lambda) = \log 2$, and uniform weights are not needed.
- The method is based on repeated applications of entropy increase under convolution and quantitatively efficient estimates.
- It was extended to Furstenberg measures (Kittle '23+).

Thank you!