Multivariate multifractal analysis of Lévy functions

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Motivation

- The purpose of multifractal analysis is to study everywhere irregular functions whose pointwise regularity exponent varies from one point to another.
- Multivariate multifractal analysis is a simultaneous multifractal analysis of several pointwise regularity exponents derived from one or several functions.
- Lévy Functions were introduced by Paul Lévy as a deterministic toy example of Lévy processes, which provide simple function examples with a dense set of discontinuities.

Motivation

- Besides considering new questions in diophantine approximation, the motivation of this work is to understand on some toy examples how the multivariate multifractal formalism is affected by translations.

  (One example is brain data (EEG and MEG) , which are recorded at different locations of the brain, so that the events captured reflect what occured at other parts of the brain with different unknown time-lags depending of the location of the event.)

Lévy function

The 1-periodic 'saw-tooth' function is defined by

$$\{x\} = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2}, & \text{if } x \notin \mathbb{Z}, \\ 0, & \text{if } x \in \mathbb{Z}, \end{cases}$$

where $\lfloor x \rfloor$ denotes the integer part of the real number x.

Let $b \geqslant 2$ be an integer. The Lévy function L_α^b, which depends on a parameter $\alpha > 0$, is defined by

$$L_\alpha^b(x) = \sum_{j=1}^{\infty} \frac{\{b^j x\}}{b^{\alpha j}}, \quad \forall x \in \mathbb{R}.$$

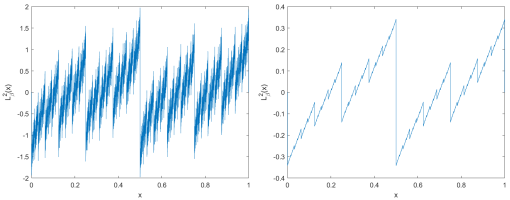

Figure: The graphs of Lévy functions with different parameter α. For the left figure, the parameter $\alpha = 0.3$ and $b = 2$; for the right figure, the parameter $\alpha = 1.3$ and $b = 2$.

I. Multifractal spectrum

- Lévy functions are bounded so that the Hölder exponent is relevant in order to analyze their regularity.

Definition (Hölder exponent)

Let $f : \mathbb{R} \to \mathbb{R}$ be a locally bounded function, α be a nonnegative real number, and $x_0 \in \mathbb{R}$. Then f belongs to $C^\alpha(x_0)$ if there exist $C > 0$, $r > 0$ and a polynomial P satisfying $\deg(P) < \alpha$ such that

$$\forall x \in B(x_0, r), \ |f(x) - P(x - x_0)| \leqslant C|x - x_0|^\alpha,$$

where $B(x_0, r)$ is a ball centered at x_0 with radius r.
The Hölder exponent of f at x_0 is defined as

$$h_f(x_0) = \sup\{\alpha \geqslant 0 : \ f \in \ C^\alpha(x_0)\}.$$

- Denote by $E_f(H)$ the level set of the points having the same pointwise regularity exponent H. That is
$$E_f(H) = \{x \in \mathbb{R} : h_f(x) = H\}.$$
- The univariate multifractal spectrum is
$$\mathcal{D}_f : H \longmapsto \dim_{\mathrm{H}} (E_f(H)),$$
where $\dim_{\mathrm{H}}$ stands for the Hausdorff dimension. By convention, $\dim_{\mathrm{H}}(\emptyset) = -\infty$.

Multivariate multifractal spectrum

- Suppose that we are dealing with two functions f_1 and f_2 defined on $\mathbb{R}$ and each associated with a pointwise regularity exponent $h_{f_1}(x)$ and $h_{f_2}(x)$ respectively. Given $H = (H_1, H_2) \in \mathbb{R}^2$, we are interested in the level sets

$$E_{f_1,f_2}(H) = \left\{x \in \mathbb{R} : h_{f_1}(x) = H_1, h_{f_2}(x) = H_2\right\}.$$

- The multivariate multifractal spectrum is the function

$$\mathcal{D}_{f_1,f_2} : H \longmapsto \dim_{\mathrm{H}} \left(E_{f_1,f_2}(H)\right).$$

The support of the multivariate multifractal spectrum consists of the vectors H, such that $E_{f_1,f_2}(H) \neq \emptyset$. By convention, $\dim_{\mathrm{H}}(\emptyset) = -\infty$.

II. Validity of the multifractal formalism

Definition (Oscillation)

Let $j \in \mathbb{Z}$ and $k \in \mathbb{Z}$. Denote by $\lambda (= \lambda(j,k))$ the dyadic interval $\left[\frac{k}{2^j}, \frac{k+1}{2^j}\right)$ and 3λ the interval of the same center and three times wider. We denote by Λ_j the collection of dyadic intervals of width 2^{-j} included in $[0,1]$.
We consider the multiresolution quantities based on the local oscillations.
At first, let

$$\Delta_f^1(x,h) = f(x+h) - f(x),$$

and for $n \geqslant 2$,

$$\Delta_f^n(x,h) = \Delta_f^{n-1}(x+h,h) - \Delta_f^{n-1}(x,h).$$

Then the n-th order oscillation is

$$d_\lambda = \sup_{[x,\ x+nh] \in 3\lambda} |\Delta_f^n(x,h)|.$$

Remark: n is fixed, and in the following, the results do not depend on n.

- The univariate multifractal structure function associated with the oscillation d_λ is defined by

$$\forall r \in \mathbb{R},\ \ S_f(r,j) = 2^{-j} \sum_{\lambda \in \Lambda_j} (d_\lambda)^r.$$

- The corresponding scaling function is

$$\eta(r) = \liminf_{j \to +\infty} \frac{\log\left(S_f(r,j)\right)}{\log\left(2^{-j}\right)}.$$

- The univariate multifractal Legendre spectrum is obtained through a Legendre transform

$$\forall H \in \mathbb{R},\ \ \mathcal{L}_f(H) = \inf_{r \in \mathbb{R}} \left(1 - \eta(r) + Hr\right).$$

- The property of the spectrum is interesting that it bounds the multifractal spectrum,

$$\mathcal{D}_f(H) \leqslant \mathcal{L}_f(H),\ \ \forall H \in \mathbb{R}.$$

Remark: Since Lévy functions have no global uniform regularity, wavelet methods cannot be employed (the upper bound result in the statement of the multifractal formalism holds no more).

Multivariate multifractal Legendre spectrum

- We consider two functions f_1 and f_2, which have oscillations d_λ^1 and d_λ^2 respectively. The multivariate multifractal structure function is

$$\forall r = (r_1, r_2) \in \mathbb{R}^2,\ \ S(r,j) = 2^{-j} \sum_{\lambda \in \Lambda_j} \left(d_\lambda^1\right)^{r_1} \left(d_\lambda^2\right)^{r_2}.$$

- Further, the scaling function is defined as

$$\zeta(r) = \liminf_{j \to +\infty} \frac{\log(S(r,j))}{\log(2^{-j})}.$$

- The multivariate multifractal Legendre spectrum is

$$\forall H = (H_1, H_2) \in \mathbb{R}^2,\ \ \mathcal{L}_{f_1,f_2}(H) = \inf_{r \in \mathbb{R}^2} (1 - \zeta(r) + H \cdot r),$$

  where $H \cdot r$ denotes the usual scalar product in $\mathbb{R}^2$.

Remark: A big difference between the univariate and bivariate multifractal formalisms is that the upper bound for the spectrum no longer holds.

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Univariate multifractal spectrum of Lévy function

Definition

Let $\alpha \geqslant 1$. A real number x is α-approximable by b-adic rationals if there exist infinitely many n such that

$$\|b^n x\| \leqslant \frac{1}{b^{n(\alpha-1)}},$$

where $\|\cdot\|$ denotes the distance to the nearest integer. For $x \in \mathbb{R}$, let

$$\Delta^b(x) = \sup\left\{\alpha \geqslant 1 : \|b^n x\| \leqslant \frac{1}{b^{n(\alpha-1)}} \text{ for infinitely many } n\right\}.$$

- The Hölder exponent of the Lévy function can be expressed in terms of the b-adic approximation properties.

Proposition (Jaffard,1997)

The Hölder exponent of L_α^b at x is given by

$$h_{L_\alpha^b}(x) = \frac{\alpha}{\Delta^b(x)}.$$

- The level set of Lévy function L_α^b can be written as

$$E_{L_\alpha^b}(H) = \left\{ x \in \mathbb{R} : \Delta^b(x) = \frac{\alpha}{H} \right\}, \ \forall H \in [0, \alpha].$$

Proposition (Jaffard, 1997)

The univariate multifractal spectrum $\mathcal{D}_{L_\alpha^b} : H \longmapsto \dim_{\mathrm{H}} \left(E_{L_\alpha^b}(H) \right)$ *of the Lévy function* L_α^b *is*

$$\mathcal{D}_{L_\alpha^b}(H) = \begin{cases} \frac{H}{\alpha}, & \text{if } H \in [0, \alpha], \\ -\infty, & \text{else.} \end{cases}$$

Multivariate multifractal spectrum of Lévy functions

For Lévy functions $L^b_{\alpha_1}$, and $L^{b,y}_{\alpha_2}$, the level sets can be written as

$$E_{L^b_{\alpha_1},L^{b,y}_{\alpha_2}}(H_1,H_2)=\left\{x\in\mathbb{R}:\Delta^b(x)=\frac{H_1}{\alpha_1},\Delta^b(x-y)=\frac{H_2}{\alpha_2}\right\},\ (H_1,H_2>0),$$

Let

$$K_{\alpha_1,\alpha_2}(y)=\left[0,\frac{\alpha_1}{\Delta^b(y)}\right]\times\left[0,\frac{\alpha_2}{\Delta^b(y)}\right]\cup\left\{(H_1,H_2):\ H_1\in[0,\alpha_1]\text{ and }H_2=\frac{\alpha_2}{\alpha_1}H_1.\right\},$$

and $A_\eta=\{y:\Delta^b(y)=\eta\},\ (\eta\geqslant 1)$.

Theorem

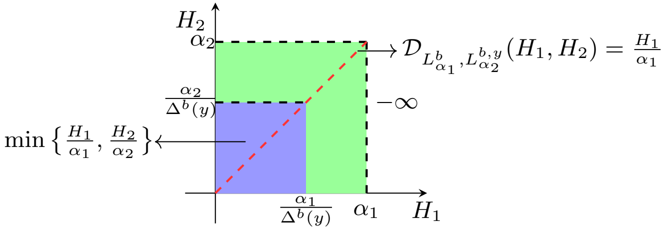
The bivariate multifractal spectrum of Lévy function $L^b_{\alpha_1}$ and the translated function $L^{b,y}_{\alpha_2}$, satisfies

$$\mathcal{D}_{L^b_{\alpha_1},L^{b,y}_{\alpha_2}}(H_1,H_2)=\begin{cases}\min\left\{\frac{H_1}{\alpha_1},\frac{H_2}{\alpha_2}\right\}, & \text{if } (H_1,H_2)\in K_{\alpha_1,\alpha_2}(y),\\ -\infty & \text{if } (H_1,H_2)\notin[0,\alpha_1]\times[0,\alpha_2].\end{cases}$$

For any $y \in \mathbb{R}$, we have the following result of the spectrum in the region $[0,\alpha_1] \times [0,\alpha_2] \setminus K_{\alpha_1,\alpha_2}(y)$.

Theorem

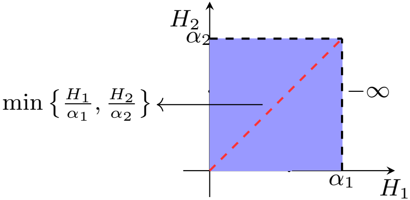
For any $(H_1, H_2) \in [0,\alpha_1] \times [0,\alpha_2] \Big\backslash K_{\alpha_1,\alpha_2}(y)$, *we have either*
$\mathcal{D}_{L^b_{\alpha_1}, L^{b,y}_{\alpha_2}}(H_1,H_2) = \min\left\{\frac{H_1}{\alpha_1}, \frac{H_2}{\alpha_2}\right\}$, *or* $\mathcal{D}_{L^b_{\alpha_1}, L^{b,y}_{\alpha_2}}(H_1,H_2) = -\infty$.



- *Remark:* For almost every y, we deduce the whole spectrum

$$\mathcal{D}_{L^b_{\alpha_1}, L^{b,y}_{\alpha_2}}(H_1, H_2) = \begin{cases} \min\left\{\frac{H_1}{\alpha_1}, \frac{H_2}{\alpha_2}\right\}, & \text{if } (H_1, H_2) \in [0, \alpha_1] \times [0, \alpha_2], \\ -\infty & \text{else.} \end{cases}$$

This is due to $\mathcal{L}eb(A_1) = \infty$.



- *The spectrum in the green part depends on the explicit expression of the b-ary expansion of y. Even if the approximable exponents are the same, the spectra might differ.*

Recall: Every real number x in $[0, 1)$ admits a b-ary expansion,

$$x = \sum_{i=1}^{\infty} \frac{\varepsilon_i}{b^i}, \quad \forall \varepsilon_i \in \{0, 1, \cdots, b-1\}.$$

Example

Let $\nu_k = 0^{n_k(\eta-1)}1$, where $\{n_k\}$ is an increasing sequence of integers satisfying

$$n_{k+1} = n_k\eta + 1, \ \forall k \geqslant 1.$$

If the b-ary expansion of y is $\nu_1\nu_2\nu_3\cdots$, that is, the b-ary expansion of y contains the following blocks

$$\cdots \underbrace{00\cdots 000}_{n_k(\eta-1)}1\underbrace{000\cdots 0000}_{n_{k+1}(\eta-1)}1\underbrace{0000\cdots 00000}_{n_{k+2}(\eta-1)}1\underbrace{00000\cdots 000000}_{n_{k+3}(\eta-1)}1\cdots$$

in this case, we have,

$$\mathcal{D}_{L^b_{\alpha_1}, L^{b,y}_{\alpha_2}}(H_1, H_2) = \begin{cases} \frac{H_1}{\alpha_1}, & \text{for} \quad \left(H_1, \frac{\alpha_2}{\alpha_1}H_1\right) \text{ with } H_1 \in [0, \alpha_1], \\ \min\left\{\frac{H_1}{\alpha_1}, \frac{H_2}{\alpha_2}\right\}, & \text{if } (H_1, H_2) \in \left[0, \frac{\alpha_1}{\eta}\right] \times \left[0, \frac{\alpha_2}{\eta}\right], \\ & \text{or } H_2 \in \left[\frac{\alpha_2 H_1}{\alpha_1 \eta}, \frac{\alpha_2 \eta H_1}{\alpha_1}\right] \text{ with } H_1 \in [0, \alpha_1], \\ -\infty & \text{else.} \end{cases}$$

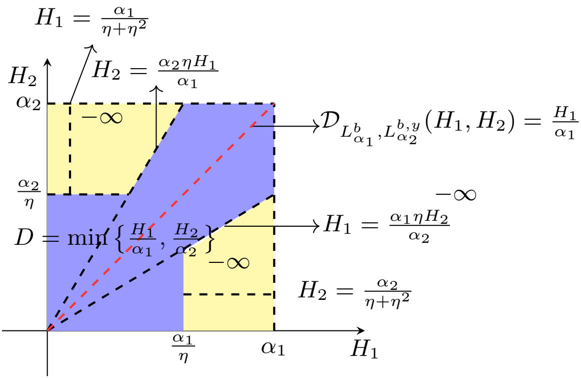

$H_1 = \frac{\alpha_1}{\eta+\eta^2}$
$H_2 = \frac{\alpha_2 \eta H_1}{\alpha_1}$
H_2
α_2
$-\infty$
$\mathcal{D}_{L^b_{\alpha_1}, L^{b,y}_{\alpha_2}}(H_1, H_2) = \frac{H_1}{\alpha_1}$
$\frac{\alpha_2}{\eta}$
$-\infty$
$H_1 = \frac{\alpha_1 \eta H_2}{\alpha_2}$
$D = \min\{\frac{H_1}{\alpha_1}, \frac{H_2}{\alpha_2}\}$
$-\infty$
$H_2 = \frac{\alpha_2}{\eta+\eta^2}$
$\frac{\alpha_1}{\eta}$
α_1
H_1

Figure:

Example

Let $\nu_k = 0^{n_k(\eta-1)}1(01)^{(n_{k+1}-n_k\eta-1)/2}$, where $\{n_k\}$ is an increasing sequence of integers satisfying

$$\lim_{n\to+\infty} \frac{n_{k-1}}{n_k} = 0.$$

If the b-ary expansion of y is $\nu_1\nu_2\nu_3\cdots$, that is, the b-ary expansion of y contains the following blocks

$$\cdots\underbrace{00\cdots0}_{n_k(\eta-1)}1\underbrace{0101010101010\cdots0101}_{n_{k+1}-n_k\eta-1}\underbrace{000\cdots00}_{n_{k+1}(\eta-1)}1\underbrace{0101010101010\cdots01010101}_{n_{k+2}-n_{k+1}\eta-1}\cdots$$

in this case, the complete spectrum is given by

$$\mathcal{D}_{L^b_{\alpha_1},L^{b,y}_{\alpha_2}}(H_1,H_2) = \begin{cases} \min\left\{\frac{H_1}{\alpha_1},\frac{H_2}{\alpha_2}\right\}, & \text{if } (H_1,H_2)\in[0,\alpha_1]\times[0,\alpha_2], \\ -\infty & \text{else.} \end{cases}$$

Proposition

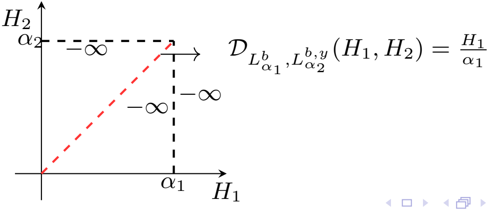
For any $y \in \mathbb{Q}$ the following dichotomy holds:

- *if y is a b-adic rational number, then the bivariate spectrum takes values on the diagonal defined by $\frac{H_1}{\alpha_1} = \frac{H_2}{\alpha_2}$, and*

$$\mathcal{D}_{L^b_{\alpha_1}, L^{b,y}_{\alpha_2}}(H_1, H_2) = \begin{cases} \frac{H_1}{\alpha_1}, & \text{if } H_2 = \frac{\alpha_2}{\alpha_1} H_1, \text{ and } H_1 \in [0, \alpha_1], \\ -\infty & \text{else.} \end{cases}$$

- *If y is a non-b-adic rational number, then the bivariate spectrum is*

$$\mathcal{D}_{L^b_{\alpha_1}, L^{b,y}_{\alpha_2}}(H_1, H_2) = \begin{cases} \min\left\{\frac{H_1}{\alpha_1}, \frac{H_2}{\alpha_2}\right\}, & \text{if } (H_1, H_2) \in [0, \alpha_1] \times [0, \alpha_2], \\ -\infty & \text{else.} \end{cases}$$



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Univariate multifractal Legendre spectrum of Lévy function

Theorem

The univariate multifractal Legendre spectrum $\mathcal{L}_{L^b_\alpha}$ *is given by*

$$\mathcal{L}_{L^b_\alpha}(H) = \begin{cases} \frac{H}{\alpha}, & \text{if } H \in [0, \alpha], \\ -\infty, & \text{else.} \end{cases}$$

- Note that $\mathcal{L}_{L^b_\alpha}(H) = \mathcal{D}_{L^b_\alpha}(H), \ \forall H \in [0, \alpha]$. The univariate multifractal formalism of Lévy function is verified.

Multivariate multifractal Legendre spectrum of Lévy functions

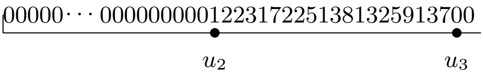
Denote $\mathcal{A} = \{0, 1, \cdots, b-1\}$. Let $u_1 = 1$. Suppose that $\varepsilon_1 = 0$ in the b-ary expansion of y and for all $k \geqslant 2$, u_{k-1} has been defined.

$$u_{2k} = \inf \{i : \ i \geqslant u_{2k-1}, \text{ and } \varepsilon_i \in \mathcal{A}\backslash\{0\}\},$$
$$u_{2k+1} = \inf \{i : \ i \geqslant u_{2k}, \text{ and } \varepsilon_i = 0\}.$$

For all $k \geqslant 1$, let

$$w_k = u_{k+1} - u_k.$$

Clearly, $\{w_k\}$ is a sequence of positive integers.



Theorem

If y is a b-adic rational number, then the bivariate multifractal Legendre spectrum of $L^b_{\alpha_1}$ *and* $L^{b,y}_{\alpha_2}$ *based on the oscillations is*

$$\mathcal{L}_{L^b_{\alpha_1},L^{b,y}_{\alpha_2}}(H_1,H_2)=\begin{cases}\frac{H_1}{\alpha_1}=\frac{H_2}{\alpha_2}, & \text{for} \quad \left(H_1,\frac{\alpha_2}{\alpha_1}H_1\right) \quad \text{with} \quad H_1\in[0,\alpha_1],\\ -\infty, & \text{else.}\end{cases}$$

If y *is not a b-adic rational number, then the following dichotomy holds:*

- *If* $\{w_k\}$ *is a bounded infinite sequence, then the bivariate multifractal Legendre spectrum based on the oscillations is*

$$\mathcal{L}_{L^b_{\alpha_1},L^{b,y}_{\alpha_2}}(H_1,H_2)=\begin{cases}\frac{H_1}{\alpha_1}+\frac{H_2}{\alpha_2}-1, & \text{if } (H_1,H_2)\in[0,\alpha_1]\times[0,\alpha_2],\\ & \text{and } \frac{H_1}{\alpha_1}+\frac{H_2}{\alpha_2}-1\geqslant 0,\\ -\infty, & \text{else.}\end{cases}$$

- *If* $\{w_k\}$ *is an unbounded infinite sequence, then the bivariate multifractal Legendre spectrum based on the oscillations is*

$$\mathcal{L}_{L^b_{\alpha_1},L^{b,y}_{\alpha_2}}(H_1,H_2)=\begin{cases}\min\left\{\frac{H_1}{\alpha_1},\frac{H_2}{\alpha_2}\right\}, & \text{if } (H_1,H_2)\in[0,\alpha_1]\times[0,\alpha_2],\\ -\infty & \text{else.}\end{cases}$$

- If y is a b-adic rational number, then the bivariate multifractal spectrum coincides with the bivariate multifractal Legendre spectrum based on the oscillations.
- If $\{w_k\}$ is an unbounded sequence, then

$$\mathcal{D}_{L^b_{\alpha_1}, L^{b,y}_{\alpha_2}}(H_1, H_2) \leqslant \mathcal{L}_{L^b_{\alpha_1}, L^{b,y}_{\alpha_2}}(H_1, H_2).$$

- But if $\{w_k\}$ is a bounded sequence, then we derive

$$\mathcal{L}_{L^b_{\alpha_1}, L^{b,y}_{\alpha_2}}(H_1, H_2) \leqslant \mathcal{D}_{L^b_{\alpha_1}, L^{b,y}_{\alpha_2}}(H_1, H_2). \tag{1}$$

Remark: We deduce that (1) holds for almost every y. This yields that the bivariate multifractal Legendre spectra fail to establish an upper bound on the bivariate multifractal spectra, even in the very special setting of a function and its translates. This phenomenon contrasts sharply with the univariate setting.

Thank you!