Multifractal properties of traces of functions in a prevalent set of inhomogeneous Besov spaces

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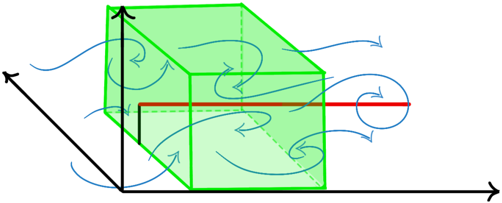
Introduction

Back to the 80s, physicists were measuring the **velocity of a turbulent fluid** and aimed at quantifying the local variation of the velocity in such environment.
The **interest in the trace** comes from the ability to **only measure one dimensional trace** of the fluid velocity across time.



Question: Can we link roughness of the 3-dimensional velocity and its 1-dimensional measurement?

Notations

For $j \in \mathbb{N}$, Λ_j^d is the **collection of closed dyadic cubes of generation j**, *i.e.* the cubes $\lambda_{j,k}^d = 2^{-j}k + 2^{-j}[0,1)^d$ where $k \in \{0, \dots, 2^j - 1\}$.

For $x \in [0,1]^d$, $\lambda_j^d(x)$ is the dyadic cubes of generation j which contains x.

For $N \in \mathbb{N}^*$ and $B = B(x,r)$, we denote $NB = B(x, Nr)$ and similarly for dyadic cubes λ gives $N\lambda$.

Definition 1

The set of ***Hölder capacities*** *is*

$$\mathcal{C}([0,1]^d) = \{\nu : \mathcal{B}([0,1]^d) \to \mathbb{R}_+ \cup \{+\infty\} \ : \ \exists C, s > 0, \ \forall E, F \in \mathcal{B}([0,1]^d), \nu(E) \leq C|E|^s \text{ and } E \subset F \Rightarrow \nu(E) \leq \nu(F)\}.$$

Example: A Borel measure is a capacity.

For a capacity μ, $s > 0$, and $E \subset [0,1]$, define

$$\mu^{(+s)} = \mu(E)|E|^s.$$

The set function $\mu^{(+s)}$ is still a capacity.

Sobolev and Besov Spaces

Let $1 \leq p, q \leq +\infty$.

- For $m \in \mathbb{N}$, **Sobolev spaces** are defined by
  $$W^{m,p}([0,1]^d) = \{f \in {}^p([0,1]^d) \ : \ \forall \alpha \in \mathbb{N}^d, |\alpha| \leq m, \ D^\alpha f \in L^p([0,1]^d)\}.$$
- For $s > 0$ with $s \notin \mathbb{N}$, $f \in W^{s,p}([0,1]^d)$ and for all $\alpha \in \mathbb{N}^d$ such that $|\alpha| = \lfloor s \rfloor$, $D^\alpha f \in C^{s - \lfloor s \rfloor}([0,1]^d)$.

For $h \in \mathbb{R}^d$ and $f : \mathbb{R}^d \to \mathbb{R}$, consider the finite difference operator $\Delta_h f : x \in \mathbb{R}^d \mapsto f(x+h) - f(x)$. Then, for $n \geq 2$, set $\Delta_h^n f = \Delta_h(\Delta_h^{n-1} f)$.

Besov spaces are generalised versions of Sobolev spaces with control on the oscillation of the function : for

$$\omega_n(f,t)_p = \sup_{0 \leq h \leq t} \|\Delta_h^n f\|_{L^p},$$

$$B^s_{p,q}([0,1]^d) = \{f \in L^p([0,1]^d) \ : \ \left\| (2^{jd/p} \omega_n(f, 2^{-j})_p)_{j \in \mathbb{N}} \right\|_{l^q(\mathbb{N})} < +\infty\}.$$

There are sharp embeddings between Sobolev and Besov spaces.

Inhomogeneous Besov Spaces

From J.Barral and S.Seuret [2], the inhomogeneous Besov spaces generalise the previously introduced Besov spaces by controlling the oscillation with a capacity ξ called a ξ-environment.

Definition 2

Let $\xi \in \mathcal{C}([0,1]^d)$ such that, for $0 < s_1 < s_2$, $|E|^{s_1} \leq \xi(E) \leq |E|^{s_2}$ and consider an integer $n \geq \left\lfloor s_2 + \frac{d}{p} \right\rfloor + 1$.

For $1 \leq p, q \leq +\infty$, the Besov space in ξ-environment $B^{\xi}_{p,q}([0,1]^d)$ is the set of the functions $f : [0,1]^d \to \mathbb{R}$ such that $\|f\|_{L^p} < +\infty$ and for

$$\omega_n^{\xi}(f,t)_p = \sup_{t/2 \leq h \leq t} \left\| \frac{\Delta_h^n f(x)}{\xi(B[x, x+nh])} \right\|_{L^p},$$

$$|f|_{B^{\xi}_{p,q}([0,1]^d)} := \left\| (2^{jd/p} \omega_n^{\xi}(f, 2^{-j})_p)_{j \in \mathbb{N}} \right\|_{l^q(\mathbb{N})} < +\infty.$$

Pointwise exponent and Spectrum

Definition 3 (pointwise Hölder exponent)

Let $f \in L^\infty_{loc}(\mathbb{R})$. Let $\alpha \in \mathbb{R}\backslash\mathbb{N}$ and $x_0 \in \mathbb{R}$. The function f belongs to $C^\alpha(x_0)$ *if there exist a polynomial P_{x_0} , $K_\alpha, r_\alpha > 0$*

$$\forall x \in [x_0 - r_\alpha, x_0 + r_\alpha], \ |f(x) - P_{x_0}(x - x_0)| \leq K_\alpha |x - x_0|^\alpha.$$

The pointwise Hölder exponent *is defined by*

$$h_f(x_0) = \sup\{\alpha \in \mathbb{R}_+ : f \in C^\alpha(x_0)\}.$$

Definition 4 (Multifractal spectrum)

*Let $f \in L^\infty_{loc}(\mathbb{R}^d)$. The **multifractal spectrum** of f is*

$$\sigma_f : \alpha \mapsto \dim_H\left(\{x \in \mathbb{R} \ : \ h_f(x) = \alpha\}\right),$$

with $\dim_H$ is the Hausdorff dimension (with the convention $\dim_H(\emptyset) = -\infty$).

Pointwise exponent and Spectrum

Definition 5 (Multifractal spectrum)

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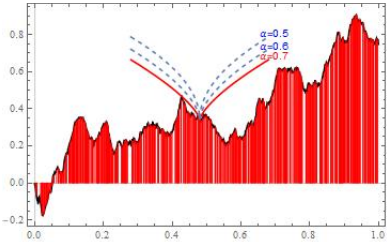

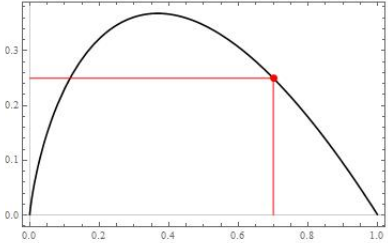

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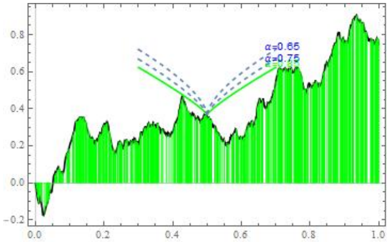
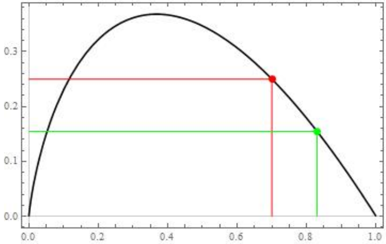




Pointwise exponent and Spectrum

Similarly for a capacity ν, one has

Definition 6

For $x \in \mathrm{supp}(\nu)$, the lower local dimension of ν at x is

$$\underline{h}_\nu(x) = \liminf_{j\to+\infty} \frac{\log_2 \nu(\lambda_j(x))}{-j}.$$

*The **multifractal spectrum** of ν is the Hausdorff dimension of the level sets*

$$\sigma_\nu \ : \ \alpha \in \mathbb{R} \mapsto \dim_H\left(\{x \in \mathbb{R} \ : \ \underline{h}_\nu(x) = \alpha\right).$$

Definition 7

The scaling function of $\nu \in \mathcal{C}([0,1]^d)$ is defined by

$$\tau_\nu(q) = \liminf_{j\to+\infty} \frac{1}{-j} \log_2\left(\sum_{\substack{I\in\Lambda_j, I\subset[0,1]^d \\ \nu(I)>0}} \nu(I)^q \right).$$

Then one always has $\sigma_\nu(\alpha) \leq \tau_\nu^*(\alpha) := \inf_{q\in\mathbb{R}}(\alpha q - \tau_\nu(q))$.

Gibbs measures

Definition 8 (Gibbs measure)

Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a $\mathbb{Z}^d$-invariant real valued Hölder continuous function. ***Gibbs measures*** *associated with the Hölder continuous potential φ are measures satisfying the following properties :*

- *Gibbs measures are doubling measure :* $\exists C_\nu > 1,\ \nu(2B) \leq C_\nu\ \nu(B)$,
- *For a Gibbs measure ν, the multifractal spectrum σ_ν is completely known,*
- *They satisfy the multifractal formalism :* $\forall \alpha \in \mathbb{R},\ \sigma_\nu(\alpha) = \tau_\nu^*(\alpha)$,
- *For a Gibbs measure ν, the function $q \mapsto \tau_\nu(q)$ is analytic.*

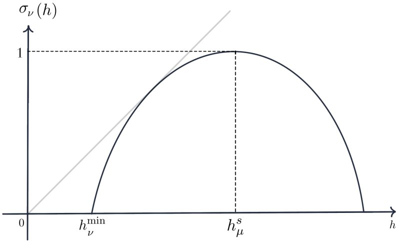
The dimension of ν defined by

$$\dim(\nu) = \inf\{\dim_H(E) : \nu(E) = 1\}$$

The minimal local dimension of ν is defined by

$$h_\nu^{\min} = \min\{\underline{h}_\nu(x) \mid x \in \operatorname{supp}(\nu)\}.$$

We denote the abscissa of the maximum of the spectrum h_ν^s.



Auxiliary measure

Definition 9 (Auxiliary measure)

Let ν be a Gibbs measure on $\mathbb{R}^d$ and $r \in \mathbb{R}$. An auxiliary measure *is a probability measure ν_r with the following properties : set $h_r = \tau_\nu'(r)$. Then :*

- *ν_r is also a Gibbs measure and satisfies the multifractal formalism,*
- $\nu_r \left(E_{\nu_r}(\sigma_\nu(h_r))\right) = 1$,
- $\dim(\nu_r) = \sigma_\nu(h_r)$.

Remark 1

for $r \in \mathbb{R}$, $h_r = \tau_\nu'(r)$ is in the support of σ_ν.

So studying $\tau_\nu'(r)$ for all $r \in \mathbb{R}$ means that one **studies all regularity values in the support of σ_ν**.

Traces

For $a \in \mathbb{R}$, my goal is to look at

$$f_a := f|_{\mathcal{H}_a} : \begin{cases} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ t & \longmapsto & f(t,a). \end{cases}$$

with $\mathcal{H}_a := \{(t,a) \mid t \in \mathbb{R}\}$ the 1-dimensional affine subspace of $\mathbb{R}^2$ passing by $(0,a)$. Consider $\xi = \mu \otimes \nu$ (μ and ν will be supposed equal here).

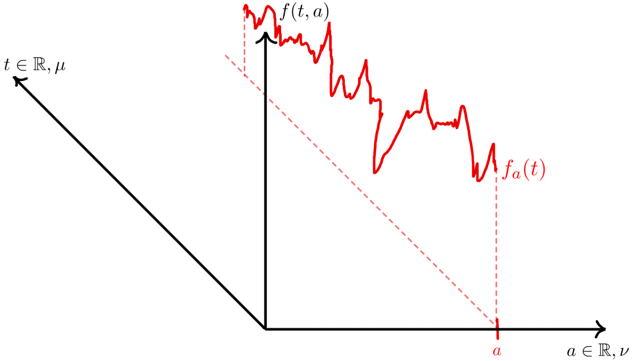


Figure: Representation of f_a for $a \in \mathbb{R}$

Trace belonging

Standard trace theorems inevitably involve a loss of regularity. For example, the trace operator $f \mapsto f_a$ maps $B^s_{p,q}([0,1]^2)$ to $B^{s-1/p}_{p,q}([0,1])$.

For standard Besov spaces, Aubry, Maman, Seuret [1] showed the following results.

Theorem 1

Let $0 < p, s < \infty$, with $s - 2/p > 0$, and $0 < q \leq \infty$.
If $f \in B^s_{p,q}([0,1]^2)$ and $q < p$ (resp. $q = p$), then for Lebesgue-almost all $a \in [0,1]$, $f_a \in B^s_{p,qp/(p-q)}([0,1])$ (resp. $B^s_{p,\infty}([0,1])$).

Theorem 2

Let $0 < p < \infty$, $0 < q \leq \infty$ and $0 < s - 1/p < +\infty$. For a prevalent set of $f \in B^s_{p,q}([0,1]^2)$, for Lebesgue-almost all $a \in [0,1]$,

$$\sigma_{f_a}(h) = \begin{cases} 1 + (h-s)p & \text{if } h \in [s-1/p, s], \\ -\infty & \text{else.} \end{cases}$$

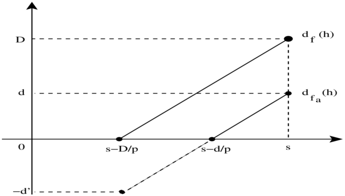


Figure: σ_f for almost all $f \in B^s_{p,q}([0,1]^2)$ and σ_{f_a} for Lebesgue-almost every $a \in [0,1]$ (Source: Aubry, Maman and Seuret).

Trace belonging

For a capacity μ, $s > 0$, and $E \subset \mathbb{R}$, define

$$\mu^{(+s)} = \mu(E)|E|^s.$$

The set function $\mu^{(+s)}$ is still a capacity.

We prove that functions in $B^{\xi}_{p,q}(\mathbb{R}^2)$ show a smaller loss of regularity than expected.

Proposition 1

Let $p, q \in [1, +\infty]$. Let μ and ν be Gibbs measures on $[0, 1]$.

Let ξ be the product capacity on $[0, 1]^2$, $\xi(\lambda^{(1)} \times \lambda^{(2)}) := \mu(\lambda^{(1)}) \cdot \nu(\lambda^{(2)})$.

Let $r \in \mathbb{R}$ and ν_r be the auxiliary measure of ν.

Then for every $f \in B^{\xi}_{p,q}([0, 1]^2)$, for ν_r-almost all $a \in [0, 1]$,

$$f_a \in \widetilde{B}^{\mu^{(+\Gamma_{\nu,r})}}_{p,q}([0, 1]) \quad \text{with } \Gamma_{\nu,r} = \tau'_\nu(r) + \frac{\dim(\nu_r)}{p}.$$

Remark 2

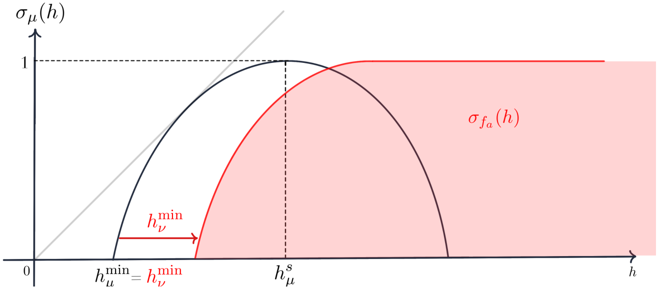
The regularity depends on the regularity of ν in a.

Upper Bounds

Proposition 2

For all $f \in B^{\xi}_{\infty,\infty}$ *and* $a \in [0,1]$, *one has*

$$\text{for } h \in \mathbb{R},\ \sigma_{f_a}(h) \leq \begin{cases} \sigma_\mu(h - h_\nu^{\min}) & \text{if } h \leq h_\nu^s + h_\nu^{\min}, \\ 1 & \text{if } h > h_\nu^s + h_\nu^{\min}. \end{cases}$$



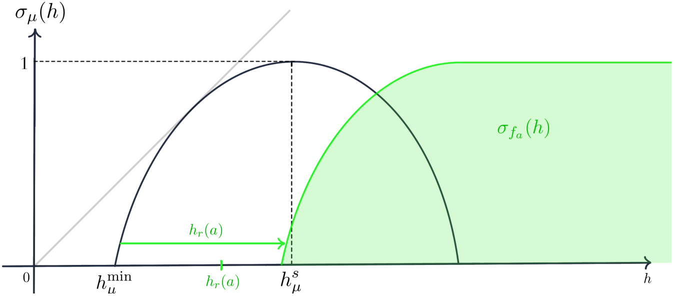
Upper Bounds

Strengthening to ν_r-almost every point a of $[0,1]$ gives the following results.

Proposition 3

Let $r \in \mathbb{R}$ and ν_r be an auxiliary measure of ν and write $h_r = \tau_\nu'(r)$. For all $f \in B^{\xi}_{\infty,\infty}$ and ν_r-almost all $a \in [0,1]$, one has

$$\text{for } h \in \mathbb{R},\ \ \sigma_{f_a}(h) \leq \begin{cases} \sigma_\mu(h - h_r) & \text{if } h \leq h_\nu^s + h_r, \\ 1 & \text{for } h > h_\nu^s + h_r. \end{cases}$$



Prevalence

Prevalence theory is proposed independently by Christensen [3] and Hunt [4].

A set is said to be **universally measurable** if it is measurable for any (completed) Borel measure.

A universally measurable set $A \subset E$ is called **shy** if there exists a Borel measure μ that is positive on some compact subset K of E and such that

$$\text{for every } x \in E, \quad \mu(A + x) = 0.$$

A set that is included in a shy universally measurable set is also shy.

The **complement of a shy subset** is called **prevalent**.

Lower Bound

Theorem 3

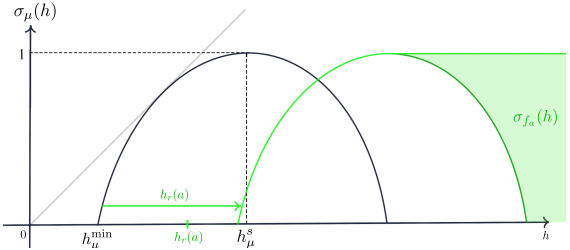
Let μ and ν be Gibbs measures on $[0,1]$.

Let ξ be the product capacity on $[0,1]^2$, $\xi(\lambda^{(1)} \times \lambda^{(2)}) := \mu(\lambda^{(1)}) \cdot \nu(\lambda^{(2)})$.

Let $r \in \mathbb{R}$ and ν_r be the auxiliary measure of ν and write $h_r = \tau_\nu'(r)$.

One has for any dense sequence $(h_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^+$, for a prevalent set of functions $f \in B^{\xi}_{\infty,\infty}([0,1]^2)$, for ν_r-almost all $a \in [0,1]$,

$$\forall n \in \mathbb{N},\ \sigma_{f_a}(h_n) \geq \begin{cases} \sigma_\mu(h_n - h_r) & \text{if } h_n \in \operatorname{supp}(\sigma_\mu) + h_r \\ -\infty & \text{else.} \end{cases}$$



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Thank you for you attention !