# On constructions of fractal spaces using replacement and the combinatorial Loewner property

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Joint work with Sylvester Eriksson-Bique

Fractal Geometry and Stochastics 7

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# Structure of the talk

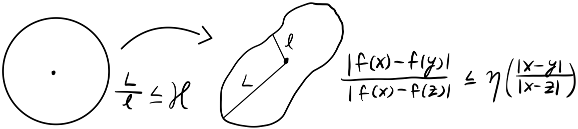
1. What is Kleiner's conjecture?
2. Background related to Kleiner's conjecture
3. Present our main result:

**A. and Eriksson-Bique '24**

Counstruction of the first counterexample to Kleiner's conjecture.

# Quasisymmetric uniformization

Quasisymmetry is a homeomorphism with controlled distortion:
Preserves **Shapes** but not necesarily **Sizes**.



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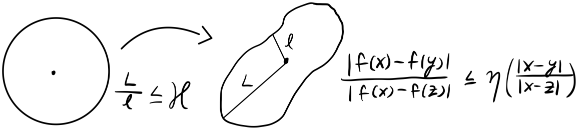


Quasisymmetric uniformization:
Find a quasisymmetrically equivalent space with more desirable properties.

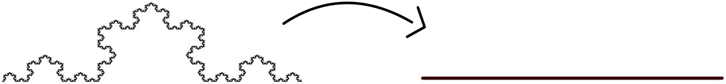

Figure: Smoothing of a snowflake is a quasisymmetric uniformization.

# What is Kleiner's conjecture?

**Kleiner's conjecture '06**

If $X$ is an approximately self-similar metric space satisfying the combinatorial Loewner property then there is a quasisymmetric uniformization of $X$ to a Loewner space.

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## Motivation

1. Loewner spaces are "optimal".
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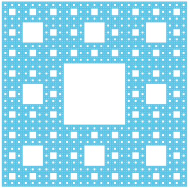
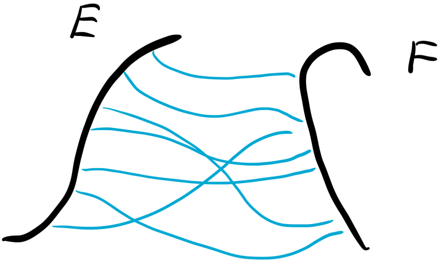


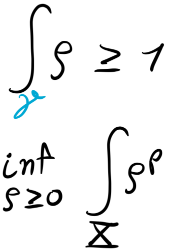
Figure: Sierpński carpet is combinatorially Loewner, but we do not the outcome of Kleiner's conjecture.

# Fundamental tool: Modulus

Let $X$ be a metric measure space and $E, F \subseteq X$ be disjoint compact sets. The *p*-modulus between $E$ and $F$ is

$$Mod_p(E,F) := \inf_{\rho \geq 0} \left\{ \int_X \rho^p : \int_\gamma \rho \geq 1 \text{ for all } \gamma \in \Gamma(E,F) \right\}.$$



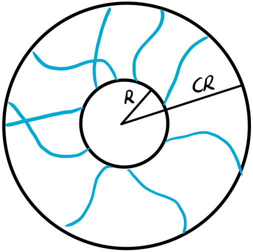



# Conformal Modulus

## Conformal modulus

*n*-modulus in $\mathbb{R}^n$ is scale invariant:

$$Mod_p(\partial B(x,R), \partial B(x,CR)) \asymp \begin{cases} \log(1/C)^{1-n} & p = n \\ \left|(CR)^{\frac{p-n}{p-1}} - R^{\frac{p-n}{p-1}}\right|^{p-1} & p \neq n. \end{cases}$$




$$\int_\gamma \rho \geq 1$$

$$\inf_{\rho \geq 0} \int_{\mathbb{R}^n} \rho^n$$

**Heinonen-Koskela** 98': Abstracted conformal modulus and introduced $Q$-Loewner spaces for $Q \in (1, \infty)$:

1. $Q$-Loewner estimates:

$$\phi(\Delta(E,F)^{-1}) \leq Mod_Q(E,F) \leq \psi(\Delta(E,F)^{-1})$$

$$\Delta(E,F) := \frac{dist(E,F)}{diam(E) \wedge diam(F)}$$

2. $Q$-Ahlfors regularity:

$$\mu(B(x,R)) \asymp R^Q.$$

# Loewner spaces and Quasiconformal geometry

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**Loewner spaces are really nice for analysis**

1. Similar theory of QC mappings as in $\mathbb{R}^n$
2. Nice Sobolev spaces
3. Rademacher-type theorem

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**Fundamental problem**: Modulus does not work for most fractals.
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## Loewner theory for fractals?

1. **Approach 1**: Uniformization? **Really hard problem**!
2. **Approach 2**: Replace modulus with something that works!

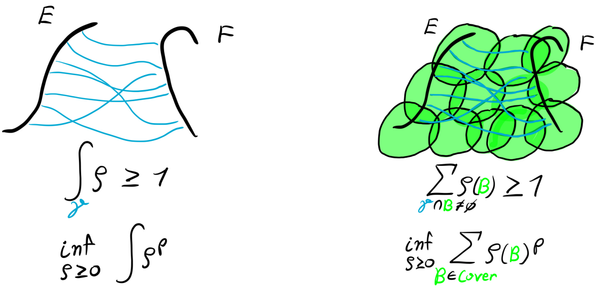


Figure: Standard modulus (Left) and Discrete modulus (Right).

# Combinatorial Loewner property

## Combinatorial Loewner property (CLP)

A space $X$ satisfies the combinatorial $Q$-Loewner property for $Q \in (1, \infty)$ if

$$\phi(\Delta(E,F)^{-1}) \leq Mod_Q^D(E,F;\mathcal{U}) \leq \psi(\Delta(E,F)^{-1})$$

where $\mathcal{U}$ is a "good" covering.

## Bourdon-Kleiner '13 (A lot of examples)

CLP is a very generic property among self-similar fractals!

# Loewner VS CLP

**Recall**: $Q$-Loewner spaces are $Q$-Ahlfors regular by definition.

## CLP and Loewner are closely related

1. If $X$ is $Q$-Ahlfors regular, then

$$Mod_Q^D \asymp Mod_Q \text{ and } Q\text{-Loewner} \iff Q\text{-CLP}.$$

2. $Q$-CLP is a quasisymmetric invariant.

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**Simplified version of Kleiner's conjecture**

**Uniformization**: $Q$-CLP to $Q$-Ahlfors regular

**Loewner estimates come for free!**

# Attainment problem

**Recall**: $\dim_H(X) = Q$ if $X$ is $Q$-Ahlfors regular.

**Carrasco-Piaggio, Keith and Laakso, Kigami,...**

If $X$ is $Q$-CLP then $Q$ is necessarily the Ahlfors regular conformal dimension of $X$:

$$d_{ARC}(X) := \inf\{Q : Y \text{ is } Q\text{-Ahlfors regular and } X \sim_{QS} Y\}$$

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**Kleiner's conjecture $\iff$ Attainment problem**

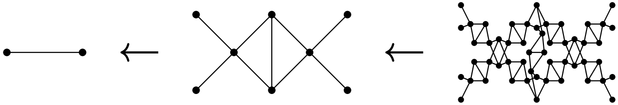
Every approximately self-similar metric space $X$ satisfying the combinatorial Loewner property attains its Ahlfors regular conformal dimension.

**CLP space is a Loewner space with non-optimal geometry!**

# Main result

**A. and Eriksson-Bique '24**

Counstruction of the first counterexample to Kleiner's conjecture.



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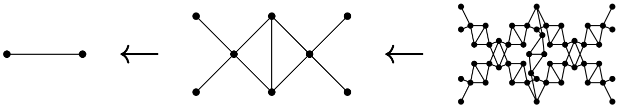

Claim: No optimal geometry!

**Punchline**

Suppose there is a subset $X \subseteq Y$ satisfying

1. $d_{ARC}(X) = d_{ARC}(Y)$
2. $X \subseteq Y$ is a porous subset

Then $Y$ does not attain its Ahlfors regular conformal dimension.

# Linear replacement rule

Proof involves two self-similar fractals which arise as limit spaces of linear replacement rules. Under very mild conditions, the limit space is combinatorially Loewner.

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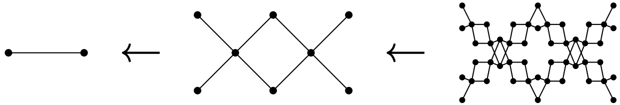

2. The limit space is the Gromov-Hausdorff limit

$$X := \lim_{n\to\infty}(G_n, 4^{-n} \cdot d_{G_n}).$$

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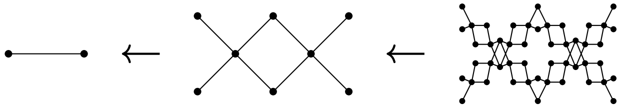


2. The limit space is the Gromov-Hausdorff limit

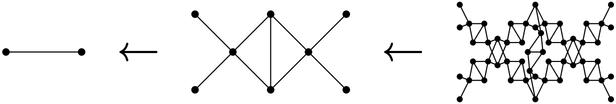
$$X := \lim_{n\to\infty}(G_n, 4^{-n}\cdot d_{G_n}).$$

**Proposition**

$X$ is $\frac{3}{2}$-Loewner and $d_{ARC}(X) = \frac{3}{2}$.

# Main example

1. Next consider the infinite sequence $\{\widetilde{G}_m\}_{m \in \mathbb{N}}$



and the limit space

$$Y := \lim_{n \to \infty} (\widetilde{G}_n, 4^{-n} \cdot d_{\widetilde{G}_n}).$$

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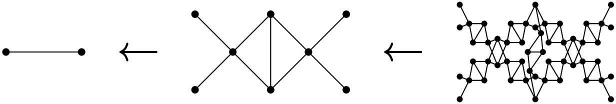
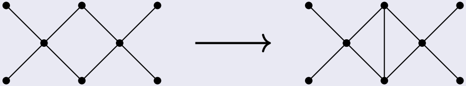


and the limit space

$$Y := \lim_{n\to\infty} (\widetilde{G}_n, 4^{-n} \cdot d_{\widetilde{G}_n}).$$

**Proposition**

- $Y$ is combinatorially Loewner.
- The natural mapping $X \to Y$ is a biLipschitz embedding onto a porous subset.

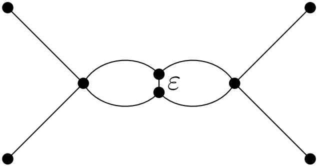


2. Only left to check that $d_{ARC}(X) = d_{ARC}(Y)$!

# $d_{ARC}(X) = d_{ARC}(Y)$

**Proposition**

$$d_{ARC}(Y) = \frac{3}{2} = d_{ARC}(X).$$

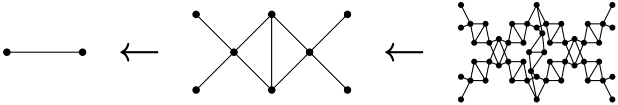



1. $\dim_H(Y)$ can be decreased arbitily close to $\frac{3}{2}$.
2. $\dim_H(Y)$ cannot be decreased below $\frac{3}{2} = d_{ARC}(X)$.
3. **Tehcnical detail**: Check that the deformation is a quasisymmetry.

# Counterexample to Kleiner's conjecture

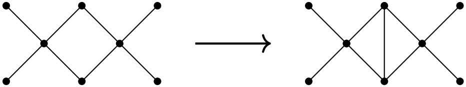
**Main result**

$Y$ is approximately self-similar and satisfies the combinatorial $\frac{3}{2}$-Loewner property, but it cannot be quasisymmetrically uniformized to a $\frac{3}{2}$-Loewner space.



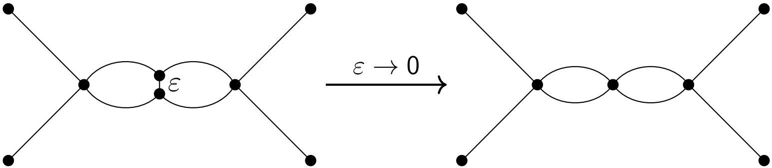
**Outline of the proof**:

1. The image of the biLipschitz embedding $X \to Y$ is a porous subset.



2. $d_{ARC}(Y) = \frac{3}{2} = d_{ARC}(X)$.

# Intuition why the uniformization fails




**Geometric intuition**

1. The porous subset $X$ has already attained its optimal geometry (it is Loewner).
2. The only way to optimize the geometry of $Y$ is to collapse the edge in middle.

**Analytic intuition**

1. Discrete modulus is supported on the a porous subset $X$.