# Construction of self-similar energy forms and self-similar energy measures on the Sierpinski carpet

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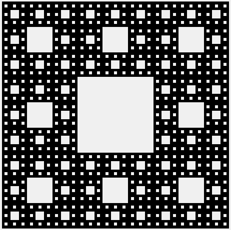
Joint work with Mathav Murugan (The University of British Columbia)

Fractal Geometry and Stochastics 7



*Supported by JSPS KAKENHI Grant Number JP20J20207, JP23KJ2011 

# $L^p$-type energy functionals $(1 < p < \infty)$

$p$-energy functional: $\mathcal{E}_p(u) := \int_{\mathbb{R}^n} |\nabla u|^p \, dx$

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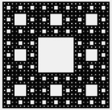
**Q.** Counterpart of $(\mathcal{E}_p, \mathcal{F}_p, \mu^p_{\langle \cdot \rangle})$ on  $=: K$ ?

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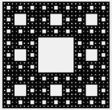
Q. Counterpart of $(\mathcal{E}_p, \mathcal{F}_p, \mu^p_{\langle \cdot \rangle})$ on  $=: K$ ?

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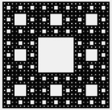
Q. Counterpart of $(\mathcal{E}_p, \mathcal{F}_p, \mu^p_{\langle \cdot \rangle})$ on  $=: K$ ?

Thm. $\forall p \in (1, \infty)$ $\exists$ nice $(\mathcal{E}_p, \mathcal{F}_p, \mu^p_{\langle \cdot \rangle})$ on $K$.

Previous results: [Kigami'23, S.'24'] for $p > d_{\mathrm{ARC}}$;

$d_{\mathrm{ARC}}$: Ahlfors regular conformal dimension

# Ahlfors regular conformal dimension

◇ $(K,d)$: metric space, $A \subset X$, $p > 0$.

$$\mathcal{H}^p_d(A) = \lim_{\varepsilon\to 0} \inf\left\{ \sum_i |U_i|_d^p \;\middle|\; A \subset \bigcup_i U_i, |U_i|_d < \varepsilon \right\},$$

where $|U_i|_d = \sup_{x,y\in U_i} d(x,y)$.

<u>**NB.**</u> $\dim_{\mathrm{H}}(K, d^{\varepsilon}) = \varepsilon^{-1}\dim_{\mathrm{H}}(K,d)$.

◇ $d_{\mathrm{ARC}}(K,d) := \inf\left\{ p \;\middle|\; \exists \theta \underset{\mathrm{QS}}{\sim} d,\, \mathcal{H}^p_\theta(B_\theta(x,r)) \asymp r^p \right\}$

◇ $d \underset{\mathrm{QS}}{\sim} \theta \overset{\text{def.}}{\iff} \exists \eta \colon \mathbb{R}_+ \to \mathbb{R}_+$ homeo. s.t.

$$\frac{\theta(x,y)}{\theta(x,z)} \le \eta\left(\frac{d(x,y)}{d(x,z)}\right) \quad \forall x,y,z \in X \text{ w/ } x \ne z.$$

# Attainment problem of $d_{\mathrm{ARC}}$

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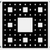
◇ $d_{\mathrm{ARC}}\big($  $\big) = ?$, but by [Bishop/Tyson'01, Keith/Laakso'04]

$1 + \frac{\log 2}{\log 3} \le d_{\mathrm{ARC}}\big($  $\big) < \frac{\log 8}{\log 3}.$

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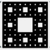
◇ $d_{\mathrm{ARC}}($$)$ $=$ ?, but by [Bishop/Tyson'01, Keith/Laakso'04]

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Q. Is $\inf$ attained? (Talk by R. Anttila)

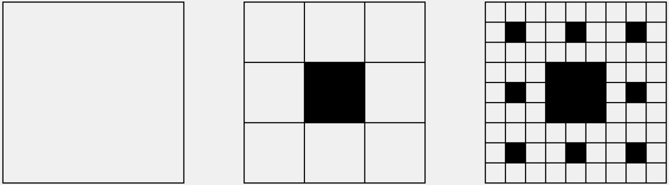
Conj. [Kajino/Murugan'23, Invent.] IF $\inf$ is attained

$\Rightarrow \exists h \in$ "$\mathcal{F}_p$": "$p$-harm. fcn." s.t. $\mathcal{H}^p_\theta \asymp$ "$\mu^p_{\langle h\rangle}$",

where $p = d_{\mathrm{ARC}}$, $\theta$: a metric attaining $d_{\mathrm{ARC}}$.

# How to get $p$-energy forms?

◇ **To get $\mathcal{F}_p$, find a candidate $\widehat{\mathcal{E}}_p$ via $\{\mathbb{G}_n\}_{n\in\mathbb{N}}$:**



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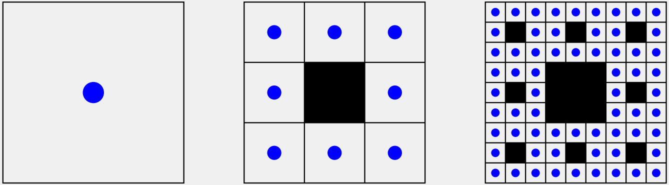

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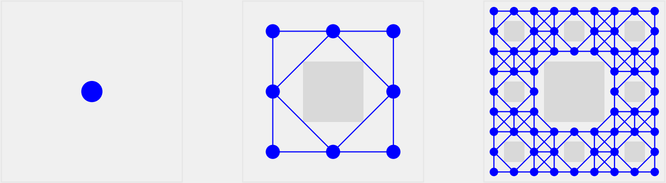

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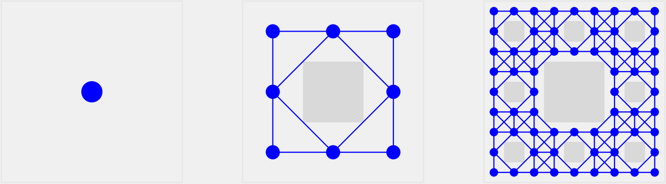

$m$: $\frac{\log 8}{\log 3}$-dim. Hausdorff measure w/ $m(K) = 1$.

$f_n(w) := \frac{1}{m(K_w)} \int_{K_w} f\,dm$, $w \in \mathbb{G}_n$, $f \in L^p(K, m)$.

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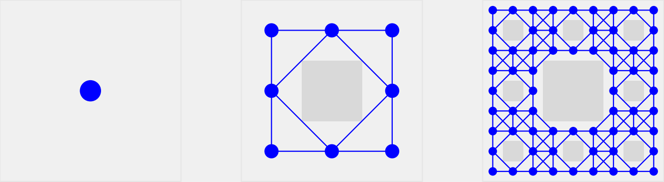

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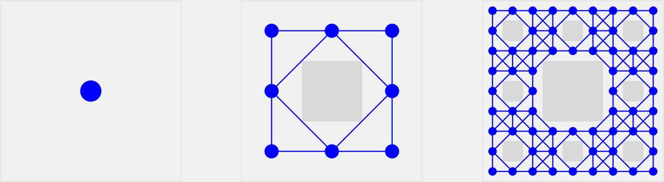

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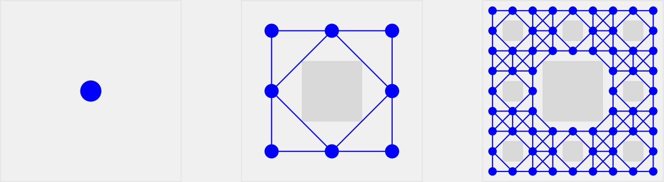

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**Idea** [Kusuoka/Zhou'92, PTRF]: **Find $\{r_n\}_{n\in\mathbb{N}}$ so that**

$$\varlimsup_{n\to\infty} r_n \mathcal{E}_p^{\mathbb{G}_n} \asymp \varliminf_{n\to\infty} r_n \mathcal{E}_p^{\mathbb{G}_n} \asymp \sup_{n\in\mathbb{N}} r_n \mathcal{E}_p^{\mathbb{G}_n}$$

# Correct scaling factor $\rho_p$

◇ $\mathcal{C}_p^{(n)} := \inf\left\{ \mathcal{E}_p^{\mathbb{G}_n}(f) \;\middle|\; f = \right.$  $\left.\right\}$, $n \in \mathbb{N}$.



[Bourdon/Kleiner'13, GGD]: $\exists C \geq 1$ s.t. $\forall k, l \in \mathbb{N}$,

$$C^{-1}\,\mathcal{C}_p^{(k)}\mathcal{C}_p^{(l)} \leq \mathcal{C}_p^{(k+l)} \leq C\,\mathcal{C}_p^{(k)}\mathcal{C}_p^{(l)}.$$

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$\rightsquigarrow$ $\rho_p := \lim_{n\to\infty} \left(\mathcal{C}_p^{(n)}\right)^{-1/n} \in (0, \infty)$ and

$$C^{-1} \rho_p^{-n} \leq \mathcal{C}_p^{(n)} \leq C \rho_p^{-n}, \quad \forall n \in \mathbb{N}.$$

<u>**Rmk.**</u> [Carrasco'13, Kigami'20]: $\rho_p > 1 \Leftrightarrow p > d_{\mathrm{ARC}}$

◇ $\widetilde{\mathcal{E}}_p^{n}(f) := \rho_p^n \mathcal{E}_p^{\mathbb{G}_n}(f_n), \quad \widehat{\mathcal{E}}_p(f) := \sup_{n\in\mathbb{N}} \widetilde{\mathcal{E}}_p^{n}(f),$

# Results of [Murugan/S.'23+]

**Hereafter, $\forall p \in (1, \infty)$ is fixed.**

**Thm.** $\widehat{\mathcal{E}}_p \asymp \varlimsup_{n\to\infty} \widetilde{\mathcal{E}}_p^n \asymp \varliminf_{n\to\infty} \widetilde{\mathcal{E}}_p^n$ **on** $L^p(K, m)$.

$\rightsquigarrow \mathcal{F}_p$ **w/** $\|\cdot\|_{L^p} + (\widehat{\mathcal{E}}_p)^{1/p}$**: ref. sep. Banach s.p.**

**Moreover,** $\mathcal{F}_p \cap C(K) \underset{\text{dense}}{\subset} C(K)$ **&** $\mathcal{F}_p$**.**

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**Rmk.** $\mathcal{F}_p \cap C(K) \underset{\text{dense}}{\subset} C(K)$ **is new for** $p \leq d_{\mathrm{ARC}}$**!**

**NB.** [Cao/Chen/Kumagai'24]: $\mathcal{F}_p \subset C(K) \Leftrightarrow p > d_{\mathrm{ARC}}$.

★ **Mod. of path-families to deal with** $p \leq d_{\mathrm{ARC}}$.

# Results of [Murugan/S.'23+] (contd.)

$d_{\mathrm{w},p} := \frac{\log(8\rho_p)}{\log 3}$ ($p$-walk dimension)

**Thm.** $d_{\mathrm{w},p} > p$ ([S.24]) and $\widehat{\mathcal{E}}_p \asymp \varlimsup_{r \downarrow 0} J_{p,r}$,

where

$$J_{p,r}(f) := \int_K \fint_{B(x,r)} \frac{|f(x) - f(y)|^p}{r^{d_{\mathrm{w},p}}}\, m(dy) m(dx)$$

Moreover, $\sup_{r>0} J_{p,r}(f) \lesssim \varliminf_{r \downarrow 0} J_{p,r}(f)$.

**NB.** $K \ni x \mapsto d_{\mathrm{Euc}}(x, x_0) \notin \mathcal{F}_p$!

# Results of [Murugan/S.'23+] (contd.)

**Thm.** $\exists(\mathcal{E}_p, \mathcal{F}_p, \mu^p_{\langle\,\cdot\,\rangle})$ on $K$:

- $\mathcal{E}_p \asymp \widehat{\mathcal{E}}_p$ & $\mathcal{E}_p(u \circ T) = \mathcal{E}_p(u)\, \forall (u, T) \in \mathcal{F}_p \times D_4$.
- [Kajino/S.'24+] $(\mathcal{E}_p, \mathcal{F}_p)$ satisfies **(GenCont)$_p$**
- $\mathcal{F}_p \cap C(K) = \{u \in C(K) \mid u \circ F_i \in \mathcal{F}_p\, \forall i \in S\}$,

& $\mathcal{E}_p(u) = \rho_p^n \sum_{w \in \{1,\dots,8\}^n} \mathcal{E}_p(u \circ F_w)$.

- $\mu^p_{\langle u \rangle}(K_w) = \rho_p^n \mathcal{E}_p(u \circ F_w)$.
- $\forall (u, \Psi) \in (\mathcal{F}_p \cap C(K)) \times C^1(\mathbb{R})$, we have

$$\Psi \circ u \in \mathcal{F}_p, \qquad d\mu^p_{\langle \Psi \circ u \rangle} = |\Psi'(u)|^p \, d\mu^p_{\langle u \rangle}.$$

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**Message: Optimal meas. should be** $p$**-energy meas.!**

**Conj.** $u$ **can be reduced to a** $p$**-harmonic fcn.**

**Issue: Nonlinearity of** $p$**-harmonic fcn.**

**cf. [Kajino/Murugan'23] for the case** $p = 2$**.**