Shrinking Targets versus Recurrence: the quantitative theory.

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Joint work with Jason Levesley (York), Bing Li (SCUT), and David Simmons (York)

General setup

Let (X, d) be a compact metric space and $(X, \mathcal{A}, \mu, T)$ be an ergodic, probability measure preserving system.

Given a real, positive function $\psi : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ let

$$R(T, \psi) := \left\{x \in X : T^n x \in B(x, \psi(n)) \text{ for i. m. } n \in \mathbb{N}\right\}$$

denote the associated **recurrent set**, and given a point $x_0 \in X$ let

$$W(T, \psi) := \left\{x \in X : T^n x \in B(x_o, \psi(n)) \text{ for i. m. } n \in \mathbb{N}\right\}$$

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  This means that the trajectories of almost all points will hit the 'constant' ball infinitely often. In view of this, it is natural to ask:

Question. What happens if the ball shrinks with time? More precisely.....

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Given $\psi : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ with that $\psi(n) \to 0$ as $n \to \infty$, what is the size of

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$+$ (convergent) Borel – Cantelli Lemma $\Longrightarrow$

$$\mu(W(T,\psi)) = 0 \quad \text{if} \quad \sum_{n=1}^{\infty} \mu(B_n) < \infty \ .$$

What happens when the sum diverges?

Shrinking Targets: a quantitative statement

Given $N \in \mathbb{N}$ and $x \in X$, consider the **counting function**

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T is a **exponentially mixing** with respect to μ; i.e. $\exists$ constants $C > 0$ and $\gamma \in (0,1)$ such that for any $n \in \mathbb{N}$, and any ball $B \in X$ and $F \in \mathcal{A}$,

$$\left|\mu(B \cap T^{-n}(F)) - \mu(B)\mu(F)\right| \leqslant C\gamma^n \mu(F). \tag{1}$$

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Theorem A

Let $(X, \mathcal{A}, \mu, T)$ be a measure-preserving dynamical system and suppose that T is exponentially mixing with respect to μ. Let $\psi : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ be a real, positive function. Then, for any given $\varepsilon > 0$, we have that

$$W(x, N) = \Phi(N) + O\left(\Phi^{1/2}(N) \, (\log \Phi(N))^{3/2+\varepsilon}\right)$$

for μ-almost all $x \in X$, where

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Shrinking Targets: main step for quantitative statement

Proposition 1

For arbitrary $a, b \in \mathbb{N}$ *with* $a < b$,

$$2 \sum_{a \leq m < n \leq b} \sum \mu(E_m \cap E_n) \leqslant \left(\sum_{n=a}^{b} \mu(E_n) \right)^2 + O\left(\sum_{n=a}^{b} \mu(E_n) \right).$$

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Since $E_n = T^{-n}(B_n)$, T is measure preserving and exponentially mixing:

$$\mu(E_m \cap E_n) = \mu\Big(T^{-m}(B_m) \cap T^{-n}(B_n)\Big) = \mu\Big(B_m \cap T^{-(n-m)}(B_n)\Big)$$

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This completes the proof since the sum involving γ is convergent.

Shrinking Targets: the zero-one criterion

Theorem A $\implies$ $\lim_{N\to\infty} W(x,N) = \infty$ for μ–almost all $x \in X$ if the measure sum $\Phi := \lim_{N\to\infty} \Phi(N)$ diverges.

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Theorem B

Let $(X, \mathcal{A}, \mu, T)$ be a measure-preserving dynamical system and suppose that T is exponentially mixing with respect to μ. Let $\psi : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ be a real, positive function. Then

$$\mu(W(T,\psi)) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} \mu(B_n) < \infty \\ 1 & \text{if } \sum_{n=1}^{\infty} \mu(B_n) = \infty. \end{cases}$$

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Under additional assumptions, analogues of Theorem B for the recurrent set $R(T,\psi)$ have been established in numerous works (eg. recently Baker-Farmer (2021), Hussian-Li-Simmons-Wang (2022), Kirsebom-Kunde-Persson (2023), He-Liao (2023), ...)

Recurrent Set: the zero-one criterion

If one of the additional assumptions (beyond exponentially mixing) is that μ is δ-**Ahlfors regular** (as in BF & HLSW) then $\mu(B_n) \asymp \psi^\delta(n)$ and the conclusion of the analogous of Theorem B for $R(T,\psi)$ reads:

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The "desirable" statement would be: for μ-almost all $x \in X$

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We establish (3) for a class of piecewise linear maps.

Our result in one-dimension

The following constitutes our main one dimensional result.

Theorem 1

Let $T : [0,1] \to [0,1]$ be a piecewise linear map sending each interval of linearity to $[0,1]$. Let $\psi : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ be a real, positive function. Then, for any given $\varepsilon > 0$, we have that

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Apparently new even for $T : x \to 2x \mod 1$.

Key properties of one-dimensional set up.

Let J_m be an interval of linearity for T^m; i.e., $J_m \in \mathcal{P}_m$ - the collection of cylinder sets of order m that partition $[0, 1]$.

Let $K_{J_m} := 1/\mu(J_m)$, where μ is Lebsegue measure. Then $T^m|_{J_m}$ is a similarity with dilatation factor K_{J_m}. In particular, $|(T^m)'(y)| = K_{J_m}$ for any $y \in J_m$.

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- T is expanding: there exists a $\lambda > 1$ such that $|T'(x)| \geqslant \lambda$ for all $x \in [0,1]$. It follows that for any $m \in \mathbb{N}$
$$K_{J_m} \geqslant \lambda^m .$$
- T is a measure preserving transformation with respect to μ; i.e. μ is a T-invariant probability measure. For any measurable set $F \subseteq J_m$
$$\mu(T^m(F)) = K_{J_m}\mu(F)$$
- T is a exponentially mixing with respect to μ; i.e. there exists a constants $C > 0$ and $0 < \gamma < 1$ such that for any $n \in \mathbb{N}$, and any ball $B \in [0,1]$ and measurable set $F \in [0,1]$,
$$\mu(B \cap T^{-n}(F)) = \mu(B)\mu(F) + O(\gamma^n)\mu(F) .$$

The setup and result in higher dimensions

Let $T : [0,1]^d \to [0,1]^d$ be a piecewise linear map sending each rectangle of linearity to $[0,1]^d$; i.e., we allow the collection $\mathcal{P}_m$ of cylinder sets J_m of order m to be rectangles.

Given $N \in \mathbb{N}$ and $\mathbf{x} = (x_1, \ldots, x_d) \in [0,1]^d$, let

$$R(\mathbf{x}, N; T, \psi) := \#\Big\{1 \leqslant n \leqslant N : \mathrm{dist}(x_i, T^n(\mathbf{x})_i) \leq \psi_i(n) \quad \forall\ 1 \leqslant i \leqslant d\Big\}.$$

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Analogue of Theorem 1: *for any given $\varepsilon > 0$, we have that*

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for μ-almost all $x \in X$, where

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I will just concentrate on one-dimensional statement.

A mechanism for establishing counting results.

A quantitative form of the (divergence) Borel-Cantelli Lemma.

Lemma H (Harman: Lemma 1.5)

Let $(X, \mathcal{A}, \mu)$ be a probability space, let $(f_n(x))_{n\in\mathbb{N}}$ be a sequence of non-negative μ-measurable functions defined on X, and $(f_n)_{n\in\mathbb{N}}$, $(\phi_n)_{n\in\mathbb{N}}$ be sequences of real numbers such that

$$0 \leq f_n \leq \phi_n \qquad (n = 1, 2, \ldots).$$

Suppose that for arbitrary $a, b \in \mathbb{N}$ with $a < b$, we have

$$\int_X \left(\sum_{n=a}^{b} (f_n(x) - f_n) \right)^2 \mathrm{d}\mu(x) \leq C \sum_{n=a}^{b} \phi_n \tag{4}$$

for an absolute constant $C > 0$. Then, for any given $\varepsilon > 0$, we have

$$\sum_{n=1}^{N} f_n(x) = \sum_{n=1}^{N} f_n + O\left(\Phi(N)^{1/2} \log^{\frac{3}{2}+\varepsilon} \Phi(N) + \max_{1\leq k\leq N} f_k \right) \tag{5}$$

for μ-almost all $x \in X$, where $\Phi(N) := \sum_{n=1}^{N} \phi_n$.

Bounding the variance is the key

In statistical terms, if the sequence f_n is the mean of $f_n(x)$; i.e.

$$f_n = \int_X f_n(x) \mathrm{d}\mu(x),$$

then the l.h.s. of (4); namely

$$\int_X \left(\sum_{n=a}^{b} (f_n(x) - f_n) \right)^2 \mathrm{d}\mu(x),$$

is simply the variance $\mathrm{Var}(Z_{a,b})$ of the random variable

$$Z_{a,b} = Z_{a,b}(x) := \sum_{n=a}^{b} f_n(x).$$

In particular,

$$\mathrm{Var}(Z_{a,b}) = \mathbb{E}(Z_{a,b}^2) - \mathbb{E}(Z_{a,b})^2 \quad \text{where} \quad \mathbb{E}(Z_{a,b}) = \int_X Z_{a,b}(x) \mathrm{d}\mu(x).$$

Proving Theorem 1

We consider Lemma H with

$$X := [0, 1]\,, \qquad f_n(x) := \chi_{A_n}(x) \qquad \text{and} \qquad f_n := \phi_n := \ \mu(A_n)\,,$$

where $\chi_{A_n}$is the characteristic function A_n. Then, by definition f_n is the mean of $f_n(x)$ and for any $x \in X$ and $N \in \mathbb{N}$ we have that the

$$\text{l.h.s. of } (5) \ = \ R(x, N)\,.$$

Also, the main term on the r.h.s. of (5) is

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Furthermore, it can be verified that for any $a, b \in \mathbb{N}$ with $a < b$

$$\text{l.h.s. of } (4) \ = \sum_{n=a}^{b} \mu(A_n) \ + \ 2 \sum\sum_{a \le m < n \le b} \mu(A_m \cap A_n) - \left(\sum_{n=a}^{b} \mu(A_n) \right)^2 \tag{6}$$

UPSHOT: in view of Lemma H, the proof boils down to 'appropriately' estimating the r.h.s. of (6) and showing $\Phi(N)$ can be replaced by $\Psi(N)$.

Proving Theorem 1: continued

Estimating the measure of the intersection of the sets A_n is where the main difficulty lies. The following is at the heart.

Proposition 1. *For arbitrary $a, b \in \mathbb{N}$ with $a < b$,*

$$2 \sum\sum_{a \leq m < n \leq b} \mu(A_m \cap A_n) \leqslant \left(\sum_{n=a}^{b} \mu(A_n) \right)^2 + O\left(\sum_{n=a}^{b} \mu(A_n) \right).$$

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With this at hand, it follows that

$$\text{r.h.s. of } (6) \ll \sum_{n=a}^{b} \mu(A_n) := \sum_{n=a}^{b} \phi_n \quad (\phi_n := f_n).$$

So (4) is satisfied and Lemma H implies: for μ-almost all $x \in X$.

$$R(x, N) = \Phi(N) + O\left(\Phi^{1/2}(N) \, (\log \Phi(N))^{3/2+\varepsilon} \right), \tag{7}$$

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The following enables us to replace $\Phi(N)$ by $\Psi(N) := 2\sum_{n=1}^{N} \psi(n)$.

Proposition 2. *For arbitrary $N \in \mathbb{N}$,*

$$\sum_{n=1}^{N} \mu(A_n) = 2\sum_{n=1}^{N} \psi(n) + O(1).$$

Filling in the "holes": proving Propositions 1 & 2

The following provides a mechanism for "locally" representing A_n as the inverse image of a ball.

Lemma 1. *Let* $B := B(z, r)$ *be a ball centred at* $z \in X$ *and radius* $r > 0$*. Then for any* $m \in \mathbb{N}$ *with* $\psi(m) > r$

$$B \cap T^{-m}\big(B\,(z, \psi(m) - r)\big) \;\subset\; B \cap A_m \;\subset\; B \cap T^{-m}\big(B\,(z, \psi(m) + r)\big)$$

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Recall: $T : X \to X$ is piecewise linear map sending intervals of linearity to X. Let J_m be an interval of linearity for T^m. Then there is a unique fixed point z_{J_m} of T^m on J_m. Let $K_{J_m} = 1/|J_m|$. The following (a special case of the Lemma 1 with $B = J_m$) is key.

Lemma 2. *For any $m \in \mathbb{N}$,*

$$J_m \cap A_m := I_{J_m} = J_m \cap B\big(z_{J_m}, (K_{J_m} \pm 1)^{-1}\psi(m)\big)\,,$$

where it is "plus" if T^m is decreasing on J_m and "minus" otherwise.