Quasi-isometric equivalence of Galton-Watson trees

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Fractal Geometry and Stochastics 7, Chemnitz

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Overview

Motivating Question

How similar are two independent realisations of a stochastically self-similar set?

Structure:

- What are quasi-isometries?
- Random graphs: Lattice & tree
- Applications to stochastically self-similar sets
- The proofs: parking cars on graphs

Krishna Athreya (1939–2023)

Jayadev Athreya

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Quasi-isometries

Quasi-isometries

Definition 1

Let (X, d), (Y, ρ) be metric spaces. We say $\phi : X \hookrightarrow Y$ is a (A, B)-**quasi-isometric embedding** if for all $x, y \in X$,

$$\frac{1}{A} \cdot \rho(\phi(x), \phi(y)) - B \leq d(x, y) \leq A \cdot \rho(\phi(x), \phi(y)) + B,$$

We say ϕ is an (A, B, C)-**quasi-isometry (QI)** if it is an (A, B)-quasi-isometric embedding and there exists $C > 0$ s.t. for all $y \in Y$ there exists $x \in X$ such that

$$\rho(\phi(x), y) \leq C.$$

A quasi-isometry is a ‘weak’ notion of geometric similarity that represents ‘coarse’ bi-Lipschitz mappings.

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Some remarks

REMARK 1

If there exists a quasi-isometry $\phi : X \to Y$ then there exists a quasi-isometry $\psi : Y \to X$.

Quasi-isometries define an equivalence relation: $X \sim_{qi} Y$.

EXAMPLE 2

Quasi-isometries are 'coarse' bi-Lipschitz maps:

- $\mathbb{Z}^d \sim_{qi} \mathbb{R}^d$
- All Delone sets $F \subset X$ satisfy $F \sim_{qi} X$.
- All bounded sets F are in the same equivalency class: $F \sim_{qi} \{x\}$

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Relevance

Quasi-isometries are a major tool in the study of groups (geometric group theory).

We may study Cayley graphs of groups $(G, \cdot)$ using quasi-isometries with the graph metric.

Many interesting properties are invariant under quasi-isometries:

- Hyperbolicity,
- Growth rate,
- Amenability,
- $\ldots$

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Random graphs

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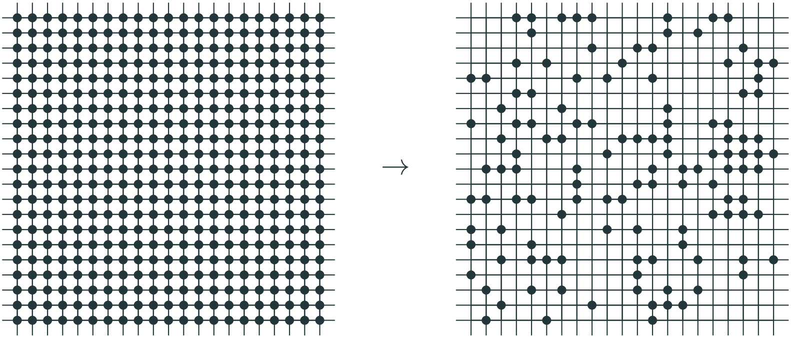
The percolated lattice

Consider $\mathbb{Z}^d$ (either as $\subset \mathbb{R}^2$ or $(\mathbb{Z}^d, +)$).

Let $p \in [0,1]$ and let $X_g \equiv \mathrm{Ber}(1-p)$, be iid random variables indexed by $g \in \mathbb{Z}^d$. X_g determines when g is discarded:

The percolated lattice is $Z(\omega) = Z_p(\omega) = \{g \in \mathbb{Z}^d : X_g = 1\}$ for $\omega \in \{0,1\}^{\mathbb{Z}^d}$.



A stunning structural statement

Some easy observations:

- For $p \in (0,1)$ there are (almost surely) arbitrarily large 'holes'. Hence $Z(\omega) \not\sim_{qi} \mathbb{Z}^d \sim_{qi} \mathbb{R}^d$ (a.s.).
- 'Density' of large holes different for different $0 < p < q < 1$. Hence $Z_p(\omega) \not\sim_{qi} Z_q(\omega')$ (a.s.) for independent ω, ω'.

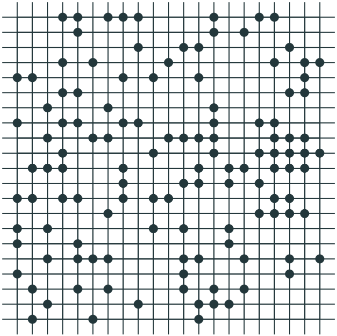
Theorem 3 (Basu-Sly '14, Basu-Sidoravicius-Sly '18)

Fix $p \in [0,1]$. Then, for a.e. choice of independent $\omega, \omega' \in \Omega$,

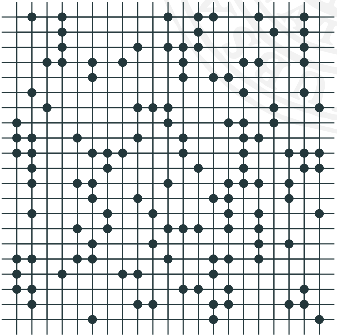
$$Z(\omega) \sim_{qi} Z(\omega').$$

Their result is much more general using "pattern matching".

Some pictures



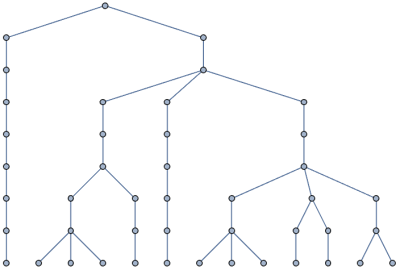
$\sim_{qi}$



Galton-Watson trees

Let X be a bounded, positive, integer-valued random variable. Write $p_j = \mathbb{P}(X = j)$.

We write $T = T(\omega)$ for the random where the 'offspring' has distribution X.



TRIVIAL FACT

Let T_n be the n-tree. Then $T_n \sim_{qi} T_m$ for all $2 \leq n \leq m$.

Hence $T(\omega) \sim_{qi} T(\omega')$ almost surely if $p_1 = 0$.

The problem is $p_1 > 0$:
Expanding trees cannot be embedded into long lines.

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Results

Theorem 4 (Athreya-T. '24+)

Let T be a Galton-Watson tree with bounded offspring distribution $((\exists N)(\forall n > N)p_n = 0)$. Conditioned on $\operatorname{diam} T = \infty$,

$$T(\omega) \sim_{qi} T(\omega') \quad \text{for a.e. independent} \quad \omega, \omega'.$$

Corollary 5

Let F be a self-similar set satisfying the SSC and let $F(\omega)$ be fractal percolation on F. Then, $F(\omega)$ and $F(\omega')$ are quasisymmetrically equivalent for a.e. ω, ω'.

Before we start: it suffices[1] to consider $p_1 + p_2 = 1$.

[1] Terms and Conditions apply.

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Parking on the binary tree

The car parking problem

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The car parking problem

THEOREM 6 (ALDOUS-CONTAT-CURIEN-HÉNARD '23)

Let $G(t)$ be the generating function of X. Suppose there exists $t_c \in (0, \infty)$ such that

$$t_c = \min\{t \geq 0 : 2(G(t) - tG'(t))^2 = t^2 G(t)G''(t)\}$$

Then the parking process is subcritical if and only if

$$(t_c - 2)G(t_c) \geq t_c(t_c - 1)G'(t_c).$$

EXAMPLE 7 (GEOMETRIC CARS)

If $\mathbb{P}(X = k) = q^k(1 - q)$ then the car parking process is critical if and only if $E(X) \leq \frac{1}{8}$.

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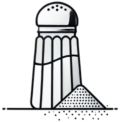
Connecting cars and quasi-isometries



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