Gaussian vs. Sub-Gaussian Behaviour of Random Conductance Models

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Fractal Geometry and Stochastics 7
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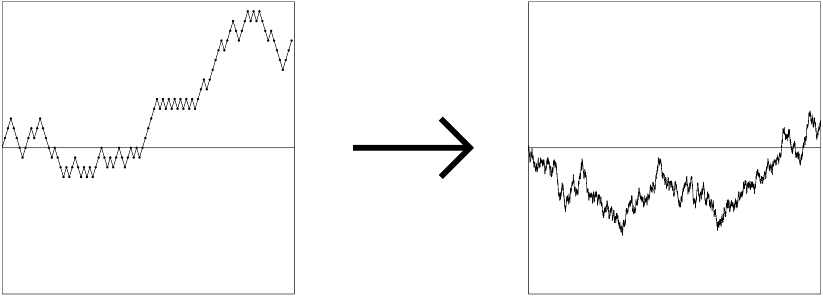
Donsker's Invariance Principle

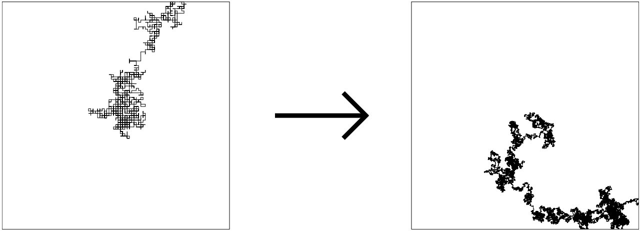


Let $(X_t)_{t\geq 0}$ be the simple random walk (linearly interpolated). Then

$$\left(n^{-1}X_{n^2 t}\right)_{t\geq 0} \underset{n\to\infty}{\Longrightarrow} \left(W_t\right)_{t\geq 0} \qquad \text{with } W \text{ Brownian motion.}$$

Scaling Limit for the Simple Random Walk

The same convergence happens in all dimensions.

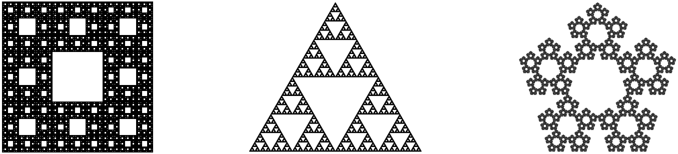


- Brownian motion is the continuous counterpart of the simple random walk.
- Brownian motion is a Gaussian process. Its transition probability densities $P_x[W_t \in dy] = p_t(x,y)\,dy$ are given by the Gaussian heat kernel

$$p_t(x,y) \;=\; \frac{1}{\sqrt{(2\pi)^d}}\, t^{-d/2} \exp\left(-\frac{|x-y|^2}{t}\right).$$

Heat kernel behaviour on fractals

For Brownian motion on fractal spaces like



we have sub-Gaussian heat kernel behaviour

$$p_t(x,y) \asymp c\, t^{-\alpha/\beta} \exp\left(-c \left(\frac{d(x,y)^\beta}{t} \right)^{\frac{1}{\beta-1}} \right), \quad \beta > 2$$

(Barlow-Perkins '88; Kumagai '93; Fitzsimmons-Hambly-Kumagai '94, Barlow-Bass '92, '99)

with

- $\alpha =$ Hausdorff-dimension
- $\beta =$ walk dimension

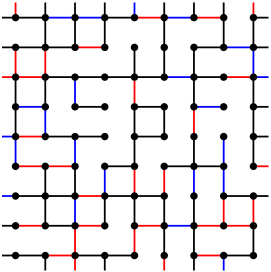
$$E_x\big[d(x, W_t)\big] \asymp t^{2/\beta}.$$

The Random Conductance Model (RCM)

Intuitive description

- Put random conductances (or weights) $\omega_e \in [0, \infty)$ on the edges of the Euclidean lattice $(\mathbb{Z}^d, E_d)$, $d \geq 2$.
- Look at a continuous time Markov chain X_t with jump probabilities proportional to the edge conductances. Then the jump probability from x to $y \sim x$ is

$$P_{xy} = \frac{\omega_{xy}}{\sum_{z \sim x} \omega_{xz}}.$$



Bond conductivities: blue $\ll 1$, black ≈ 1, red $\gg 1$.

Definitions

Environment. Let

- $\Omega = [0, \infty)^{E_d}$ be the space of environments,
- $\mathbb{P}$ be a probability law on Ω which makes the coordinates $(\omega_e)_{e \in E_d}$ stationary ergodic random variables.

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Random walk. For $\omega \in \Omega$ and $x \in \mathbb{Z}^d$ let P_x^ω be the law of the random walk $(X_t)_{t\geq 0}$ on $\mathbb{Z}^d$ starting in x with generator

$$\mathcal{L}^\omega f(x) = \frac{1}{\mu_x^\omega} \sum_{y\sim x} \omega_{xy} \big(f(y) - f(x)\big), \qquad \mu_x^\omega := \sum_{y\sim x} \omega_{xy}.$$

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Heat kernel. Let

$$p_t^\omega(x,y) = \frac{P_x^\omega(X_t = y)}{\mu_y^\omega} = p_t^\omega(y,x).$$

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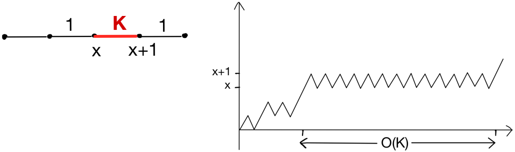

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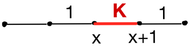

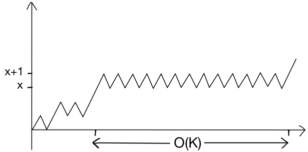

Problems

- Quenched invariance principle (QIP): For $\mathbb{P}$-a.a. ω, under P_0^ω, $\left(n^{-1}X_{n^2t}\right)_{t\geq 0} \underset{n\to\infty}{\Longrightarrow} \left(\Sigma \cdot W_t\right)_{t\geq 0}$ with W Brownian motion on $\mathbb{R}^d$.
- Quenched local limit theorem:.

$$n^d p^\omega_{n^2t}(0, \lfloor nx \rfloor) \underset{n\to\infty}{\longrightarrow} P_0\left[\Sigma \cdot W_t \in dx\right] / \mathbb{E}[\mu_0^\omega], \qquad \mathbb{P}\text{-a.s.}$$

- Gaussian bounds: $p_t^\omega(x,y) \asymp c\, t^{-d/2} exp\left(-c|x-y|^2/t\right)$.

Results in the i.i.d. case

Theorem (A.-Barlow-Deuschel-Hambly '13)

Let $d \geq 2$ and $(\omega_e)_{e \in E_d}$ be i.i.d. with $\omega_e \geq 0$ $\mathbb{P}$-a.s. and $\mathbb{P}(\omega_e > 0) > p_c$. Then, QIP holds with $\Sigma = \sigma \,\mathrm{Id}$ and $\sigma > 0$ iff $\mathbb{E}\big[\omega_e\big] < \infty$.

Previous results: Sidoravicius-Sznitman '04; Berger-Biskup '07; Mathieu-Piatnitski '07; Biskup-Prescott '07; Mathieu '08; Barlow-Deuschel '10.

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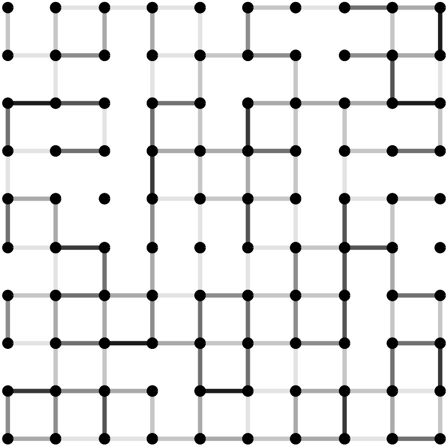
Gaussian bound and a local limit theorem hold e.g. in the case of

- 'Uniformly elliptic'conductances: $0 < c_1 \leq \omega_e \leq c_2 < \infty$ (Delmotte '99; Barlow-Hambly '09).
- SRW on i.i.d. percolation clusters (Barlow '04; Barlow-Hambly '09)

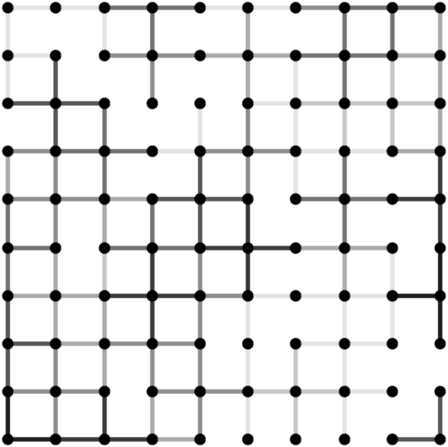
But: For $\omega_e \in [0,1]$ i.i.d., sub-Gaussian heat kernel decay can occur due to trapping effects, so

Gaussian bounds and local limit theorem may fail!

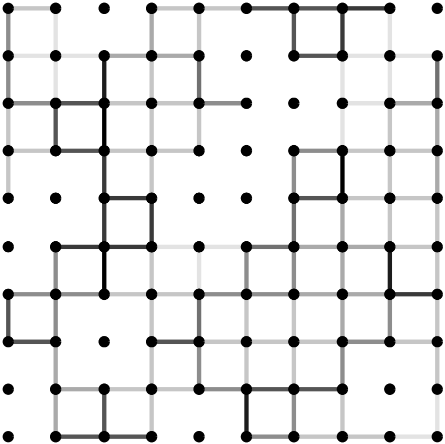
(Berger-Biskup-Hoffmann-Kozma '08; Boukhadra-Kumagai-Mathieu '14)



$\omega_e \in [0,1]$ i.i.d. with $\mathbb{P}[\omega_e \leq t] \sim t$



$\omega_{xy} = \lambda(x) \vee \lambda(y)$, $\lambda(x) \in [0,1]$ i.i.d. with $\mathbb{P}[\lambda(x) \le t] \sim t$



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QIP for ergodic environments

We need moment conditions!

- QIP if $\mathbb{E}[\omega_e] < \infty$ and $\mathbb{E}[\omega_e^{-1}] < \infty$ in $d = 2$ (Biskup '11).
- Example with $\mathbb{E}[\omega_e^p \vee \omega_e^{-p}] < \infty$, $p < 1$, for which the QIP fails (Barlow-Burdzy-Timár '13).

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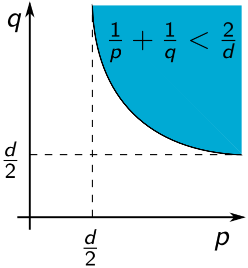
Theorem (A.-Deuschel-Slowik '15)

Assume $\mathbb{E}\big[(\omega_e)^p\big] < \infty$ *and* $\mathbb{E}\big[(\omega_e)^{-q}\big] < \infty$ *for* $p, q \in (1, \infty]$ *such that* $1/p + 1/q < 2/d$. *Then QIP holds.*



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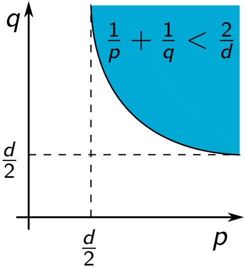

- Improved moment condition $1/p + 1/q < 2/(d-1)$ (Bella-Schäffner '20).

Quenched local limit theorems

Suppose $\mathbb{E}\big[(\omega_e)^p\big] < \infty$ and $\mathbb{E}\big[(\omega_e)^{-q}\big] < \infty$ with $1/p + 1/q < 2/d$.

- Quenched local limit theorem (A.-Deuschel-Slowik, '16; A.-Taylor '21; Bella-Schäffner '22)
  - The moment condition is sharp!
  - The proof requires
    - (i) QIP,
    - (ii) Hölder regularity of the heat kernel, deduced from a parabolic Harnack inequality.
- Quantitative local limit theorem with optimal rates of convergence on i.i.d. percolation clusters. (Dario-Gu '21)
- Upper Gaussian bounds (A.-Deuschel-Slowik '16,'19)

Results for ergodic time-dynamic environments

Moment condition:

Suppose $\mathbb{E}[\omega_t(e)^p] < \infty$ and $\mathbb{E}[\omega_t(e)^{-q}] < \infty$, for any $e \in E_d$, $t \in \mathbb{R}$, with $p, q \in (1, \infty]$ satisfying $\frac{1}{p-1} \cdot \frac{q+1}{q} + \frac{1}{q} < \frac{2}{d}$.

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Theorem (QIP; A.-Chiarini-Deuschel-Slowik '18)

Under the above moment condition the QIP holds with a deterministic, non-degenerate covariance matrix Σ^2.

Further results:

- $0 \leq \omega_t(e) \leq 1$ (Biskup-Rodriguez '18).
- $0 < c_1 \leq \omega_t(e) \leq c_2$ and mixing (A. '14).

Theorem (Quenched local limit theorem; A.-Chiarini-Slowik '21)

Under the above moment condition, for all $K > 0$ and $0 < T_1 \leq T_2$,

$$\lim_{n\to\infty} \sup_{|x|\leq K} \sup_{t\in[T_1,T_2]} \left| n^d p^\omega_{0,n^2 t}(0, \lfloor nx \rfloor) - p^\Sigma_{\mathrm{BM}}(t, 0, x) \right| = 0, \qquad \mathbb{P}\text{-a.s.}$$

Application: $\nabla\phi$-Interface Models

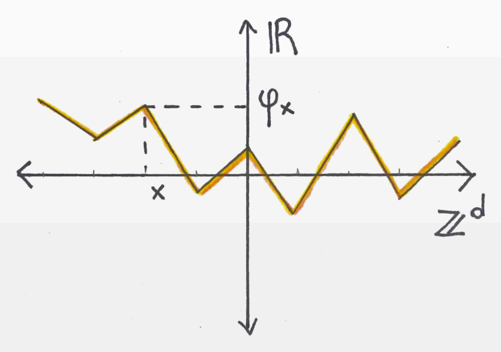
Interfaces are ubiquitous in statistical physics:

- seperation of media (water-oil solution),
- separation of phases (water-ice at freezing temperature)
- alloys consisting of two types of metal
- ...

Confluence of the Rhone and Arve Rivers (Geneva, Switzerland)

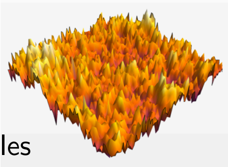
picture provided by A. Chiarini

Mathematical model



- A d-dimensional interface is the graph of a function $\varphi : \mathbb{Z}^d \to \mathbb{R}$.
- $\varphi_x = \varphi(x)$ is the height of the interface at $x \in \mathbb{Z}^d$.

Ginzburg-Landau $\nabla\phi$-interface model



- The interface is specified by a field of height variables $\phi_t(x)$, $x \in \mathbb{Z}^d$, $t \geq 0$, given by

$$d\phi_t(x) = -\sum_{y \sim x} V'(\phi_t(x) - \phi_t(y))\, dt + \sqrt{2} dw_t(x),$$

with

  - $\{w(x), x \in \mathbb{Z}^d\}$ collection of independent Brownian motions,
  - potential $V \in C^2(\mathbb{R}, \mathbb{R}_+)$ even and strictly convex $0 < c_- \leq V'' \leq c_+$.

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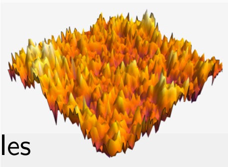

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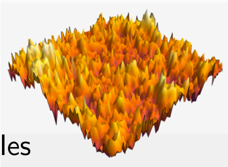

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- Formal Gibbs measure

$$\mu = \frac{1}{Z} \exp(-H(\phi)) \prod_{x\in\mathbb{Z}^d} d\phi(x), \qquad \text{on } \mathbb{R}^{\mathbb{Z}^d}$$

  with formal Hamiltonian $H(\phi) = \frac{1}{2}\sum_{x\sim y} V(\phi(x) - \phi(y))$.

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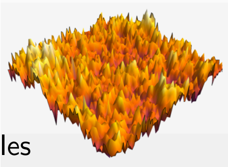

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- Example: $V(x) = \frac{1}{2}x^2$ discrete Gaussian free field

Scaling limit for the space-time covariances

Helffer-Sjöstrand representation:

$$\mathrm{cov}_\mu(\phi_0(0), \phi_t(x)) = \int_0^\infty \mathbb{E}_\mu\big[p^{\nabla\phi}_{0,t+s}(0,x)\big]\, ds.$$

where $p^{\nabla\phi}$ denotes the heat kernel of the dynamic RCM with

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Theorem (A.-Taylor '21)

Let $d \geq 3$ and $V'' \geq c_-$. There exists $p \in (1,\infty)$ such that if $\mathbb{E}_\mu[V''(\nabla\phi_t(e))^p] < \infty$,

$$\lim_{n\to\infty} n^{d-2}\,\mathrm{cov}_\mu\big(\,\phi_0(0),\, \phi_{n^2 t}(\lfloor nx \rfloor)\,\big) = \int_0^\infty p^{\Sigma}_{\mathrm{BM}}(t+s, 0, x)\, ds.$$

Example. Anharmonic crystal potential $V(x) = x^2 + \lambda x^4$ (see Bricmont-Fontaine-Lebowitz-Spencer '80, '81).

RCM with long-range jumps

Moment condition: There exist $p, q \in (1, \infty]$ satisfying $1/p + 1/q < 2/d$ such that

$$\mathbb{E}\Big[\Big(\sum_{x\in\mathbb{Z}^d} \omega(0,x)\,|x|^2\Big)^p\Big] < \infty \quad \text{and} \quad \mathbb{E}\big[\omega(0,x)^{-q}\big] < \infty \text{ whenever } |x| = 1.$$

In particular, $\omega(x, y) > 0$ $\mathbb{P}$-a.s. for all $x \sim y$.

Results:

- QIP (Biskup-Chen-Kumagai-Wang '21)
- Quenched local limit theorem (Chen-Kumagai-Wang '24, A.-Slowik '24+)
- RCMs with stable-like jumps, i.e. conductances i.i.d. of the form

$$\omega(x,y) = \frac{\tilde{\omega}(x,y)}{|x-y|^{d+\alpha}}, \quad \alpha \in (0,2).$$

Convergence towards symmetric α-stable Lévy process (Crawford-Sly '13, Chen-Kumagai-Wang '21, Berger-Tokushige '24)

New direction: RCM on fractals

- Annealed Scaling limit for random walks on (pre-)Sierpinski gasket graph under uniformly elliptic i.i.d. conductances uniformly bounded from below (Kumagai-Kusuoka '96)
- improved to i.i.d. conductances with upper moment condition and bounded from below in (Croydon-Hambly-Kumagai '17)
- Quenched two-sided subdiffusive heat kernel bounds with polylogarithmic corrections (Kajino-Slowik-Wille '24+)

Open problems:

- Quenched scaling limits.
- Quenched local limit theorems
- ...

Thank you for your attention!